CHAPTER VIII

DIMENSION AND COHOMOLOGY

The purpose of this chapter is to investigate dimension theory from the point of view of algebraic topology. This point of view is originally due to P. Alexandroff [2] and has been greatly developed by his school [1]. Here we shall not try to cover the extensive developments in this field which could easily be the subject of another book. We shall only aim at a characterization of dimension in terms of cohomology adopting the method due to C. H. Dowker [1] which goes back to E. Čech [3]. Recent investigations on the subject show that for non-compact spaces cohomology is more practical than homology, and that is the reason why we deal mainly with cohomology in this chapter. The final section will contain a brief account of homology in dimension theory. Throughout the chapter every space considered will be a paracompact T_2-space unless the contrary is explicitly stated, though some discussions are valid for more general spaces. [2]

VIII. 1. Homology group and cohomology group of a complex

Definition VIII. 1. *Let* E *be a set of elements which are called (abstract) vertices and* K *a collection of finite subsets of* E *such that every subset of a*

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[1] See P. Alexandroff [6], [8].
[2] K. Morita [8] gave a cohomological characterization of dimension of general topological spaces.

set belonging to K *also belongs to* K; *then we call* K *an (abstract) complex. We call a set of* $n+1$ *vertices* $a_0,\dots,a_n$ *belonging to* K *an (abstract) n-simplex or n-dimensional simplex of* K *and denote it by* $|a_0,\dots,a_n|$[3]. *An m-simplex* $|a_{i_0},\dots,a_{i_m}|$ *whose vertices are chosen from* $\{a_0,\dots,a_n\}$ *is called an m-face of* $|a_0,\dots,a_n|$. *If* K *contains an n-simplex, but no n'-simplex for* $n'>n$, *then* K *is called an n-complex or n-dimensional complex. A complex* K *is called a finite complex if it contains only finitely many simplices and is called a star-finite complex if every simplex of* K *is a face of at most finitely many simplices of* K.

Example VIII. 1. Let K_0 be a geometrical complex [4] in a Euclidean space and E the set of the vertices of K_0.

We say that $n+1$ vertices of E form an n-simplex if and only if they are the vertices of an n-dimensional (geometrical) simplex belonging to K_0. Then we get an abstract complex K. Generally every finite abstract n-complex K is *isomorphic* to a geometrical complex K_0 in E^{2n+1}, i.e. the vertices of K are mapped onto the vertices of K_0 by a one-to-one mapping which gives rise to a one-to-one correspondence between the simplices of K and those of K_0. K_0 is called an n-*polyhedron* if it is regarded as a point set in the Euclidean space. In the case where K is infinite we can realize K as a metric space in the following way. Let V be the set of all vertices of K. Put $K_0=\{x \mid x$ is a real-valued function on V such that $0\leq x\leq 1$, $\sum_{p\in V}x(p)=1$, $\{p\in V \mid x(p)>0\}$ is a simplex of $K\}$. The distance between two points x and y of K_0 is defined by $\sum_{p\in V}|x(p)-y(p)|$. We shall call K_0 the *realization* of K. For each $x\in K_0$, $\{x(p) \mid p\in V\}$ is called the *barycentric coordinate* of x. Let $s=|a_0,\dots,a_n|\in K$. Then the subset $K_0(s)=\{x\in K_0 \mid x(p)=0$ if $p\neq a_0,\dots,a_n\}$ of K_0 is homeomorphic to the geometrical simplex $[a_0,\dots,a_n]$, and $K_0=\bigcup\{K_0(s) \mid s\in K\}$. Where there is no fear of confusion the same notation is used for an abstract complex and for its realization.

[3] On the other hand, we shall denote by $[a_0,\dots,a_n]$ the geometrical simplex with the vertices $a_0,\dots,a_n$, where $a_0,\dots,a_n$ are points of E^m in a general position and $[a_0,\dots,a_n]=\{x\in E^m \mid x=\sum_{i=0}^n \lambda_i a_i,\ \sum_{i=0}^n\lambda_i=1,\ 0\leq\lambda_i\leq 1$ for $i=0,\dots,n\}$.

[4] A finite set K_0 of geometrical simplices in Euclidean space is called a *geometrical complex* if (i) every face of a simplex belonging to K_0 also belongs to K_0. (ii) the intersection of two simplices belonging to K_0 is a face of each of those simplices.

Example VIII. 2. Let U be a covering of a topological space R. We regard the members of U as the vertices and define that $U_0, \ldots, U_n \in U$ will make up an n-simplex if and only if $U_0 \cap \ldots \cap U_n \neq \emptyset$. Then we obtain an abstract complex which is called the *nerve* of U and denoted by $N(U)$ through this chapter. As a matter of fact essentially we used this concept in the proof of the imbedding theorem. The nerve of a finite or star-finite covering is a finite or star-finite complex respectively.

Definition VIII. 2. *Let* $|a_0, \ldots, a_n|$ *be an n-simplex belonging to a complex* K. *We consider an oriented n-simplex with the vertices* $a_0, \ldots, a_n$ *in this order and denote it by* $(a_0, \ldots, a_n)$. *For an oriented n-simplex* x^n *we denote by* $|x^n|$ *the non-oriented simplex consisting of the vertices of* x^n *and call it the absolute simplex of* x^n. *We define* $(a_{i_0}, \ldots, a_{i_n}) = (a_0, \ldots, a_n)$ *or* $(a_{i_0}, \ldots, a_{i_n}) = -(a_0, \ldots, a_n)$ *according to whether the permutation* $\begin{pmatrix} 0 \ldots n \\ i_0 \ldots i_n \end{pmatrix}$ *is even or odd. For* $n = 0$ *we consider two oriented 0-simplices* a_0 *and* $-a_0$.

Definition VIII. 3. *Let us denote by* G *and* K *a commutative group and an abstract complex, respectively. We shall write the operation of* G *additively. Let us denote by* $\{|x_\alpha^n| \mid \alpha \in A\}$ *the totality of the abstract simplices of* K. *Then an* (n,G)*-chain of* K *is a linear form*

$$\varphi^n = \sum_{\alpha \in A} g^\alpha x_\alpha^n$$

of the n-simplices x_α^n, $\alpha \in A$ *of* K *with the coefficients* $g^\alpha \in G$. Let $\varphi^n = \sum_\alpha g^\alpha x_\alpha^n$ *and* $\psi^n = \sum_\alpha h^\alpha x_\alpha^n$ *be two* (n,G)*-chains of* K; *then we define the sum of* φ^n *and* ψ^n *by*

$$\varphi^n + \psi^n = \sum_\alpha (g^\alpha + h^\alpha) x_\alpha^n .$$

We denote by 0 *the* (n,G)*-chain* $\sum_\alpha g^\alpha x_\alpha$ *with* $g^\alpha = 0$ *for every* α. *Then the* (n,G)*-chains of* K *form a commutative group denoted by* $C_n(K,G)$.

Definition VIII. 4. *Let* x_α^n *and* x_β^{n-1} *be oriented simplices of a complex* K. *Then we define the incidence number* η_β^α *between them as follows:*

$$\eta_\beta^\alpha = 0 \quad \text{if} \quad x_\beta^{n-1} \text{ is no face of } x_\alpha^n ,$$
$$\eta_\beta^\alpha = (-1)^i \varepsilon \quad \text{if} \quad x_\alpha^n = (a_0, \ldots, a_n) , \; x_\beta^{n-1} = \varepsilon(a_0, \ldots, \hat{a}_i, \ldots, a_n) ,$$

where $\varepsilon=+1$ *or* -1, *and* $(a_0,\dots,\hat{a}_i,\dots,a_n)=(a_0,\dots,a_{i-1},a_{i+1},\dots,a_n)$. *To define the incidence number for* $n=0$, *we consider one -1-simplex* x^{-1} *containing no vertex and belonging to* K. *We define the incidence number between* a *and* x^{-1} *as* 1 *and the incidence number between* $-a$ *and* x^{-1} *as -1. Let* $\varphi^n=\sum_{\alpha\in A}g^\alpha x^n_\alpha$ *be an* (n,G)*-chain of* K. *We express by*

$$\{\,|x^{n+1}_\gamma|\mid\gamma\in C\,\},\ \{\,|x^n_\alpha|\mid\alpha\in A\,\},\ \{\,|x^{n-1}_\beta|\mid\beta\in B\,\},$$

the totalities of the absolute $(n+1)$, n, $(n-1)$*-simplices of* K, *respectively. Then the boundary* $\partial\varphi^n$ *of* φ^n *is the* $(n-1,G)$*-chain defined by*

$$\partial\varphi^n=\sum_{\beta\in B}\Big(\sum_{\alpha\in A}g^\alpha\eta^\alpha_\beta\Big)x^{n+1}_\beta\ ,$$

where we assume that K *is star-finite.*[5] *The coboundary* $\delta\varphi^n$ *of* φ^n *is the* $(n+1,G)$*-chain defined by*

$$\delta\varphi^n=\sum_{\gamma\in C}\Big(\sum_{\alpha\in A}g^\alpha\eta^\gamma_\alpha\Big)x^{n+1}_\gamma\ ,$$

where K *is not necessarily star-finite.* ($\delta\varphi^n=0$ *if* K *contains no* $(n+1)$*-simplex.*)

A) $\partial\partial\varphi^n=0$, $\delta\delta\varphi^n=0$.

Proof. Assume $\varphi^n=\sum_\alpha g^\alpha x^n_\alpha$. Then

$$\partial\varphi^n=\sum_{\alpha,\beta}g^\alpha\eta^\alpha_\beta x^{n-1}_\beta\ ,\ \text{and}\ \ \partial\partial\varphi^n=\sum_{\alpha,\beta,\gamma}g^\alpha\eta^\alpha_\beta\eta'^\beta_\gamma x^{n-2}_\gamma\ ,$$

where η'^β_γ denotes the incidence number between x^{n-1}_β and x^{n-2}_γ. Suppose

$$x^n_\alpha=(a_0,\dots,a_n)\ ,$$
$$x^{n-1}_\beta=\varepsilon(a_0,\dots,\hat{a}_i,\dots,a_n)\ ,\ x^{n-1}=\varepsilon'(a_0,\dots,\hat{a}_k,\dots,a_n)\ ,$$
$$x^{n-2}_\gamma=\varepsilon''(a_0,\dots,\hat{a}_i,\dots,\hat{a}_k,\dots,a_n)\ .$$

[5] In case all but a finite number of the g^α's vanish, K need not be star-finite.

Then

$$g^{\alpha}\eta^{\alpha}_{\beta}\eta'^{\beta}_{\gamma} = \varepsilon(-1)^{i}\varepsilon\varepsilon''(-1)^{k-1}g^{\alpha} = \varepsilon''(-1)^{i+k-1}g^{\alpha}$$

and

$$g^{\alpha}\eta^{\alpha}_{\beta'}\eta'^{\beta'}_{\gamma} = \varepsilon'(-1)^{k}\varepsilon'\varepsilon''(-1)^{i}g^{\alpha} = \varepsilon''(-1)^{i+k}g^{\alpha}$$

cancel each other. Therefore $\sum_{\beta} g^{\alpha}\eta^{\alpha}_{\beta}\eta'^{\beta}_{\gamma} = 0$ for every α, γ, which implies $\partial\partial\varphi^{n} = 0$. $\delta\delta\varphi^{n} = 0$ can be proved in a similar way.

Definition VIII. 5. *An (n,G)-chain is called an (n,G)-cycle if its boundary is equal to zero. An (n,G)-chain is called an (n,G)-cocycle if its coboundary is equal to zero. We denote by $Z_{n}(K,G)$ and $Z^{n}(K,G)$ the set of the (n,G)-cycles and of the (n,G)-cocycles of K, respectively. It is easily seen that they are both subgroups of $C_{n}(K,G)$.*

Furthermore, by A), the (n,G)-chains which are the boundaries of $(n+1,G)$-chains form a subgroup of $Z_{n}(K,G)$ which is denoted by $B_{n}(K,G)$. The chains in $B_{n}(K,G)$ are called bounding cycles. The (n,G)-chains which are the coboundaries of $(n-1,G)$-chains form a subgroup of $Z^{n}(K,G)$ which is denoted by $B^{n}(K,G)$. The chains in $B^{n}(K,G)$ are called cobounding cocycles.

Now we call the difference group

$$H_{n}(K,G) = Z_{n}(K,G) - B_{n}(K,G)$$

the n-dimensional homology group of K with the coefficient group G and the difference group

$$H^{n}(K,G) = Z^{n}(K,G) - B^{n}(K,G)$$

the n-dimensional cohomology group of K with the coefficient group G. The elements of $H_{n}(K,G)$ and $H^{n}(K,G)$ are called n-dimensional homology classes and n-dimensional cohomology classes, respectively. Two cycles (cocycles) belonging to the same homology (cohomology) classes are called homologous (cohomologous). We denote by $\varphi^{n} \sim \psi^{n}$ in K the fact that two cocycles φ^{n} and ψ^{n} of K are cohomologous.

Definition VIII. 6. *Let* K *be a complex and* L *a subcollection of* K *. If every face of a simplex of* L *belongs to* L *, then we call* L *a subcomplex of* K *. Suppose* $\varphi^n = \sum_{\alpha \in A} g^\alpha x_\alpha^n$ *is an* (n,G)*-chain of* K *. Then we denote by* $\{ | x_\beta^n | \mid \beta \in B \}$ *the totality of the absolute n-simplices of* L *. Now, to* φ^n *there corresponds an* (n,G)*-chain:* $\psi^n = h_L \varphi^n$ *of* L *such that*

$$\psi^n = h_L \varphi^n = \sum_{\beta \in B} g^\beta x_\beta^n .$$

We call ψ^n *the restriction of* φ^n *to* L *and* φ^n *an extension of* ψ^n *over* K *. Conversely to an* (n,G)*-chain* $\psi^n = \sum_{\beta \in B} g^\beta x_\beta^n$ *of* L *there corresponds an* (n,G)*-chain* $\varphi^n = h_K \psi^n$ *of* K *such that*

$$\varphi^n = h_K \psi^n = \sum_{\alpha \in A} g^\alpha x_\alpha^n ,$$

where $g^\alpha = 0$ *for* $\alpha \notin B$ *.*

B) h_L induces a homomorphism also denoted by h_L of $H^n(K,G)$ into $H^n(L,G)$. h_K induces a homomorphism also denoted by h_K of $H_n(L,G)$ into $H_n(K,G)$.

Proof. It is clear that h_L is a homomorphism of $C_n(K,G)$ into $C_n(L,G)$ and satisfies $h_L \delta \varphi^n = \delta h_L \varphi^n$. Hence h_L maps every cocycle of K to a cocycle of L and every cobounding cocycle of K to a cobounding cocycle of L . Therefore h_L induces a homomorphism of $H^n(K,G)$ into $H^n(L,G)$. The proof for h_K is similar.

Definition VIII. 7. *The homomorphisms* h_L *and* h_K *in B) are called the natural homomorphisms. Suppose* h_L *maps* $\tilde{e} \in H^n(K,G)$ *to* $e \in H^n(L,G)$ *; then* $\tilde{e}$ *is called an extension of* e *, and* e *is said to be extendible over* K *.*

C) Let $\tilde{e} \in H^n(K,G)$ be an extension of $e \in H^n(L,G)$. If ψ^n is a cocycle of L which represents e , then there exists a cocycle φ^n of K which represents $\tilde{e}$ and is an extension of ψ^n , i.e. $h_L \varphi^n = \psi^n$.

Proof. If $\tilde{e} = 0$, then $e = 0$. Hence ψ^n is a cobounding cocycle of L , i.e. $\psi^n = \delta \psi^{n-1}$ for some $(n-1,G)$-chain ψ^{n-1} of L . Then $\delta h_K \psi^{n-1}$ is a cobounding cocycle of K satisfying

$$h_L \delta h_K \psi^{n-1} = \delta h_L h_K \psi^{n-1} = \delta \psi^{n-1} = \psi^n .$$

Thus $\delta h_K \psi^{n-1} = \varphi^n$ is the desired extension of ψ^n. If $\tilde{e}$ is an arbitrary element, we take a cocycle $\varphi_1^n \in \tilde{e}$. Then $h_L \varphi_1^n \in e$, and hence $\psi^n - h_L \varphi_1^n$ is a cobounding cocycle of L. Hence, as above, there exists a cobounding cocycle φ_2^n of K which is an extension of $\psi^n - h_L \varphi_1^n$ over K. Put $\varphi^n = \varphi_1^n + \varphi_2^n$; then since $\varphi^n \in \tilde{e}$, and $h_L \varphi^n = h_L \varphi_1^n + h_L \varphi_2^n = \psi^n$, φ^n is the desired extension.

Definition VIII. 8. *Let K and L be two complexes and f a mapping which maps each vertex of K to a vertex of L. f is called a simplicial mapping if it satisfies the following condition: If vertices $a_0, \ldots, a_n$ form a simplex of K, then $f(a_0), \ldots, f(a_n)$ form a simplex of L, where some of $f(a_0), \ldots, f(a_n)$ may coincide. If for an n-simplex $x^n = (a_0, \ldots, a_n)$, $(f(a_0), \ldots, f(a_n))$ is also an n-simplex, then we denote it by $f(x^n)$.*

Let f be a simplicial mapping of K into L. For an (n,G)-chain $\psi^n = \sum_\beta h^\beta y_\beta^n$ of L, we define an (n,G)-chain φ^n of K by

$$\varphi^n = f^* \psi^n = \sum_\alpha \Big(\sum_\beta h^\beta f_\beta^\alpha\Big) x_\alpha^n ,$$

where

$$f_\beta^\alpha = \begin{cases} 1 & \text{if } f(x_\alpha^n) = y_\beta^n \\ -1 & \text{if } f(x_\alpha^n) = -y_\beta^n, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose for every y_β^n of L, there exist only finitely many simplices x_α^n of K such that $f(x_\alpha^n) = \pm y_\beta^n$. Then for an (n,G)-chain $\varphi^n = \sum_\alpha g^\alpha x_\alpha^n$ of K we define an (n,G)-chain ψ^n of L by

$$\psi^n = f\varphi^n = \sum_\beta \Big(\sum_\alpha g^\alpha f_\beta^\alpha\Big) y_\beta^n .$$

D) $\delta f^* \psi^n = f^* \delta \psi^n$, $\partial f \varphi^n = f \partial \varphi^n$.

Proof. We shall prove the first formula only; the second one can be proved in a similar way. Let $\psi^n = \sum_\beta h^\beta y_\beta^n$. Then

$$\delta f^* \psi^n = \delta \sum_\alpha \Big(\sum_\beta h^\beta f_\beta^\alpha\Big) x_\alpha^n = \sum_{\alpha,\gamma} \Big(\sum_\beta h^\beta f_\beta^\alpha\Big) \eta_\alpha^\gamma x_\gamma^{n+1} =$$

$$= \sum_\gamma \Big(\sum_{\alpha,\beta} h^\beta f_\beta^\alpha \eta_\alpha^\gamma\Big) x_\gamma^{n+1} = \sum_\gamma \Big(\sum_{\gamma',\beta} h^\beta f_{\gamma'}^\gamma \eta_\beta^{\gamma'}\Big) x_\gamma^{n+1} = f^* \delta \psi^n ,$$

because $\sum_{\alpha,\beta} h^\beta f^\alpha_\beta \eta^\gamma_\alpha = \sum_{\gamma',\beta} h^\beta f^\gamma_{\gamma'} \eta^{\gamma'}_\beta$ is easy to check.

Since f^* is easily seen to be a homomorphism of $C_n(L,G)$ into $C_n(K,G)$, we can prove, by use of D) and an argument like the one in B), that

E) f^* induces a homomorphism of $H^n(L,G)$ into $H^n(K,G)$. f induces a homomorphism of $H_n(K,G)$ into $H_n(L,G)$.[6]

From now on we shall be chiefly concerned with cohomology with coefficient group I, the additive group of integers, though some of the following results may be true even for general coefficient groups. Accordingly, we mean by an n-chain, n-cocycle, $H^n(K)$ etc. an (n,I)-chain, (n,I)-cocycle, $H^n(K,I)$ etc. respectively.

Definition VIII. 9. *Let K be a complex with the not necessarily countable set of vertices $a_1, a_2, \ldots$. In addition we consider the vertices $b_1, b_2, \ldots$ corresponding to $a_1, a_2, \ldots$. We define $(b_{i_0}, b_{i_1}, \ldots, b_{i_\lambda}, a_{i_\lambda}, \ldots, a_{i_n})$ $(0 \leq \lambda \leq n)$ to be a simplex if $(a_{i_0}, a_{i_1}, \ldots, a_{i_\lambda}, \ldots, a_{i_n})$ is an n-simplex of K. We denote by PK the complex which consists of those simpleces and their faces. For an n-simplex $x^n = (a_{i_0}, \ldots, a_{i_n})$ of K we define an $(n+1)$-chain Px^n of PK by*

$$Px^n = \sum_{\lambda=0}^{n} (-1)^\lambda (b_{i_0}, b_{i_1}, \ldots, b_{i_\lambda}, a_{i_\lambda}, \ldots, a_{i_n}),$$

and more generally

$$P\varphi^n = \sum_\alpha g^\alpha P x^n_\alpha$$

for a chain $\varphi^n = \sum_\alpha g^\alpha x^n_\alpha$ of K.

F) Let $x^n = (a_0, \ldots, a_n)$ be an n-simplex of a complex K and $y^n = (b_0, \ldots, b_n)$ the n-simplex of PK which consists of the vertices $b_0, \ldots, b_n$ corresponding to $a_0, \ldots, a_n$, respectively. Then $\partial P x^n = x^n - y^n - P\partial x^n$.

[6] As a matter of fact B) is a special case of E) in which f is the identity mapping of L into K.

Proof. We put, for brevity.

$$z_j^{n+1} = (b_0, \ldots, b_j, a_j, \ldots, a_n) ,$$
$$z_j^n[b_i] = (b_0, \ldots, \hat{b}_i, \ldots, b_j, a_j, \ldots, a_n) ,$$
$$z_j^n[a_k] = (b_0, \ldots, b_j, a_j, \ldots, \hat{a}_k, \ldots, a_n) ,$$
$$x_i^{n-1} = (a_0, \ldots, \hat{a}_i, \ldots, a_n) .$$

Then

$$Px^n = \sum_{j=0}^{n} (-1)^j z_j^{n+1}$$

$$\partial z_j^{n+1} = \sum_{i=0}^{j} (-1)^i z_j^n[b_i] - \sum_{i=j}^{n} (-1)^i z_j^n[a_i] ,$$

$$\partial x^n = \sum_{i=0}^{n} (-1)^i x_i^{n-1} ,$$

$$Px_i^{n-1} = \sum_{j=0}^{i-1} (-1)^j z_j^n[a_i] - \sum_{j=i+1}^{n} (-1)^j z_j^n[b_i] .$$

Therefore we obtain

$$\begin{aligned}\partial P x^n + P\partial x^n &= \sum_{j=0}^{n} (-1)^j \Big(\sum_{i=0}^{j} (-1)^i z_j^n[b_i] - \sum_{i=j}^{n} (-1)^i z_j^n[a_i] \Big) + \\ &\quad \sum_{i=0}^{n} (-1)^i \Big(\sum_{j=0}^{i-1} (-1)^j z_j^n[a_i] - \sum_{j=i+1}^{n} (-1)^j z_j^n[b_i] \Big) = \\ &= \sum_{i=0}^{n} (-1)^{2i} z_i^n[b_i] - \sum_{i=0}^{n} (-1)^{2i} z_i^n[a_i] = \\ &= z_0^n[b_0] + \Big(\sum_{i=1}^{n} z_i^n[b_i] - \sum_{i=0}^{n-1} z_i^n[a_i] \Big) - z_n^n[a_n] = \\ &= x^n - y^n ,\end{aligned}$$

because

$$z_{i+1}^n[b_{i+1}] = z_i^n[a_i].$$

This implies the desired relation.

G) Let φ^n be an n-cocycle of PK. We denote by g_0^α the coefficient of x_α^n in the linear form φ^n and by g_1^α that of y_α^n, where x_α^n and y_α^n denote simplices which correspond to each other as mentioned in F). Then the n-chains $\varphi_0^n = \sum_\alpha g_0^\alpha x_\alpha^n$ $(= h_K \varphi^n)$ and $\varphi_1^n = \sum_\alpha g_1^\alpha x_\alpha^n$ of K are cohomologous in K.

Proof. Generally, for an n-finite chain $\psi_1^n = \sum_\mu a^\mu z_\mu^n$ and an n-chain $\psi_2^n = \sum_\mu b^\mu z_\mu^n$ of PK we define

$$I(\psi_1^n, \psi_2^n) = \sum_\mu a^\mu b^\mu ,$$

where we mean by a finite chain a chain only finitely many of whose coefficients do not vanish. Then, for an $(n+1)$-finite chain $\psi_1^{n+1} = \sum_\lambda a^\lambda z_\lambda^{n+1}$ and an n-chain $\psi_2^n = \sum_\mu b^\mu z_\mu^n$ we obtain

$$I(\partial\psi_1^{n+1}, \psi_2^n) = \sum_{\lambda,\mu} a^\lambda \eta_\mu^\lambda b^\mu = I(\psi_1^{n+1}, \delta\psi_2^n) .$$

Hence, we obtain

$$I(\partial\psi^{n+1}, \varphi^n) = I(\psi^{n+1}, \delta\varphi^n) = 0$$

for the cocycle φ^n and an arbitrary $(n+1)$-finite chain ψ^{n+1} because $\delta\varphi^n = 0$. Thus, it follows from F) that

$$0 = I(\partial P x_\alpha^n, \varphi^n) = I(x_\alpha^n, \varphi^n) - I(y_\alpha^n, \varphi^n) - \sum_\beta \eta_\beta^\alpha I(P x_\beta^{n-1}, \varphi^n) .$$

Since

$$I(x_\alpha^n, \varphi^n) = g_0^\alpha \quad \text{and} \quad I(y_\alpha^n, \varphi^n) = g_1^\alpha ,$$

putting

$$I(P x_\beta^{n-1}, \varphi^n) = g^\beta ,$$

we obtain

$$g_0^\alpha - g_1^\alpha - \sum_\beta \eta_\beta^\alpha g^\beta = 0 .$$

We consider an $(n-1)$-chain $\varphi^{n-1} = \sum_\beta g^\beta x_\beta^{n-1}$ of K. Then it follows from the above formula that in K

$$\delta\varphi^{n-1} = \sum_{\alpha,\beta} g^\beta \eta_\beta^\alpha x_\alpha^n = \sum_\alpha (g_0^\alpha - g_1^\alpha) x_\alpha^n = \sum_\alpha g_0^\alpha x_\beta^n - \sum_\alpha g_1^\alpha x_\alpha^n = \varphi_0^n - \varphi_1^n$$

proving that φ_0^n and φ_1^n are cohomologous in K.

Example VIII. 3. The oriented n-simplex $(-1)^k (a_0,\dots,\hat{a}_k,\dots,a_{n+1})$ is called a (positively) oriented face of the oriented $(n+1)$-simplex $(a_0,\dots,a_{n+1})$, i.e. the oriented faces are oriented so that the incidence numbers are equal to 1.

Let T^{n+1} be the complex consisting of an $(n+1)$-simplex $x_0^{n+1} = (a_0, a_1,\dots,a_{n+1})$ and its faces. Let $\varphi^n = \sum_{k=0}^{n+1} g^k x_k^n$ be an n-cocycle where $x_k^n = (-1)^k (a_0,\dots,\hat{a}_k,\dots,a_{n+1})$, $k = 0,\dots,n+1$, are the oriented faces of x_0^{n+1}. Then

$$\delta\varphi^n = \sum_{k=0}^{n+1} g^k x_0^{n+1} = 0 \text{ , i.e. } \sum_{k=0}^{n+1} g^k = 0 .$$

Let $x_{k\ell}^{n-1} = (a_0,\dots,\hat{a}_k,\dots,\hat{a}_\ell,\dots,a_{n+1})$, $\psi^{n-1} = \sum h^{k\ell} x_{k\ell}^{n-1}$, where $h^{k\,k+1} = g^0 + g^1 + \dots + g^k$ and $h^{k\ell} = 0$, otherwise. Then we easily see that $\delta\psi^{n-1} = \varphi^n$.

Therefore we obtain $H^n(T^{n+1}) = 0$. As a matter of fact, we can show that $H^m(T^{n+1}) = 0$ for every m.

On the other hand, we easily see that if K is a union of disjoint complexes K_α, then $H^n(K)$ $(n > 0)$ is the direct sum of $H^n(K_\alpha)$. Hence, if K consists of disjoint simplices and their faces, then $H^n(K) = 0$ for $n > 0$.

Example VIII. 4. Let K be an abstract complex. Then we define the *barycentric subdivision* K' of K as follows. We associate a vertex $a(x^n)$ with each absolute simplex $|x^n|$ of K. Geometrically, $a(x^n)$ is the barycenter of the simplex. Then K' consists of all the simplices

$$|a(x_{\alpha_0}^{n_0}), a(x_{\alpha_1}^{n_1}),\dots,a(x_{\alpha_i}^{n_i})|$$

where $n_0 > n_1 > \dots > n_i$, and $x_{\alpha_h}^{n_h}$ is a face of $x_{\alpha_{h-1}}^{n_{h-1}}$. If the vertices $a(x_{\alpha_h}^{n_h})$ are all barycenters of $|x^n|$ or of its faces, then this simplex is called a *subsimplex* of $|x^n|$. If $x^n = (a_0,\dots,a_n)$ is an oriented simplex, then we can decide the orientations of the n-subsimplices of x^n by deciding just one of them as

$$(a((a_0,\dots,a_n)); a((a_1,\dots,a_n)),\dots,a((a_n)))$$

which is called the *oriented n-subsimplex* of x^n. (For example, in case of $n=2$

$$|a((a_0,a_1,a_2)),a((a_1,a_2)),a((a_1))|$$

is oriented as

$$-(a((a_0,a_2,a_1)),a((a_2,a_1)),a((a_1))$$

because $(a_0,a_2,a_1)=-(a_0,a_1,a_2)$.) Suppose the n-simplices of K and K' are oriented as above. To each n-chain $\psi^n=\sum_\beta g^\beta y_\beta^n$ of K' we assign an n-chain $\varphi^n=\pi\psi^n$ of K by

$$\varphi^n=\pi\psi^n=\sum_{\alpha,\beta} g^\beta\xi_\beta^\alpha x_\alpha^n,$$

where

$$\xi_\beta^\alpha=\begin{cases}1 & \text{if } y_\beta^n \text{ is an oriented } n\text{-subsimplex of } x_\alpha^n,\\ 0 & \text{otherwise.}\end{cases}$$

Then we can show that π satisfies $\pi\delta\varphi^n=\delta\pi\varphi^n$ and, moreover, that π induces an isomorphism of $H^n(K')$ onto $H^n(K)$.[7]

Repeating this process of barycentric subdivision we obtain complexes K', K'', K''',... whose cohomology groups are all isomorphic to that of K. We call them *successive barycentric subdivisions.*

Let L be a subcomplex of K. If, in constructing a barycentric subdivision of K, we choose as the additional vertices only the barycenters of the simplices which do not belong to L, then we obtain the *barycentric subdivision* K' of K *mod* L. For such a barycentric subdivision we can also show that the above mapping induces an isomorphism of $H^n(K')$ onto $H^n(K)$.

Example VIII. 5. Let S^n be the complex consisting of all the faces of $\dim\leq n$ of an $(n+1)$-simplex. We call S^n the *elementary n-sphere*. If we suppose the original simplex is oriented and its faces are its oriented faces, then we can speak of the *oriented elementary n-sphere* S^n.

[7] A detailed proof can be found in W. Hurewicz and H. Wallman [1].

Let us denote by S^n the oriented elementary n-sphere. Then every n-chain $\varphi^n = \sum g^k x_k^n$ is clearly a cocycle. φ^n is cobounding in S^n if and only if $h_{T^{n+1}}\varphi^n$ is cobounding in T^{n+1}, and, as seen in Example VIII.3, $h_{T^{n+1}}\varphi^n$ is cobounding if and only if $\sum g^k = 0$. Thus each cohomology class in $H^n(S^n)$ is characterized by the value of $\sum g^k$, and hence $H^n(S^n)$ is isomorphic to the coefficient group I. In view of Example VIII.4 this assertion, including the fact that φ^n is cobounding if and only if $\sum g^k = 0$, is also true for any of the successive barycentric subdivisions of S^r.

VIII. 2. Cohomology group of a topological space

Definition VIII. 10. *Let $\mathfrak{U}_\alpha$ and $\mathfrak{U}_\beta$ be open coverings of a topological space R such that $\mathfrak{U}_\beta < \mathfrak{U}_\alpha$. For every member U of $\mathfrak{U}_\beta$ we choose a member U' of $\mathfrak{U}_\alpha$ such that $U' \supset U$. Then we obtain a simplicial mapping of $N(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$ which is called a projection of $N(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$ and denoted by π_α^β. Then, by 1 E), $\pi_\alpha^{*\beta}$ induces a homomorphism of $H^n(N(\mathfrak{U}_\alpha))$ into $H^n(N(\mathfrak{U}_\beta))$.*

A) Let us consider two projections f_1 and f_2 of $N(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$, where $\mathfrak{U}_\beta < \mathfrak{U}_\alpha$. Then they induce the same homomorphism of $H^n(N(\mathfrak{U}_\alpha))$ into $H^n(N(\mathfrak{U}_\beta))$.

Proof. We denote by $a_1, a_2, \dots$ the not necessarily countably many vertices of $N(\mathfrak{U}_\beta)$ and by $b_1, b_2, \dots$ vertices which correspond to $a_1, a_2, \dots$ in a one-to-one manner. We construct the complex $PN(\mathfrak{U}_\beta)$ by use of Definition VIII.9. Let us define a mapping f by $f(a_i) = f_1(a_i)$ and $f(b_i) = f_2(a_i)$.

Then f is obviously a simplicial mapping of $PN(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$. We denote by $N'(\mathfrak{U}_\beta)$ the subcomplex of $PN(\mathfrak{U}_\beta)$ consisting of the vertices $b_1, b_2, \dots$. It is clear that the restrictions of f to $N(\mathfrak{U}_\beta)$ and $N'(\mathfrak{U}_\beta)$ turn out to be f_1 and f_2, respectively.

Now, let φ^n be an n-cocycle of $N(\mathfrak{U}_\alpha)$. It is easy to see that

$$f_1^*\varphi^n \sim f_2^*\varphi^n \quad \text{in } N(\mathfrak{U}_\beta).$$

For $f^*\varphi^n$ is an n-cocycle of $PN(\mathfrak{U}_\beta)$, and $f^*\varphi^n$ restricted to $N(\mathfrak{U}_\beta)$ and $N'(\mathfrak{U}_\beta)$ turns out to be $f_1^*\varphi^n$ and $f_2^*\varphi^n$, respectively.[8] Hence, by 1 G), the desired result follows. Thus A) is proved.

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[8] Occasionally, we have regarded f_2 as a mapping of $N'(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$, since $N(\mathfrak{U}_\beta)$ and $N'(\mathfrak{U}_\beta)$ are isomorphic. But this should cause no confusion.

Definition VIII. 11. *Let* A *be a directed set, i.e. a set between some of whose elements the order is defined such that* $\alpha<\beta$ *and* $\beta<\gamma$ *implies* $\alpha<\gamma$ *and such that for every two elements* α,β *there is a third element* γ *with* $\gamma>\alpha$, $\gamma>\beta$. *Then we consider a collection* $\{G_\alpha \mid \alpha\in A\}$ *of commutative groups* G_α. *Suppose for each* $\alpha,\beta\in A$ *with* $\alpha<\beta$ *a homomorphism* $\overset{*}{\pi}{}^{\beta}_{\alpha}$ *of* G_α *into* G_β *is defined such that*

$$\overset{*}{\pi}{}^{\gamma}_{\beta}\overset{*}{\pi}{}^{\beta}_{\alpha}=\overset{*}{\pi}{}^{\gamma}_{\alpha} \quad \text{for} \quad \alpha<\beta<\gamma .$$

Then we call $\{G_\alpha,\overset{*}{\pi}{}^{\beta}_{\alpha} \mid \alpha,\beta\in A,\ \alpha<\beta\}$ *a directed spectrum. We call* $g_\alpha\in G_\alpha$ *and* $g_\beta\in G_\beta$ *equivalent in this directed spectrum if there exists* $\gamma\in A$ *such that*

$$\gamma>\alpha,\ \gamma>\beta,\ \overset{*}{\pi}{}^{\gamma}_{\alpha}g_\alpha=\overset{*}{\pi}{}^{\gamma}_{\beta}g_\beta \quad \text{in} \quad G_\gamma .$$

Classifying all the elements of $\cup\{G_\alpha \mid \alpha\in A\}$ by this equivalence relation we obtain a collection G of equivalence classes. We define the sum of two elements g and g' of G as follows. Let g_α and g'_β be representatives of the classes g and g' respectively. Taking $\gamma\in A$ with $\gamma>\alpha$, $\gamma>\beta$, we define the sum $g+g'$ as the class represented by the element

$$\overset{*}{\pi}{}^{\gamma}_{\alpha}g_\alpha+\overset{*}{\pi}{}^{\gamma}_{\beta}g_\beta .$$

It is clear that such a class is uniquely determined by g and g'. In this way G becomes a commutative group called the *limit group* of the directed spectrum $\{G_\alpha,\overset{*}{\pi}{}^{\beta}_{\alpha}\}$.

Definition VIII. 12. *Let* $\{\mathfrak{U}_\alpha \mid \alpha\in A\}$ *be the collection of the locally finite open coverings of a space* R. *We define that* $\alpha<\beta$ *if and only if* $\mathfrak{U}_\alpha>\mathfrak{U}_\beta$. *Then* A *is a directed set. We denote by* $\overset{*}{\pi}{}^{\beta}_{\alpha}$ *the homomorphism of* $H^n(N(\mathfrak{U}_\alpha))$ *into* $H^n(N(\mathfrak{U}_\beta))$ *dealt with in A). Then* $\overset{*}{\pi}{}^{\gamma}_{\beta}\overset{*}{\pi}{}^{\beta}_{\alpha}=\overset{*}{\pi}{}^{\gamma}_{\alpha}$ *holds obviously for every* α,β,γ *with* $\alpha<\beta<\gamma$. Hence

$$\{H^n(N(\mathfrak{U}_\alpha)),\ \overset{*}{\pi}{}^{\beta}_{\alpha} \mid \alpha,\beta\in A,\ \alpha<\beta\}$$

is a directed spectrum. Thus we can define by use of Definition VIII.11 its limit group $H^n(R)$, *called the n-dimensional cohomology group of* R.[9]

[9] The definition can also be used to define the cohomology group $H^n(R,G)$ more generally for every topological space R and every commutative coefficient group G.

Let R and S be two spaces and f a continuous mapping of R into S. Let $\{V_\alpha \mid \alpha \in A\}$ be the collection of the locally finite open coverings of S. Then $U_\alpha = f^{-1}(V_\alpha) = \{f^{-1}(V) \mid V \in V_\alpha\}$, $\alpha \in A$ are locally finite open coverings of R. Mapping $f^{-1}(V) \in U_\alpha$ to $V \in V_\alpha$ we obtain a simplicial mapping of $N(U_\alpha)$ into $N(V_\alpha)$ which we shall also denote by f. Now we can prove the following:

B) Let $e^n = \{e^n_\alpha\}$ be an element of $H^n(S)$. Then the element of $H^n(R)$ containing $f^* e^n_\alpha$ is uniquely determined by e^n, where $e^n_\alpha \in H^n(N(U_\alpha))$ is a representative of e^n. We shall denote this element of $H^n(R)$ by $f^* e^n$.

Proof. Let $V_\gamma < V_\alpha$; then we put

$$\tilde{\pi}^\gamma_\alpha = f^{-1} \pi^\gamma_\alpha f$$

for the mapping f of the vertices of $N(U_\gamma)$ onto the vertices of $N(V_\gamma)$, the projection π^γ_α of $N(V_\gamma)$ into $N(V_\alpha)$ and the mapping f^{-1} of the vertices of $N(V_\alpha)$ onto the vertices of $N(U_\alpha)$. Then $\tilde{\pi}^\gamma_\alpha$ is easily seen to be a projection of $N(U_\gamma)$ into $N(U_\alpha)$. We now assert that

(1) $$\overset{*}{\tilde{\pi}}{}^\gamma_\alpha f^* \varphi^n_\alpha = f^* \overset{*}{\pi}{}^\gamma_\alpha \varphi^n_\alpha \quad \text{in } N(U_\gamma)$$

for every n-cocycle φ^n_α of $N(V_\alpha)$. This obviously follows from the definition of $\tilde{\pi}^\gamma_\alpha$, i.e. $f \tilde{\pi}^\gamma_\alpha = \pi^\gamma_\alpha f$, and this implies the desired relation.

Let e^n_α, $e^n_\beta \in e^n$; then for some γ with $U_\gamma < U_\alpha \wedge U_\beta$ we obtain

$$\overset{*}{\pi}{}^\gamma_\alpha e^n_\alpha = \overset{*}{\pi}{}^\gamma_\beta e^n_\beta .$$

Therefore

$$f^* \overset{*}{\pi}{}^\gamma_\alpha e^n = f^* \overset{*}{\pi}{}^\gamma_\beta e^n_\beta .$$

By use of the assertion (1) we conclude

$$\overset{*}{\tilde{\pi}}{}^\gamma_\alpha f^* e^n_\alpha = \overset{*}{\tilde{\pi}}{}^\gamma_\beta f^* e^n_\beta ,$$

i.e. $f^* e^n_\alpha$ and $f^* e^n_\beta$ represent the same element of $H^n(R)$.

The easy proof of the following assertion is left to the reader.

C) The mapping f^* in B) is a homomorphism of $H^n(S)$ into $H^n(R)$.

Definition VIII. 13. *Let* C *be a closed subset of a space* R . *The homomorphism* h^* *of* $H^n(R)$ *into* $H^n(C)$ *induced by the identity mapping (imbedding)* h *of* C *into* R *is called the natural homomorphism. If an element* e^n *of* $H^n(C)$ *is the image of an element* $\tilde{e}^n$ *of* $H^n(R)$ *by the natural homomorphism, then* $\tilde{e}^n$ *is called the extension of* e^n *and* e^n *is said to be extendible over* R .

Let $e^n \in H^n(R)$, and let $e^n_\alpha \in H^n(N(V_\alpha))$ be a representative of e^n , where V_α is a locally finite open covering of R . Then $U_\alpha = \{ U \cap C \mid U \in V_\alpha \}$ is a locally finite open covering of C such that $U_\alpha = h^{-1}(V_\alpha)$. Put, for brevity, $K = N(V_\alpha)$, $L = N(U_\alpha)$; then L is a subcomplex of K . Now h^*e^n is the element of $H^n(C)$ represented by $h_L e^n_\alpha$ for the natural homomorphism h_L of $H^n(K)$ into $H^n(L)$ since $h_L e^n_\alpha = h^* e^n_\alpha$.

Example VIII. 6. Let R be a polyhedron and K the complex associated with R . Let $K_1, K_2, \dots$ be the successive barycentric subdivisions of K . For a vertex p of K_i we denote by $S(p,K_i)$ the open star around p , i.e. the set of the points of (the realized) K_i whose barycentric coordinate for p is positive. Then $U_i = \{ S(p,K_i) \mid p:$ vertices of $K_i \}$ is a finite open covering. We note that $N(U_i)$ is isomorphic to K_i . Since $\{ U_i \mid i = 1,2,\dots \}$ is a *cofinal subset* [10] of the set of all locally finite open coverings of R , we can easily see that $H^n(R)$ is isomorphic to the limit group of $\{ H^n(N(U_i)) , \pi^{*i}_j \mid i < j ; i,j = 1,2,\dots \}$. It is also easy to see that the homomorphism π^{*i}_j is an isomorphism of $H^n(N(U_j)) = H^n(K_j)$ onto $H^n(N(U_i)) = H^n(K_i)$. Hence $H^n(R)$ is isomorphic to $H^n(K_i)$ for each i and accordingly to $H^n(K)$. Thus we can say that the cohomology group of a polyhedron is isomorphic to the cohomology group of the associated complex.

VIII. 3. Dimension and cohomology

Definition VIII. 14. *Let* U *be an open covering of a space* R *and* N *the realization of the nerve* $N(U)$. *Let* φ *be a continuous mapping of* R *into* N . *We denote by* $x(U_\alpha)$ *the vertex of* N *corresponding to* $U_\alpha \in U$ *and by* S_α *the open star around* $x(U_\alpha)$. *If* $\varphi^{-1}(S_\alpha) \subset U_\alpha$ *for every* $U_\alpha \in U$, *then we call* φ *a canonical mapping relative to* U .

[10] Let B be a subset of a directed set A . If for every element a of A there is an element b of B satisfying $b > a$, then we call B a *cofinal subset* of A.

A) Let φ be a continuous mapping of a space R into the realized nerve N of a point-finite open covering $\mathfrak{U}$ of R. Then φ is canonical relative to $\mathfrak{U}$ if and only if for every point p of R $\varphi(p) \in \overline{S(p)}$, where $S(p)$ denotes the geometrical simplex spanned by the vertices $\{x(U_\alpha) \mid p \in U_\alpha \in \mathfrak{U}\}$.

Proof. Let φ be a canonical mapping and p a point of R. Let $\varphi(p) \in S$ for a simplex S of N and denote by $x(U_{\alpha_i})$, $i = 1,\dots,k$ its vertices. Then

$$\varphi(p) \in S_{\alpha_i}, \quad i = 1,\dots,k,$$

which implies

$$p \in \varphi^{-1}(S_{\alpha_i}) \subset U_{\alpha_i}, \quad i = 1,\dots,k$$

by the definition of canonical mapping. Therefore

$$S \subset \overline{S(p)}, \text{ i.e. } \varphi(p) \in \overline{S(p)}.$$

Conversely, if φ satisfies $\varphi(p) \in \overline{S(p)}$ for every point p of R, then we assume that p is a given point of $\varphi^{-1}(S_\alpha)$, where $U_\alpha \in \mathfrak{U}$. Then

$$\varphi(p) \in S_\alpha \cap \overline{S(p)}.$$

Hence $x(U_\alpha)$ is a vertex of $S(p)$, and hence $p \in U_\alpha$, proving that φ is canonical.

B) Let $\mathfrak{U}$ be a locally finite open covering of a space R; then there exists a canonical mapping φ of R into the realized nerve N of $\mathfrak{U}$.

Proof. Let $\mathfrak{U} = \{U_\alpha \mid \alpha \in A\}$ and denote by $x(U_\alpha)$ the vertex of N corresponding to U_α. For each choice of $\alpha_1,\dots,\alpha_k \in A$ satisfying $U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \neq \emptyset$ we define a closed set $F(\alpha_1,\dots,\alpha_k)$ of R by

$$F(\alpha_1,\dots,\alpha_k) = R - \bigcup\{U_\alpha \mid \alpha \neq \alpha_1,\dots,\alpha_k\}.$$

Let $p \in F(\alpha_1,\dots,\alpha_k)$; then we define as follows a point

$$\varphi(p) = \{x_1(\alpha_1,\dots,\alpha_k,p),\dots,x_k(\alpha_1,\dots,\alpha_k,p)\}$$

belonging to the simplex $[x(U_{\alpha_1}),\dots,x(U_{\alpha_k})]$ by induction on the number k, where we denote by $x_1(\alpha_1,\dots,\alpha_k,p),\dots,x_k(\alpha_1,\dots,\alpha_k,p)$ the barycentric coordinates of the point $\varphi(p)$ associated with the vertices $x(U_{\alpha_1}),\dots,x(U_{\alpha_k})$, respectively.

For $k=1$ we define

$$\varphi(p)=\{x_1(\alpha_1,p)\}$$
$$x_1(\alpha_1,p)=1 \quad \text{for} \quad p\in F(\alpha_1)\ .$$

Namely

$$\varphi(p)=x(\alpha_1) \quad \text{for} \quad p\in F(\alpha_1)\ .$$

Assume that we have defined $\varphi(p)$ for $k=\ell$. Let $p\in F(\alpha_1,\dots,\alpha_l,\alpha_{l+1})$. Then we put

$$\begin{aligned} y_1(\alpha_1,\dots,\alpha_l,\alpha_{l+1},p) &= x_1(\alpha_1,\hat\alpha_2,\dots,\alpha_{l+1},p) \quad \text{for } p\in F(\alpha_1,\hat\alpha_2,\dots,\alpha_{l+1}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= x_1(\alpha_1,\dots,\alpha_l,\hat\alpha_{l+1},p) \quad \text{for } p\in F(\alpha_1,\dots,\alpha_l,\hat\alpha_{l+1}) \\ &= 0 \quad \text{for } p\in F(\hat\alpha_1,\alpha_2,\dots,\alpha_{l+1})\ . \end{aligned}$$

Since $y_1(\alpha_1,\dots,\alpha_l,\alpha_{l+1},p)$ is a continuous function defined on a closed subset of $F(\alpha_1,\dots,\alpha_{l+1})$, by Tietze's extension theorem we can continuously extend it over $F(\alpha_1,\dots,\alpha_{l+1})$. Note that we assume R to be paracompact T_2 and accordingly normal.

We define the continuous functions

$$y_2(\alpha_1,\dots,\alpha_{l+1},p),\dots,y_l(\alpha_1,\dots,\alpha_{l+1},p)$$

over $F(\alpha_1,\dots,\alpha_{l+1})$ in a similar way. Moreoever, we define a continuous function $y_{l+1}(\alpha_1,\dots,\alpha_{l+1},p)$ such that

$$\begin{aligned} y_{l+1}(\alpha_1,\dots,\alpha_{l+1},p) &= x_{l+1}(\hat\alpha_1,\alpha_2,\dots,\alpha_{l+1},p) \quad \text{for } p\in F(\hat\alpha_1,\alpha_2,\dots,\alpha_{l+1})\ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= x_{l+1}(\alpha_1,\dots,\hat\alpha_l,\alpha_{l+1},p) \quad \text{for } p\in F(\alpha_1,\dots,\hat\alpha_l,\alpha_{l+1})\ , \\ &= 0 \quad \text{for } p\in F(\alpha_1,\dots,\alpha_l,\hat\alpha_{l+1})\ , \\ &= 1 \quad \text{if } y_1(\alpha_1,\dots,\alpha_{l+1},p)=\dots=y_l(\alpha_1,\dots,\alpha_{l+1},p)=0\ . \end{aligned}$$

Putting for $i = 1,\dots,l+1$

$$x_i(\alpha_1,\dots,\alpha_{l+1},p) = \frac{y_i(\alpha_1,\dots,\alpha_{l+1},p)}{y_1(\alpha_1,\dots,\alpha_{l+1},p)+\dots+y_{l+1}(\alpha_1,\dots,\alpha_{l+1},p)}$$

we define a mapping

$$\varphi(p) = \{ x_1(\alpha_1,\dots,\alpha_{l+1},p),\dots,x_{l+1}(\alpha_1,\dots,\alpha_{l+1},p) \}$$

of $F(\alpha_1,\dots,\alpha_{l+1})$ into $[x(U_{\alpha_1}),\dots,x(U_{\alpha_{l+1}})]$ which coincides with the previously defined mapping on

$$F(\alpha_1,\dots,\alpha_l) \cup \dots \cup F(\alpha_2,\dots,\alpha_{l+1}) .$$

Thus we have a mapping $\varphi(p)$ of R into N which is continuous on each $F(\alpha_1,\dots,\alpha_k)$. Since U is locally finite, for every point p of R there is a neighbourhood $U(p)$ which meets only finitely many members $U_{\alpha_1},\dots,U_{\alpha_k}$ of U. Then since $\varphi(p)$ is continuous on $F(\alpha_1,\dots,\alpha_k)$, $\varphi(p)$ is continuous at p i.e. it is a continuous mapping.

Let $S(p) = [x(U_{\alpha_1}),\dots,x(U_{\alpha_k})]$ be the simplex determined by p as in A). Then it is clear that $p \in F(\alpha_1,\dots,\alpha_k)$, and hence $\varphi(p) \in \overline{S(p)}$. Therefore, by A), φ is canonical.

C) Let U be a locally finite open covering of a space R and N its realized nerve. Let φ be a canonical mapping of R into N and f a mapping of N into a space S which is continuous on each finite subcomplex of N. Then $f\varphi$ is a continuous mapping of R into S.

Proof. Let p be a given point of R. Take a neighbourhood $U(p)$ of p which meets only finitely many members $U_1,\dots,U_k$ of U. Then $\varphi(U(p))$ is contained in the finite subcomplex of N spanned by $x(U_1),\dots,x(U_k)$ and continuously mapped into S by f. Thus $f\varphi$ is continuous at p.

Let K, L be two complexes and K_0, L_0 be the realizations of K, L, respectively. We consider a simplicial mapping f of K into L. Let $[a_0,\dots,a_n]$ be a simplex of K_0 and p a point in it with barycentric coordinates $\{x_0,\dots,x_n\}$. Then we define a point $f(p)$ in the simplex $[f(a_0),\dots,f(a_n)]$ as the point having the barycentric coordinates $\{x_0,\dots,x_n\}$. Then f is a mapping of K_0 into L_0 which is continuous on the closure of each simplex of K_0. We shall make no distinction between the mapping of K into L and the induced mapping of K_0 into L_0.

Definition VIII. 15. *Let either of K and S^n be an oriented elementary n-sphere or one of its successive barycentric subdivisions. We fix a positively oriented simplex y_0^n of S^n. Let f be a simplicial mapping of K into S^n. Then we define $d_f(K)$ by*

$$d_f(K) = \sum_\alpha f_0^\alpha ,$$

where $\sum_\alpha$ denotes the sum over all positively oriented simplices x_α^n of K, and

$$f_0^\alpha = \begin{cases} 1 \text{ if } f(x_\alpha^n) = y_0^n , \\ -1 \text{ if } f(x_\alpha^n) = -y_0^n , \\ 0 \text{ otherwise} . \end{cases}$$

D_n) Let either of K and S^n be an oriented n-sphere or one of its successive barycentric subdivisions. Suppose f is a simplicial mapping of K into S^n. If $d_f(K) = 0$, then f is homotopic to 0.

E_n) Let L be a subcomplex of an $(n+1)$-complex K and S^n an oriented elementary n-sphere. Suppose f is a simplicial mapping of L into S^n. For a fixed positively oriented simplex y_0^n of S^n and n-simplices x_α^n of L we define f_0^α as in Definition VIII. 15. If the n-cocycle

$$\varphi^n = f^* y_0^n = \sum_{\alpha \in A} f_0^\alpha x_\alpha^n , \text{ [11]}$$

of L is the restriction of a cocycle of K, then f can be extended to a mapping F of K into S^n which is continuous on the closure of every simplex of K.

Proof. We shall prove D_n) and E_n) simultaneously by induction on the number n.

Proof of D_1). Let us consider the 1-cocycle

$$\varphi^1 = f^* y_0^1 = \sum_\alpha f_0^\alpha x_\alpha^1$$

[11] We denote by y_0^n the n-chain $1 y_0^n$.

of K for a fixed 1-simplex $y_0^1 = (q_0, q_1)$ of S^1. Then since

$$d_f(K) = \sum_\alpha f_0^\alpha = 0 ,$$

by Example VIII.5 φ^1 is cobounding, i.e. there is a 0-chain ψ^0 of K such that $\delta\psi^0 = \varphi^1$.

Let $\psi^0 = \sum_{i=1}^{k+1} g^i x_i^0$. Then the 0-simplex x_i^0 is also a vertex of K. We suppose (x_i^0, x_{i+1}^0), $i = 1, \ldots, k$ $(x_{k+1}^0 = x_1^0)$ are the 1-simplices of S^1. Then $g^{i+1} - g^i$ is the coefficient of (x_i^0, x_{i+1}^0) in $\delta\psi^0$. Hence

$$g^{i+1} - g^i = \begin{cases} 1 & \text{if } f((x_i^0, x_{i+1}^0)) = (q_0, q_1) , \\ -1 & \text{if } f((x_i^0, x_{i+1}^0)) = -(q_0, q_1) , \\ 0 & \text{otherwise.} \end{cases}$$

We regard S^1 as the space R^1 of reals mod 1 identifying the segment $y_0^1 = (q_0, q_1)$ in S^1 with the segment $(0, \frac{1}{2})$ in R^1. For a real number x we denote by (x) the congruence class mod 1 of x. We define a real valued function $g(x)$ over K by

$$(g(x)) = f(x) ,$$
$$\tfrac{1}{2}(g^i + g^{i+1}) - \tfrac{1}{2} \leq g(x) < \tfrac{1}{2}(g^i + g^{i+1}) + \tfrac{1}{2} \quad \text{for} \quad x \in \overline{[x_i^0, x_{i+1}^0]} .$$

It is easily seen that $g(x)$ is continuous. Now $f(x,t) = (tg(x))$ gives a homotopy between f and a constant mapping. Therefore f is homotopic to 0.

Proof that D_n) implies E_n). Assume that

$$\varphi^n = \sum_{\alpha \in A} f_0^\alpha x_\alpha^n$$

is the restriction to L of the cocycle

$$\psi^n = \sum_{\beta \in B} g^\beta x_\beta^n$$

of K. Then

$$g^{\beta} = f_0^{\beta} \quad \text{if} \quad \beta \in A\,,$$

$$\sum_{\beta \in B} \eta_{\beta}^{\gamma} g^{\beta} = 0 \quad \text{for every} \quad (n+1)\text{-simplex} \quad x_{\gamma}^{n+1} \quad \text{of} \quad K\,.$$

Let us denote by K_n and L_n the complexes consisting of the simplices with $\dim \leq n$ of K and L respectively. By use of the successive barycentric subdivisions of $K_n \bmod L_n$, we obtain a complex K'_n such that each simplex x_{β}^n of $K_n - L_n$ contains an oriented n-subsimplex $x_{\beta}'^{n}$ belonging to K'_n such that the *star* [12] of $x_{\beta}'^{n}$ in K'_n has no vertex in common with x_{β}^n. For x_{β}^n we denote by X_{β} the subcomplex of K'_n consisting of the n-subsimplices of x_{β}^n and their faces. We apply the successive barycentric subdivisions to $X_{\beta} \bmod X_{\beta} - \{x_{\beta}'^{n}\}$ to obtain a complex X'_{β} such that there are g^{β} subsimplices

$$x_{\beta k}^n\,, \quad k = 1, \ldots, g^{\beta} \quad \text{of} \quad x_{\beta}'^{n}$$

belonging to X'_{β} such that the stars of $x_{\beta k}^n$ in X'_{β} have no common vertex with each other, where we are assuming $g^{\beta} \geqq 0$ without loss of generality. Put

$$K''_n = \cup \{X'_{\beta} \mid \beta \in B - A\}\ .$$

Then note that X'_{β} is a complex obtained by triangulating x_{β}^n, and thus $K''_n \cup L_n$ is a complex obtained from K_n by bestowing a finer triangulation. Further note that $x_{\beta k}^n$, $k = 1, \ldots, g^{\beta}$ are oriented n-subsimplices of x_{β}^n belonging to K''_n such that the stars of $x_{\beta k}^n$ have no vertices in common either with x_{β}^n or with each other.

Suppose

$$x_{\beta k}^n = (a_1^{\beta k} \ldots a_{n+1}^{\beta k})\,, \quad k = 1, \ldots, g^{\beta}\,,$$

and denote by $b_0, b_1, \ldots, b_{n+1}$ the vertices of S^n assuming that $y_0^n = (b_1 \ldots b_{n+1})$. Then we extend f to a simplicial mapping g of $K''_n \cup L$ into S^n by

$$g(a_l^{\beta k}) = b_l\,, \quad l = 1, \ldots, n+1\,,$$

$$g(a) = b_0 \quad \text{for the other vertices} \quad a \quad \text{of} \quad K''_n\,.$$

---

[12] Let x^n be a simplex belonging to a complex K. Then we mean by the star of x^n in K the collection of the simplices of K which have a common vertex with x^n.

We note that $x^n_{\beta k}$, $k = 1, \ldots, g^\beta$ are the only simplices of K''_n which are mapped by g to y^n_0. Thus each X'_β contains exactly g^β n-simplices which are mapped by g to y^n_0.

Let x^{n+1}_γ be an arbitrary $(n+1)$-simplex of K and K_γ the complex consisting of the simplices which belong to $K''_n \cup L_n$ and are on the boundary of x^{n+1}_γ. Then it is easily seen that

$$d_g(K_\gamma) = \sum_{\beta \in B} \eta^\gamma_\beta g^\beta = 0 \ .$$

Thus it follows from D_n) that g is homotopic to 0 on the boundary of x^{n+1}_γ. Hence, by Borsuk's theorem, we can extend g over $\bar{x}^{n+1}_\gamma$. In consequence, f can be extended over K such that the extension is continuous on the closure of every simplex of K.

Proof that E_n) implies D_{n+1}). Let us consider a fixed simplex y^{n+1}_0 of S^{n+1} and denote by L the complex consisting of the $(n+1)$-simplices of K which are mapped onto y^{n+1}_0 by f, and their faces. We may assume that the $(n+1)$-simplices of L have no vertex in common. Because, if they have, then we can subdivide K and S^{n+1} such that there appears a subsimplex y^{n+1}_1 of y^{n+1}_0 which has no vertex in common with y^{n+1}_0. If we replace y^{n+1}_0 by y^{n+1}_1 then it satisfies the desired condition keeping the value of $d_f(K)$ at 0.

Assume

$$\varphi^{n+1} = f^* y^{n+1}_0 \ .$$

Then, since

$$\varphi^{n+1} = \sum f^\alpha_0 x^{n+1}_\alpha \quad \text{and} \quad d_f(K) = \sum f^\alpha_0 = 0 \ ,$$

by Example VIII.5 φ^{n+1} is cobounding, i.e.,

$$(1) \qquad \varphi^{n+1} = \delta \psi^n$$

for some n-chain ψ^n of K. We denote by L_n the complex consisting of the simplices of L of $\dim \leqq n$. Then we regard the restriction f_0 of f to L_n as a mapping of L_n into the elementary n-sphere S^n, the boundary of y^{n+1}_0. Moreover, we denote by f_1 the restriction of f to L and regard f_1 as a mapping of L into the complex T^{n+1} consisting of y^{n+1}_0 and its faces.

Writing

$$\psi_1^n = f_1^* y_0^n$$

for the n-chain y_0^n of T^{n+1} we obtain an n-chain ψ_1^n of L. Since

$$\delta y_0^n = y_0^{n+1}$$

in T^{n+1}, we obtain, using 1 D),

$$(2) \qquad \delta\psi_1^n = \delta f_1^* y_0^n = f_1^* \delta y_0^n = f_1^* y_0^{n+1} = \tilde{\varphi}^{n+1} ,$$

where $\tilde{\varphi}^{n+1}$ denotes the restriction of φ^{n+1} to L. On the other hand, by $\tilde{\psi}^n$ we denote the restriction of ψ^n to L. Then it follows from (1) that

$$\tilde{\varphi}^{n+1} = \delta\tilde{\psi}^n \quad \text{in } L ,$$

and hence by (2)

$$\delta(\psi_1^n - \tilde{\psi}^n) = 0 .$$

Therefore, by Example VIII.3, we obtain $\psi_1^n - \tilde{\psi}^n \sim 0$ in L, and accordingly

$$(3) \qquad h_{L_n} \psi_1^n \sim h_{L_n} \tilde{\psi}^n \quad \text{in } L_n .$$

We consider the complex $K' = (K - L) \cup L_n$. Then, since the coefficient f_0^α of φ^{n+1} vanishes for $x_\alpha^{n+1} \notin L$, we obtain

$$\delta\psi^n = h_{K'} \varphi^{n+1} = 0 \quad \text{in } K' ,$$

regarding ψ^n as a chain of K'. Thus ψ^n, or more precisely $h_{K'} \psi^n$ is a cocycle of K'. Hence the element of $H^n(L_n)$ represented by $h_{L_n} \tilde{\psi}^n$ is extendible over K'. Therefore, by 1 C) and (3),

$$h_{L_n} \psi_1^n = f_0^* y_0^n$$

is the restriction of a cocycle of K'. Thus it follows from E_n) that f_0 can be extended to a mapping F of K' into the boundary of y_0^{n+1} which is continuous on the closure of every simplex of K' and accordingly on K'.

Defining

$$g(x) = \begin{cases} f(x) & \text{for } x \in L , \\ F(x) & \text{for } x \notin L , \end{cases}$$

we obtain a continuous mapping g of K into $\overline{y_0^{n+1}}$. We may assume that y_0^{n+1} is a hemisphere of S^{n+1} . Therefore $f(x)$ and $g(x)$ are never antipodal, and hence f and g are homotopic. Since $g(K) \subset \overline{y_0^{n+1}}$, g is homotopic to zero. Thus f is also homotopic to zero.

Theorem VIII. 1 (Hopf's Extension Theorem). *Let R be a (paracompact T_2)-space of* $\dim R \leqq n+1$ *and C a closed set of R . Suppose f is a continuous mapping of C into S^n . Then f can be extended to a continuous mapping F of R into S^n if and only if $f^*e \in H^n(C)$ is extendible over R for every element $e \in H^n(S^n)$.*

Proof. Necessity. Let us denote by h the identity mapping of C into R and by h^* the associated natural homomorphism of $H^n(R)$ into $H^n(C)$. Since $f = Fh$, we obtain $f^*e = h^*F^*e$ for every element e of $H^n(S^n)$. Hence f^*e is the image of the element F^*e of $H^n(R)$ by h^* , i.e. it is extendible over R .

Sufficiency. We regard S^n as an oriented elementary n-sphere as well as a topological space. We denote by $S(a)$ for a vertex a of S^n the union of the simplices of S^n having a as a vertex. Put $V_0 = \{ S(a) \mid a \text{ is a vertex of } S^n \}$.

Then $N(V_0)$ is a complex isomorphic to S^n itself. We denote by y_0^n an oriented n-simplex of S^n or of $N(V_0)$. Let $e_0^n = \{y_0^n\}$ be the element of $H^n(N(V_0))$ represented by y_0^n . Put $U_0' = f^{-1}(V_0)$; then $f^*e_0^n = d_0^n$ is an element of $H^n(N(U_0'))$. By the assumption the element of $H^n(C)$ represented by d_0^n is extendible over R . Hence there exist locally finite open coverings U_β' , U_γ' of C and $d_\gamma^n \in H^n(N(U_\gamma'))$ satisfying

(i) d_γ^n is extendible over $N(U_\gamma)$ for an extended locally finite open covering U_γ of U_γ' over R ,

(ii) $U_\beta' < U_\gamma' \wedge U_0'$,

(iii) $\pi_\gamma^{*\beta} d_\gamma^n = \pi_0^{*\beta} d_0^n$,

where we denote by π_γ^β , π_0^β the projections of $N(U_\beta')$ into $N(U_\gamma')$, of $N(U_\beta')$ into $N(U_0')$, respectively.

Denote by U_β an extended locally finite open covering of U_β' over R such that $U_\beta < U_\gamma$. Then since R is a paracompact space of $\dim R \leqq n+1$, there is a locally finite open covering $\mathfrak{U}_\alpha$ such that

$$\text{ord } \mathfrak{U}_\alpha \leq n+2 \ , \ \mathfrak{U}_\alpha < \mathfrak{U}_\beta \ .$$ [13]

Now

$$\overset{*}{\pi}{}^\alpha_\beta \overset{*}{\pi}{}^\beta_\gamma d^n_\gamma = \overset{*}{\pi}{}^\alpha_\beta \overset{*}{\pi}{}^\beta_0 d^n_0 \ ;$$

hence

$$\overset{*}{\pi}{}^\alpha_\gamma d^n_\gamma = \overset{*}{\pi}{}^\alpha_0 d^n_0 \ .$$

Since d^n_γ is extendible,

$$d^n_\gamma = h^* \tilde{d}^n_\gamma$$

for some $\tilde{d}^n_\gamma \in H^n(N(\mathfrak{U}_\gamma))$ and the natural mapping h^* of $H^n(N(\mathfrak{U}_\gamma))$ into $H^n(N(\mathfrak{U}'_\gamma))$. Hence

$$\overset{*}{\pi}{}^\alpha_\gamma h^* \tilde{d}^n_\gamma = \overset{*}{\pi}{}^\alpha_0 d^n_0 \ ,$$

which implies, as seen in the proof of 2 B),

$$h^* \overset{*}{\tilde{\pi}}{}^\alpha_\gamma \tilde{d}^n_\gamma = \overset{*}{\pi}{}^\alpha_0 d^n_0 \ ,$$

where $\tilde{\pi}^\alpha_\gamma$ denotes the projection of $N(\mathfrak{U}_\alpha)$ into $N(\mathfrak{U}_\gamma)$. Therefore, by 1 C), $(f\pi^\alpha_0)^* y^n_0$ is the restriction of an n-cocycle of $N(\mathfrak{U}_\alpha)$. Since $N(\mathfrak{U}_\alpha)$ is an $(n+1)$-complex, by use of E_n) we can extend $f\pi^\alpha_0$ to a mapping π of $N(\mathfrak{U}_\alpha)$ into S^n which is continuous on the closure of each simplex of $N(\mathfrak{U}_\alpha)$.

By virtue of B) we can construct a canonical mapping φ of R into $N(\mathfrak{U}_\alpha)$. It follows from C) that $\pi\varphi$ is a continuous mapping of R into S^n. Now we denote by f_0 the restriction of $\pi\varphi$ to C. Since φ is canonical, for every $x \in C$, $\varphi(x)$ belongs to the closed simplex of $N(\mathfrak{U}'_\alpha)$ determined by x as seen in A). Therefore, for $x \in C$

$$f_0(x) = \pi\varphi(x) = f\pi^\alpha_0 \varphi(x)$$

---

[13] Here we use Theorem II.6 for a paracompact space. Though Theorem II.6 is stated for metric spaces, it has been proved essentially for every paracompact space R.

is contained in the same closed simplex of S^n which contains $f(x)$. Hence f is homotopic to f_0 in C . Since f_0 is extendible over R , by Borsuk's theorem [14] we can extend f over R .

Theorem VIII. 2. *Let R be a space of finite dimension. Then R has $\dim \leqq n$ if and only if for every integer $m \geqq n$ and every closed set C of R, the natural homomorphism of $H^m(R)$ into $H^m(C)$ is an onto mapping.*

Proof. Necessity. Let $e \in H^m(C)$ and U'_α be a locally finite open covering of C such that e has a representative e_α in $H^m(N(U'_\alpha))$. We extend the covering U'_α to a locally finite open covering U_α of R . Choose a locally finite open covering U_β of R such that $U_\beta < U_\alpha$, and ord $U_\beta \leqq n+1$.

By U'_β we denote the restriction of U_β to C . Then $N(U_\beta)$ is a complex of $\dim \leqq n$ and $N(U'_\beta)$ is a subcomplex of $N(U_\beta)$. e has a representative e_β in $H^m(N(U'_\beta))$. Since every m-chain of $N(U'_\beta)$ is an m-chain of $N(U_\beta)$ and also an m-cocycle, e_β has an extension $\tilde{e}_\beta$ in $H^m(N(U_\beta))$. Thus the element $\tilde{e} \in H^m(R)$ represented by $\tilde{e}_\beta$ is an extension of e , i.e. $\tilde{e}$ is mapped by the natural homomorphism to e .

Sufficiency. Let $\dim R = m+1 > n$. Then $m \geqq n$, and by Theorem VII.10 there is a closed set C of R and a continuous mapping f of C into S^m which cannot be extended over R . Thus by Hopf's extension theorem $f^*e \in H^m(C)$ for some $e \in H^m(S^m)$ is not extendible over R , i.e. the natural homomorphism of $H^m(R)$ into $H^m(C)$ is not onto.

VIII. 4. Dimension and homology

For dimension theory in compact spaces the concept of homology is also useful. It seems to be more convenient to intuitive understanding than the concept of cohomology. In this final section we shall give a brief description, without proof, of the relation between homology and dimension. Throughout the section all spaces are assumed to be compact T_2-spaces unless the contrary is explicitly stated.

[14] We have proved Borsuk's theorem only in case R is metric. But it is easily seen that the same is true for every paracompact T_2-space R . The only problem is to show that $[0,1] \times R$ is normal if R is paracompact T_2 . But this proposition was proved by C. H. Dowker [4] for every countably paracompact normal space R .

Definition VIII. 16. *Let A be a directed set. Suppose $\{G_\alpha \mid \alpha \in A\}$ is a collection of commutative groups. If for every $\alpha, \beta \in A$ with $\alpha > \beta$ a homomorphic mapping π^α_β of G_α into G_β is given such that $\pi^\alpha_\gamma = \pi^\alpha_\beta \pi^\beta_\gamma$, then we call the collection $\{G_\alpha ; \pi^\alpha_\beta \mid \alpha, \beta \in A, \alpha > \beta\}$ an inverse spectrum.*

Let $\{G_\alpha ; \pi^\alpha_\beta\}$ be an inverse spectrum. We consider an element $\{g_\alpha \mid \alpha \in A\}$ of the cartesian product $\prod_\alpha G_\alpha$ such that $\pi^\alpha_\beta g_\alpha = g_\beta$ whenever $\alpha > \beta$. We define the sum $g + g'$ of two such elements $g = \{g_\alpha\}$ and $g' = \{g'_\alpha\}$ by

$$g + g' = \{g_\alpha + g'_\alpha \mid \alpha \in A\}.$$

Then the collection of all such elements turns out to be a commutative group G which is called the *limit group* of the inverse spectrum $\{G_\alpha ; \pi^\alpha_\beta\}$.

We consider the collection $\{U_\alpha \mid \alpha \in A\}$ of all finite open coverings of a space R. If $U_\alpha < U_\beta$, then, by definition, $\alpha > \beta$; in this way A becomes a directed set. Denote by π^α_β a projection of $N(U_\alpha)$ into $N(U_\beta)$. Then, by 1 E), π^α_β induces a homomorphism of $H_n(N(U_\alpha), G)$ into $H_n(N(U_\beta), G)$. Moreover, we can prove that all projections of $N(U_\alpha)$ into $N(U_\beta)$ induce the same homomorphism $\hat{\pi}^\alpha_\beta$ of $H_n(N(U_\alpha), G)$ into $H_n(N(U_\beta), G)$.

Definition VIII. 17. *The limit group of the inverse spectrum*

$$\{H_n(N(U_\alpha), G) ; \hat{\pi}^\alpha_\beta \mid \alpha, \beta \in A, \alpha > \beta\}$$

is denoted by $H_n(R,G)$ and is called the n-dimensional (Čech) homology group of R.

In this definition we often adopt the additive group R_1 of the real numbers mod 1 as the coefficient group G and denote the homology group by $H_n(R)$.

As easily seen, we obtain the same limit group if we replace A by a cofinal subset A' of A in Definition VIII.12 or Definition VIII.16. Hence for compact spaces we may take the collection $\{U_\alpha \mid \alpha \in A\}$ of all finite open coverings, instead of locally finite open coverings, to define the cohomology group of R by means of Definition VIII.12 [15]. For homology and cohomology groups founded on the finite coverings there exists a duality which enables us to translate theorems on

[15] For non-compact spaces homology and cohomology groups based on the finite coverings are not practical. As shown by C. H. Dowker [1] even the 1-dimensional cohomology group of E^1 has a complicated structure.

cohomology to those on homology. As a matter of fact $H_n(R)$ is the character group of $H^n(R)$ [16].

Let C be a closed set of R. Suppose $e=\{e_\alpha\}$ is an element of $H_n(C)$ represented by $e_\alpha \in H_n(N(U'_\alpha))$. Let U_α be an extension of the finite open covering U'_α over R. Then, since $N(U'_\alpha)$ is a subsimplex of $N(U_\alpha)$, we can consider the image $he_\alpha \in H_n(N(U_\alpha))$ of e_α under the natural homomorphism. We denote by $\tilde{e}$ the element of $H_n(R)$ represented by he_α. Then, mapping e to $\tilde{e}$, we obtain the *natural homomorphism* of $H_n(C)$ into $H_n(R)$. As for the relation between homology and dimension of compact spaces it is known that

Theorem VIII. 3. *A finite dimensional space R has $\dim \leq n$ if and only if for every integer $m \geq n$ and for every closed set C of R, the natural homomorphism of $H_m(C)$ into $H_m(R)$ is an isomorphism.*

To characterize homologically the dimension of compact metric spaces the original method due to P. S. Alexandroff [2] is also useful.

Definition VIII. 18. *Let $a_0,\dots,a_n$ be points of a metric space R. If the diameter of the set $\{a_0,\dots,a_n\}$ is less than $\varepsilon>0$, then we call $\{a_0,\dots,a_n\}$ an n-dimensional ε-simplex in R. We denote by K the abstract complex of the ε-simplices in R.*

Let G be a commutative group. We mean by an ε-chain (or more precisely (n,G)-ε-chain) a finite (n,G)-chain of K and by an ε-cycle a finite (n,G)-cycle of K.

Let φ^n be an (n,G)-ε-cycle. If $\partial\psi^{n+1}=\varphi^n$ for some $(n+1, G)$-δ-chain ψ^{n+1}, then we write $\varphi^n \underset{\delta}{\sim} 0$.

Let $\{\varepsilon_i \mid i=1,2,\dots\}$ be a sequence of positive numbers such that $\lim \varepsilon_i=0$, and φ_i^n be an ε_i-cycle. Then we call the sequence $\Phi^n=(\varphi_1^n,\varphi_2^n,\dots)$ of those ε_i-cycles a true cycle or more precisely (n,G)-true cycle.

If $\varphi_i^n \underset{\delta_i}{\sim} 0$, $i=1,2,\dots$ and $\lim \delta_i=0$, then we call Φ^n homologous to zero in R and denote it by $\Phi^n \sim 0$.

A true cycle Φ^n of R may also be a true cycle for some closed set C of R. We call such a closed set C a carrier of Φ^n. Φ^n is called essential if it has a carrier in which $\Phi^n \sim 0$ does not hold.

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[16] See W. Hurewicz and H. Wallman [1].

Now the following theorem holds.

Theorem VIII. 4. *A finite dimensional compact metric space* R *has* $\dim \leq n$ *if and only if every* (n, R_1)*-true cycle* Φ^n *homologous to* 0 *in* R *is not essential* [17].

Referring to compact subsets of a Euclidean space, one can characterize their dimension in connection with their complementary spaces as demonstrated in the following obstruction theorem due to P. Alexandroff.

Theorem VIII. 5. *Let* R *be an r-dimensional compact subset of* E^n. *Then there is* $\delta > 0$ *such that for every* $\varepsilon > 0$ *one can find an* $(n-(r+1), 1)$*-cycle* φ^{n-r-1} *lying in* $E^n - R$ *having diameter* $< \varepsilon$ *but being no boundary of a chain in* $E^n - R$ *of diameter* $< \delta$. *On the other hand, for* $k > r+1$ *every* $(n-k, 1)$*-cycle of diameter* $< \varepsilon$ *lying in* $E^n - R$ *is the boundary of a chain in* $E^n - R$ *of diameter* $< \varepsilon$ [18].

The homological method provides also a useful tool for finding conditions in order that $\dim (R \times S) = \dim R + \dim S$. Here we quote a famous problem posed by P. Alexandroff: To determine a finite dimensional compact metric space R such that

$$\dim (R \times S) = \dim R + \dim S$$

for every compact metric space S.

This problem was solved by V. Boltyanski [2] [3] by means of homological methods. Y. Kodama [1] [4] solved this problem for more general spaces. [19]

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[17] In this theorem we may replace true cycle with *convergent true cycle.* If a true cycle $\Phi^n = (\varphi_1^n, \varphi_2^n, \dots)$ satisfies $\varphi_i^n - \varphi_{i+1}^n \underset{\eta_i}{\sim} 0$ for a sequence $\{\eta_i\}$ of positive numbers such that $\lim \eta_i = 0$, then it is called a convergent true cycle. This theorem can be deduced from Theorem VIII.3 by use of the relation between Čech's homology theory and Vietori's homology theory concerning convergent true cycles; see S. Lefschetz [1].

[18] In this theorem the concepts of chain and cycle are based on geometrical simplices, so a diameter of a chain means the diameter of the point set which is the union of all simplices occurring in the chain with non-null coefficients. This theorem was further developed by K. Sitnikov [2].

[19] This problem is related with another problem, to look for conditions in order that the homology or cohomology dimension coincide with the covering dimension. See E. Dyer [1], Y. Kodama [2].
M. F. Bokštein [1] established an interesting theory to show that dim of the topological product of compact metric spaces R and S is determined by the homological dimensions of R and S.

Several dimension functions, *homology dimensions* and *cohomology dimensions* can be defined in connection with the homological and cohomological concepts.

Definition VIII. 19. *Let* G *be a commutative group and* R *a space. We denote by* $D_C(R,G)$ *the greatest integer* n *such that the natural homomorphism of* $H^{n-1}(R,G)$ *into* $H^{n-1}(C,G)$ *is not onto for some closed set* C *of* R [20]. *Similarly,* $D_h(R,G)$ *will be the greatest integer* n *such that the natural homomorphism of* $H_{n-1}(C,G)$ *into* $H_{n-1}(R,G)$ *is not isomorphic for some closed set* C *of* R. *By* $D_t(R,G)$ *we denote the greatest integer* n *such that there exists an essential* $(n-1,G)$*-true cycle in* R *which is homologous to* 0. *In each of these three definitions, we define* $D_C(R,G)=0$, $D_h(R,G)=0$ *or* $D_t(R,G)=0$ *respectively if there is no* n *satisfying the condition. If for every* n *the condition is satisfied, then the respective dimension is* ∞, *and it is* -1 *if* $R=\emptyset$.

The theorems VIII.2.3.4. imply that for a finite dimensional compact metric space R

$$D_C(R,I)=D_h(R,R_1)=D_t(R,R_1)=\dim R\,.$$ [21]

Generally, we can show that

$$\dim R \geqq D_t(R,G) \geqq D_h(R,G)$$

for every G and every compact metric space.

While we have so many different definitions of dimension, [22] it is an interesting problem to characterize a certain dimension as a function (defined on a class of spaces) satisfying certain axioms. Let us denote by d a function on a class C of spaces taking values in the set $\{-1,0,1,2,\dots\}$. P. Alexandroff considered the following axioms to characterize dim on the class of all compact metric spaces.

A$_1$) $d(T^n)=n$ for the n-dimensional simplex T^n $(T^{-1}=\emptyset)$.

A$_2$) If R and S are homeomorphic, then $d(R)=d(S)$.

A$_3$) If $R=R_1\cup R_2$ for closed sets R_1, R_2 of R, then $d(R)=\max(d(R_1), d(R_2))$.

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[20] The dimension $D_C(R,G)$ and a similar dinension was investigated by H. Cohen [1], A. Okuyama [1] and others.

[21] In case $\dim R=\infty$ it is unknown whether this relation remains still valid.

[22] There are more attempts to find out good new dimension functions, which were not discussed in this book. See, e.g. O. V. Lokucievskiǐ [2], J. M. Aarts [3], J. M. Aarts - T. Nishiura [1], [2], V. I. Ponomarev [2], V. A. Valiev [1].

A_4) If $R \subset E^n$, then there exists $\varepsilon > 0$ such that for every continuous mapping f of R onto $S \subset E^n$ satisfying $\rho(p, f(p)) < \varepsilon$ the inequality $d(R) \leqq d(S)$ is implied.

Obviously dim satisfies these axioms. On the other hand L. S. Pontrjagin [1] proved that every $D_t(R, I_m)$ $(m \geqq 2)$ also satisfies the same conditions though it is different from dim, where I_m is the additive group of integers mod m. He also proved that $D_t(R, I_m)$ for different prime numbers m are all different from each other. [23]

Further reasonable axioms for d are

A_5) If $d(R) = n$, then there is a finite open covering V of R such that for every V-mapping f from R into S, $d(f(R)) \geqq n$ holds.

A_6) If R contains more than one point, and $\dim R = n$, then there is a closed set F of R such that $d(F) < d(R)$, and $R - F$ is disconnected.

P. S. Alexandroff [2] proved that for the class C of all finite dimensional compact metric spaces, dim is the only function satisfying A_1, A_2, A_3, A_5 and A_6. But the same conditions are not sufficient to characterize dim for metric spaces.

E. Ščepin [1] replaced condition A_3 with

A_3') If $R = \bigcup_i R_i$ for at most countably many closed sets R_i, then $d(R) = \sup_i d(R_i)$,

and, by use of A_1, A_2, A_3', A_5, A_6, he succeeded to characterize dim on the class of all finite-dimensional metric spaces (and also on the class of all finite-dimensional separable metric spaces). O. V. Lokucievskiĭ [3] characterized dim on the class of finite-dimensional compact T_2-spaces by modifying A_6.

K. Menger [2] posed the problem to characterize dim for separable metric spaces by A_1, A_2, A_3', M_7 and M_8, where

M_7) If $S \subset R$, then $d(S) \leqq d(R)$.

M_8) For every R, there is a compact metric space $S \supset R$ such that $d(R) = d(S)$.

According to V. I. Kuzminov [1], I. Śvedov solved Menger's problem in the negative when d is a function on the class C of all subsets of Euclidean spaces. If C is the class of all subsets of E^n and if $n \leqq 2$, then dim is the only function on C satisfying Menger's five conditions. But the problem is not yet answered if $n \geqq 3$.

J. M. Aarts [2], S. Sakai [1], T. Nishiura [1] considered further axioms to characterize dim for metric spaces and separable metric spaces.

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[23] Relations between $D_t(R, G)$ for various G were studied by M. Bokštein [1].