CHAPTER VII

DIMENSION OF NON-METRIZABLE SPACES

Throughout the preceding chapters we have considered only metric spaces except in Chapter I. Dimension theory for non-metrizable spaces has been greatly developed in these years, but still it is not as satisfactory as the theory for metric spaces. In the present chapter we shall discuss some of the principal results ever obtained in this field. Recall that, as noted before, all normal spaces and Tychonoff spaces are assumed to be T_1 (and are accordingly T_2) in this book. At some points of this chapter proofs may be somewhat sketchy to avoid bothering the reader with arguments similar to those used previously.

VII. 1. Sum theorem and subspace theorem for dim

A) A topological space X has $\dim \leq n$ if and only if for every finite open covering $\mathcal{U} = \{ U_i \mid i = 1, \ldots, k \}$ there is an open covering $\mathcal{V} = \{ V_i \mid i = 1, \ldots, k \}$ such that $V_i \subset U_i$, $i = 1, \ldots, k$ (namely $\mathcal{U}$ shrinks to $\mathcal{V}$), and such that $\operatorname{ord} \mathcal{V} \leq n + 1$.

Proof. The easy proof is left to the reader.

We owe the following theorem to P. A. Ostrand [1].

Theorem VII. 1. *Let* $\{ F_\gamma \mid \gamma \in \Gamma \}$ *be a locally finite closed covering of a topological space* X *such that* $\dim F_\gamma \leq n$ *for all* $\gamma \in \Gamma$. *Then* $\dim X \leq n$.

Proof. Well-order the given locally finite closed covering and denote it by $\{F_\gamma \mid 0 \leqq \gamma < \tau\}$. We may assume that $F_0 = \emptyset$.

Now, let $\mathcal{U} = \{U_i \mid i = 1,\dots,k\}$ be a given open covering of X. We shall define open coverings $\mathcal{V}_\gamma = \{V(\gamma,i) \mid i = 1,\dots,k\}$, $0 \leqq \gamma < \tau$, such that

(1) if $\beta < \delta$, $\mathcal{V}_\beta$ shrinks to $\mathcal{V}_\delta$,

(2) if $\beta < \delta$, $V(\beta,i) - V(\delta,i) \subset \cup\{F_\gamma \mid \beta < \delta \leqq \delta\}$,

(3) if $x \in F_\gamma$, $\operatorname{ord}_x \mathcal{V}_\gamma \leqq n+1$.

We shall define $\mathcal{V}_\gamma$, $0 \leqq \gamma < \tau$, by use of induction on γ.

First we put $\mathcal{V}_0 = \mathcal{U}$.

Now assume that $\mathcal{V}_\gamma$ have been defined for all $\gamma < \delta$. Then we put

$$W_i = \cap\{V(\gamma,i) \mid \gamma < \delta\} \quad \text{and} \quad \mathcal{W} = \{W_i \mid i = 1,\dots,k\}.$$

Let $\beta < \delta$; then by the induction hypothesis (2) we have

(4)
$$V(\beta,i) - W_i = \cup\{V(\beta,i) - V(\gamma,i) \mid \beta < \gamma < \delta\} \subset$$
$$\subset \cup\{\cup\{F_{\gamma'} \mid \beta < \gamma' \leqq \gamma\} \mid \beta < \gamma < \delta\} = \cup\{F_\gamma \mid \beta < \gamma < \delta\}.$$

We can prove that each W_i is an open set. For this purpose assume $x \in W_i$, $x \in F_{\gamma_i}$, $i = 1,\dots,p$, and $\gamma_1 < \gamma_2 < \dots < \gamma_p < \delta$, while $x \notin F_\gamma$ if $\gamma \neq \gamma_1,\dots,\gamma_p$ and $\gamma < \delta$.

Then note that $x \in V(\gamma_p, i)$ follows from $V(\gamma_p,i) \supset W_i$. Put

$$F = \cup\{F_\gamma \mid x \notin F_\gamma\};$$

then $P = V(\gamma_p,i) \cap (X - F)$ is an open neighbourhood of x. Since it follows from (4) that

$$V(\gamma_p,i) - W_i \subset \cup\{F_\gamma \mid \gamma_p < \gamma < \delta\} \subset F,$$

we obtain

$$P \subset V(\gamma_p,i) \cap [X - (V(\gamma_p,i) - W_i)] \subset W_i.$$

Thus W_i is an open set.

Next we shall prove that $\mathcal{W}$ is a covering of X. Let $x \in X$ be given. For each $\gamma < \delta$ we can find $i(\gamma)$ such that $x \in V(\gamma, i(\gamma))$. Assume that the set $\{\gamma \mid i(\gamma) = i_0\}$ is cofinal in the set $\{\gamma \mid \gamma < \delta\}$. (Namely for each element $\gamma < \delta$, there is γ' such that $i(\gamma') = i_0$ and $\gamma \leqq \gamma' < \delta$.) Then $x \in V(\gamma, i_0)$ for all $\gamma < \delta$, because of the induction hypothesis (1). Thus $x \in W_{i_0}$ follows, proving our assertion.

Using A) there is an open collection $\mathcal{W}' = \{W_i' \mid i = 1, \dots, k\}$ in X such that

$$W_i' \subset W_i, \quad i = 1, \dots, k, \quad \bigcup_{i=1}^{k} W_i' \supset F_\delta, \quad \text{and}$$

$$\operatorname{ord}_x \mathcal{W}' \leqq n+1 \quad \text{at each } x \in F_\delta,$$

because $\dim F_\delta \leqq n$. Put

$$V(\delta, i) = (W_i - F_\delta) \cup W_i' \quad \text{and} \quad \mathcal{V}_\delta = \{V(\delta, i) \mid i = 1, \dots, k\}.$$

Then $\mathcal{V}_\delta$ obviously satisfies (1), (2), (3). (Note that (2) follows from (4) and the definition of $V(\delta, i)$.)

Finally we put

$$V_i = \cap\{V(\gamma, i) \mid 0 \leqq \gamma < \tau\} \quad \text{and} \quad \mathcal{V} = \{V_i \mid i = 1, \dots, k\}.$$

Then a similar argument with the above leads us to the conclusion that $\mathcal{V}$ is an open covering of X to which $\mathcal{V}_0 = \mathcal{U}$ shrinks. It follows from (3) that $\operatorname{ord} \mathcal{V} \leqq n+1$. Hence $\dim X \leqq n$.

B) Let $\mathsf{F} = \{F_i \mid i = 1, \dots, k\}$ be a closed collection of a normal space X and $\{U_i \mid i = 1, \dots, k\}$ an open collection such that $F_i \subset U_i$, $i = 1, \dots, k$. Then there is an open collection $\mathcal{W} = \{W_i \mid i = 1, \dots, k\}$ such that $F_i \subset W_i \subset U_i$, $i = 1, \dots, k$, and $\operatorname{ord} \mathsf{F} = \operatorname{ord} \mathcal{W}$.

Proof. If $k = \operatorname{ord} \mathsf{F}$, then the proposition is obviously true.

Assume $k > \operatorname{ord} \mathsf{F} = n$. Then $\mathcal{V}' = \{V(x) \mid x \in X\}$ is an open covering of X, where

$$V(x) = X - \cup\{F_i \mid x \notin F_i\} \quad \text{for each } x \in X.$$

Since X is normal, there is an open covering $\mathcal{P} = \{P_j \mid j = 1, \dots, \ell\}$ of X such that $\mathcal{P}^\Delta < \mathcal{V}$. Put

$W_i = S(F_i, P) \cap U_i$ for $i = 1, \dots, k$ and $\mathcal{W} = \{ W_i \mid i = 1, \dots, k \}$.

Then it is obvious that $F_i \subset W_i \subset U_i$, $i = 1, \dots, k$.
To prove ord $\mathcal{W}$ = ord F , let $x \in X$. Then $S(x, P) \subset V(y)$ for some $y \in X$. Note that because of its definition $V(y)$ intersects at most n members of F . Hence $S(x, P)$, too, intersects at most n members of F , which implies that $\operatorname{ord}_x \mathcal{W} \leqq n$. Therefore we obtain ord $\mathcal{U}$ = ord F .

C) A normal space X has $\dim \leqq n$ if and only if every finite open covering of X can shrink to a closed covering of $\operatorname{ord} \leqq n+1$.

Proof. The proposition follows directly from B).

Theorem VII. 2. *Let* $\{ G_i \mid i = 1, 2, \dots \}$ *be a closed covering of a normal space* X *such that* $\dim G_i \leqq n$, $i = 1, 2, \dots$. *Then* $\dim X \leqq n$.[1]

Proof. We may assume

(1) $\qquad G_1 \subset G_2 \subset \dots ,$

because the sum theorem holds for every finite sum (Theorem VII.1.) Let $\mathcal{U}$ = $\{ U_i \mid i = 1, \dots, k \}$ be a given open covering of X . Since X is normal, we can construct an open covering $\mathcal{U}_1 = \{ U_i^1 \mid i = 1, \dots, k \}$ of X such that $\bar{U}_i^1 \subset U_i$. By B) and C) there is an open covering $\mathcal{V}_1 = \{ V_i^1 \mid i = 1, \dots, k \}$ of G_1 such that

$$V_i^1 \subset U_i^1 \quad \text{and} \quad \operatorname{ord} \bar{\mathcal{V}}_1 \leqq n+1 .$$

Now construct an open covering $\mathcal{U}_2 = \{ U_i^2 \mid i = 1, \dots, k \}$ of X such that

$$\bar{U}_i^2 \subset V_i^1 \cup (U_i^1 - G_1) .$$

[1] This theorem was proved by E. Čech [2] and others. K. Morita [2] proved the following more general theorem: Let $\{ G_\gamma \mid \gamma < \tau \}$ be a closed covering of a normal space X and $\{ P_\gamma \mid \gamma < \tau \}$ an open covering such that $\dim G_\gamma \leqq n$, $G_\gamma \subset P_\gamma$ for every $\gamma < \tau$, and such that $\{ P_\gamma \mid \gamma < \delta \}$ is locally finite for each $\delta < \tau$. Then $\dim X \leqq n$. (Here τ denotes a not necessarily countable ordinal number.)

Again by use of B) and C) we obtain an open covering $V_2 = \{ V_i^1 \mid i = 1,\dots,k\}$ of G_2 such that

$$V_i^2 \subset U_i^2 \quad \text{and} \quad \text{ord}\ \bar{V}_2 \leqq n+1 .$$

Continue the same process to define open coverings $U_j = \{ U_i^j \mid i = 1,\dots,k\}$ and $V_j = \{ V_i^j \mid i = 1,\dots,k\}$ of X and of G_j, respectively, satisfying

(2) $\quad \text{ord}\ \bar{U}_j \leqq n+1$

(3) $\quad \bar{U}_i^{j+1} \subset V_i^j \cup (U_i^j - G_j)$,

(4) $\quad V_i^j \subset U_i^j$,

and accordingly

(5) $\quad U_i^{j+1} \subset U_i^j$.

Put $F_i = \overline{(V_i^1 \cup V_i^2 \cup \dots)}$. Then it is obvious that $F_i \subset \bar{U}_i^1 \subset U_i$, because of (4) and (5). On the other hand, since $\bigcup_{i=1}^k V_i^j = G_j$, $\bigcup_{i=1}^k F_i = X$ follows. Namely U shrinks to the closed covering $F = \{ F_i \mid i = 1,\dots,k\}$.

To prove $\text{ord}_x F \leqq n+1$ at each point $x \in X$, let $x \in G_j - G_{j-1}$ be given. (Assume $G_0 = \emptyset$.) Then if $x \notin \bar{V}_i^j$ for some i with $1 \leqq i \leqq k$, then $x \notin \bar{U}_i^{j+1}$ follows from (3). Thus

$$x \notin \overline{(U_i^{j+1} \cup U_i^{j+2} \cup \dots)} = \bar{U}_i^{j+1}$$

follows from (5). (4) implies

$$x \notin \overline{(V_i^{j+1} \cup V_i^{j+2} \cup \dots)} .$$

On the other hand

(7) $\quad x \notin G_{j-1} = \bar{G}_{j-1} \supset \bar{V}_i^1 \cup \dots \cup \bar{V}_i^{j-1}$

follows from (1). Combine (6), (7) and the original assumption $x \notin \bar{V}_i^j$ to get $x \notin F_i$. Now $\text{ord}_x F \leqq n+1$ follows from the above argument and (2). Thus by C) we conclude $\dim X \leqq n$.

Corollary. *Let* $\{F_\alpha \mid \alpha \in A\}$ *be a locally countable closed covering of a paracompact* T_2*-space* X . *If* $\dim F_\alpha \leqq n$ *for all* $\alpha \in A$, *then* $\dim X \leqq n$.

Proof. Combine Theorems VII.1, VII.2 and the following proposition D).

D) Let F be a closed subset of a topological space X . Then $\dim F \leqq \dim X$, and $\operatorname{Ind} F \leqq \operatorname{Ind} X$.

Proof. Obvious.

Definition VII. 1. *A normal space* X *is called totally normal if for every open set* U *of* X , *there is an open covering* $\mathcal{U}$ *of* U *such that*

(i) *each member of* $\mathcal{U}$ *is an* F_σ*-set (= a countable sum of closed sets) in* X ,

(ii) $\mathcal{U}$ *is locally finite in* U .

E) Every perfectly normal space and every hereditarily paracompact T_2-space are totally normal.

Proof. Omitted.

Remark. 1) Every F_σ-set in a normal space is normal.

2) A T_2-space X is normal if and only if every finite open covering of X shrinks to an open covering by cozero sets. (See I.1 A). The easy proofs of the remarks are left to the reader.)

F) Every totally normal space X is hereditarily normal. In fact every subset of a totally normal space is totally normal.

Proof. Let U be an open set of X . Then there is a covering $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of U such that $\mathcal{U}$ is locally finite in U, and each U_α is a cozero set in X. Then by Remark 1) U_α is normal. Now suppose that $\{V_i \mid i = 1,\dots,k\}$ is a given open covering of U . Then by Remark 2) $\{V_i\}$ can shrink to a covering $\{V_i^\alpha \mid i = 1,\dots,k\}$ of U_α , where each V_i^α is a cozero set in U_α . Since U_α is a cozero set in U , so is V_i^α . Define

$$W_i = \cup\{V_i^\alpha \mid \alpha \in A\} .$$

Then W_i is a cozero set in U satisfying $W_i \subset V_i$. Hence $\{V_i\}$ shrinks to the covering $\{W_i \mid i = 1,\dots,k\}$ of U by cozero sets. Hence by Remark 2) U is normal. Thus X is hereditarily normal by I.1 B). Now the second statement of the proposition is almost obvious.[2]

Theorem VII. 3.[3] *Let X be a totally normal space and Y a subset of X. Then* $\dim Y \leqq \dim X$.

Proof. Let $U = \{U_i \mid i = 1,\dots,k\}$ be an open covering of Y. Then $U_i = V_i \cap Y$ for some open sets V_i, $i = 1,\dots,k$, of X. Define $V = \bigcup\{V_i \mid i = 1,\dots,k\}$. Then, since X is totally normal, there is an open covering $\{W_\alpha \mid \alpha \in A\}$ of V such that $\{W_\alpha\}$ is locally finite in V and such that each W_α is F_σ in X, i.e. $W_\alpha = \bigcup_{i=1}^{\infty} F_{\alpha i}$ for closed sets $F_{\alpha i}$ in X.

Assume that $\dim X = n$; then by D) $\dim F_{\alpha i} \leqq n$ holds. Put $F_i = \bigcup\{F_{\alpha i} \mid \alpha \in A\}$; then $\{F_{\alpha i} \mid \alpha \in A\}$ is a locally finite closed covering of F_i, and hence by Theorem VII.1, we obtain $\dim F_i \leqq n$.

Since $\{F_i \mid i = 1,2,\dots\}$ is a closed covering of V, and V is normal, from Theorem VII.2 it follows that $\dim V \leqq n$. Thus $\{V_i \mid i = 1,\dots,k\}$ can shrink to an open covering $\{V_i' \mid i = 1,\dots,k\}$ of V with $\operatorname{ord} \leqq n+1$. Then U shrinks to $V' = \{V_i' \cap Y \mid i = 1,\dots,k\}$, where V' is an open covering of Y such that $\operatorname{ord} V' \leqq n+1$. This proves, because of A), that $\dim Y \leqq n$.

It is impossible to extend this theorem to hereditarily normal spaces. V. V. Filippov [3] and E. Pol - R. Pol [1] proved that there is a hereditarily normal, zero-dimensional space X which contains subspaces X_n, $n = 1,2,\dots$ with $\dim X_n = \operatorname{Ind} X_n = n$.[4]

---

[2] The space of all ordinal numbers $\leqq \omega_1$ with the order topology is hereditarily normal but not totally normal. For an example of a totally normal space which is neither perfectly normal nor paracompact see R. H. Bing [1] (Example G).

[3] This theorem is due to C. H. Dowker [3].

[4] In fact V. V. Filippov proved this theorem assuming a set-theoretical hypothesis while E. Pol - R. Pol showed that the assumption could be dropped. Observe that this theorem implies that the subspace theorem does not hold for a compact T_2-space and its normal subspaces. Because, as easily seen, $\dim \beta X = \dim X$ and $\operatorname{Ind} \beta X = \operatorname{Ind} X$ hold for every normal space X. (See Section 6 of the present chapter.)

C. H. Dowker [3] defined the *local dimension* $\operatorname{locdim} X$ of a space X as the least number n such that every point of X has a closed neighbourhood $\bar{U}$ with $\dim \bar{U} \leqq n$ and studied relations between local dimension and the subset theorem.

K. Morita [2] proved that if X is a normal space and Y a subset of X with the star-finite property, then $\dim Y \leqq \dim X$.

VII. 2. Dimensions of non-metrizable spaces

Although we proved sum theorems in the previous section for considerably general spaces, it is not so easy to establish a satisfactory dimension theory for non-metrizable spaces. Perhaps a reason of the difficulty lies in the fact that different dimension functions do not easily coincide when the space is non-metrizable. In the present section we will study relations between different definitions of dimension on non-metrizable spaces.

Generally it is obvious that ind $X \leqq \text{Ind } X$ holds for every T_1-space X. Conditions under which ind and Ind coincide were studied by quite a few authors, among whom Y. Katsuta [1] proved that ind $X = \text{Ind } X$ holds whenever X is hereditarily paracompact T_2 and has the star-finite property.[5]

On the other hand V. V. Filippov [1] constructed a compact T_2-space X such that ind $X = 2$, Ind $X = 3$.

Concerning the relationships between dim X and Ind X the following assertion can be proved.[6]

A) For every normal space X : dim $X \leqq \text{Ind } X$.

Proof. Let us show by induction on the number n that Ind $X \leqq n$ implies dim $X \leqq n$.

[5] T. Mizokami [1] proved the same for every totally normal space X satisfying the following condition: For each open base $\mathcal{U}$ of X there is an ordered open covering $(\mathcal{V},<)$ of X such that (i) for each $V \in \mathcal{V}$, $\{V' \in \mathcal{V} \mid V' < V\}$ is locally finite in V, (ii) for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$, $B(V) \subset B(U)$. This theorem unifies various other theorems including the above-mentioned result of Y. Katsuta and others due to K. Morita, R. Ford, B. Fitzpatrick and K. Nagami. See also L. F. McAuley [1], J. A. French [1] and V. V. Fedorčuk [2].

Perhaps another reason of the difficulty of general dimension theory is that there is no convenient theorem like Theorem II.2 to characterize dimension by a special base if the space is non-metrizable. J. Nagata [12] and T. Mizokami [2] proved Ind $X \leqq n$ if and only if X has a σ-closure preserving base $\mathcal{U}$ such that Ind $B(U) \leqq n-1$ for all $U \in \mathcal{U}$, under certain conditions on X. The latter's condition is particularly satisfied if X is the image of a metric space under a closed continuous mapping. However, it is unknown yet if the same theorem holds whenever X is a regular space with a σ-closure-preserving base.

[6] This assertion is due to N. B. Vedenissoff [1].

Since the validity of the assertion is clear for $n = -1$, we shall assume it for $\operatorname{Ind} X \leq n-1$. Now, assume $\operatorname{Ind} X \leq n$, and suppose $\mathcal{U} = \{ U_i \mid i = 1, \dots, k \}$ is a finite open covering of X. Then by virtue of the normality of X and I.1 A), there exists an open covering $\mathcal{V} = \{ V_i \mid i = 1, \dots, k \}$ such that $\bar{V}_i \subset U_i$ and $\operatorname{Ind} B(V_i) \leq n-1$. Using the induction hypothesis we obtain $\dim B(V_i) \leq n-1$. Hence by Theorem VII.2 $B = \cup_{i=1}^{k} B(V_i)$ is a closed set with $\dim B \leq n-1$. Therefore there exists an open covering $\mathcal{W} = \{ W_i \mid i = 1, \dots, k \}$ of B such that $\bar{W}_i \subset U_i$ and $\operatorname{ord} \bar{\mathcal{W}} \leq n$.

Now we can easily construct a finite open covering $\mathcal{N}$ of X such that

$$\mathcal{N} < \wedge \{ \{ U_i, X - \bar{W}_i \} \mid i = 1, \dots, k \},$$

and each member of $\mathcal{N}$ intersects at most n members of $\bar{\mathcal{W}}$. Since X is normal, by I.1 C) we can find a finite open covering $\mathcal{P}$ satisfying $\mathcal{P}^{\Delta} < \mathcal{N}$. Put

$$Q_i = S(\bar{W}_i, \mathcal{P}) \quad \text{and} \quad \mathcal{Q} = \{ Q_i \mid i = 1, \dots, k \} .$$

Then it is easy to see that $\mathcal{Q}$ is an open collection satisfying

$$\mathcal{Q} < \mathcal{U}, \ \operatorname{ord} \mathcal{Q} \leq n, \ \text{and} \ \cup \{ Q \mid Q \in \mathcal{Q} \} \supset B .$$

On the other hand, we define

$$\mathcal{U}' = \wedge \{ \{ U_i, X - \bar{U}_i \} \mid i = 1, \dots, k \};$$

then $\mathcal{U}'$ is an open collection which covers $X - B$ and satisfies $\operatorname{ord} \mathcal{U}' \leq 0$. This implies $\operatorname{ord} \mathcal{Q} \cup \mathcal{U}' \leq n+1$. Since $\mathcal{Q} \cup \mathcal{U}'$ is an open refinement of $\mathcal{U}$, we conclude that $\dim X \leq n$.

In a similar way we can easily show the following assertion which was first proved by P. S. Alexandroff.

B) For every compact T_2-space X, $\dim X \leq \operatorname{ind} X$.

On the other hand I. M. Leĭbo [1] proved that $\dim X = \operatorname{Ind} X$ holds whenever X is the image of a metric space by a closed continuous mapping and also that $\dim X = \operatorname{Ind} X = \operatorname{ind} X$ holds whenever X is the image of a separable metric space by a closed continuous mapping. B. Pasynkov [2], [5] also obtained important results in this aspect. In [2] he proved coincidence of the three dimension functions for every locally compact group. [7] In the following we shall discuss a remarkable result obtained in [5].

C) Let f be a continuous mapping from a normal space X onto a metric space Y and g a continuous mapping from Y onto a metric space Z such that $\varphi = g \circ f$ is a closed mapping satisfying $\dim \varphi^{-1}(z) \leqq 0$ for all $z \in Z$. If $\dim Y \leqq n$, then $\operatorname{Ind} X \leqq n$.

Proof. The proof will be carried out by induction on n.

If $n = -1$, then the proposition is obviously true.

Assume that it is true if $\dim Y \leqq n-1$. First we remark that for each $z \in Z$ and each open neighbourhood U of $\varphi^{-1}(z)$ in X, the set $X - f(X-U)$ is a neighbourhood of $g^{-1}(z)$ in Y (i.e. the interior of the former set contains the latter), because the mapping φ is closed and

$$g^{-1}(Z - \varphi(X - U)) \subset X - f(X - U) .$$

Now, let F and G be given disjoint closed sets in X. Then for each $z \in Z$ there are open sets U_z and V_z in X such that

(1) $$\varphi^{-1}(z) \subset U_z \cup V_z , \quad U_z \cap V_z = \emptyset , \quad U_z \cap G = \emptyset , \quad V_z \cap F = \emptyset .$$

because $\dim \varphi^{-1}(z) \leqq 0$. Put $W_z = U_z \cup V_z$. The remark above implies that $Q_z = Y - f(X - W_z)$ is a neighbourhood of $g^{-1}(z)$ in Y.

Hence there is a locally finite open covering $\mathcal{Q}$ of Y such that $\mathcal{Q} < \{ Q_z \mid z \in Z \}$ and

(2) $$\dim B(Q) \leqq n-1 \quad \text{for all} \quad Q \in \mathcal{Q} ,$$

because $\dim Y \leqq n$ for the metric space Y. Let $Q \in \mathcal{Q}$; then $Q \subset Q_z$ for some $z \in Z$. This implies that $f^{-1}(Q) \subset W_z$.

[7] In fact he proved more, namely $\dim X = \operatorname{Ind} X = \operatorname{ind} X = \operatorname{ind} G - \operatorname{ind} H$ if X is the factor space G/H of a locally compact group G by a closed subgroup H. K. Nagami [5] obtained a similar result. See also P. Alexandroff - V. Ponomarev [1] and V. V. Fedorčuk [3] for conditions implying the coincidence of different dimension functions.

Now define

(3) $U'(Q) = U_z \cap f^{-1}(Q)$ and $V'(Q) = V_z \cap f^{-1}(Q)$.

Observe that

(4) $f(B(f^{-1}(Q))) \subset B(Q)$.

Note that $\{U'(Q) \mid Q \in \mathcal{Q}\}$ is a locally finite open collection covering F and that $U'(Q) \cap G = \emptyset$ for all $Q \in \mathcal{Q}$.

Put $P = \bigcup\{U'(Q) \mid Q \in \mathcal{Q}\}$.

Then P is an open set of X such that $F \subset P \subset X - G$ and

$$B(P) \subset \bigcup\{B(U'(Q)) \mid Q \in \mathcal{Q}\} \subset \bigcup\{B(f^{-1}(Q)) \mid Q \in \mathcal{Q}\} .$$

The last relation follows from the local finiteness of $\{U'(Q) \mid Q \in \mathcal{Q}\}$ and from the fact that $B(U'(Q)) \subset B(f^{-1}(Q))$, which easily follows from (1) and (3). Hence

(5) $$f(B(P)) \subset f(\bigcup\{B(f^{-1}(Q)) \mid Q \in \mathcal{Q}\}) \subset$$
$$\subset \bigcup\{f(B(f^{-1}(Q)) \mid Q \in \mathcal{Q}\} \subset \bigcup\{B(Q) \mid Q \in \mathcal{Q}\} = Q' .$$

The last part of the relation follows from (4). Since $\mathcal{Q}$ is locally finite, $\dim Q' \leqq n-1$ follows from (2) by use of the sum theorem. Thus $\dim f(B(P)) \leqq n-1$ is implied by (5). Hence by the induction hypothesis we obtain $\operatorname{Ind} B(P) \leqq n-1$. Therefore $\operatorname{Ind} X \leqq n$.

D) Let $\mathcal{U}$ be a locally finite open covering of a normal space X with $\dim X \leqq n$. Then $\mathcal{U}$ can be shrunk to an open covering $\mathcal{V}$ such that there is a locally finite open covering $\mathcal{V}'$ each of whose members intersects at most $n+1$ members of $\mathcal{V}$.

Proof. Use I.1 A) and the technique of the proof of Theorem II.6 to shrink $\mathcal{U}$ to $\bar{\mathcal{V}}$ where $\mathcal{V}$ is an open covering such that $\operatorname{ord} \bar{\mathcal{V}} \leqq n+1$. Suppose

$$\mathcal{U} = \{U_\alpha \mid \alpha \in A\} \text{ and } \mathcal{V} = \{V_\alpha \mid \alpha \in A\} ,$$

where $\bar{V}_\alpha \subset U_\alpha$. Then for each $x \in X$ we define

$$V(x) = (\cap\{U_\alpha \mid x \in \bar{V}_\alpha\}) \cap (\cap\{X - \bar{V}_\alpha \mid x \notin \bar{V}_\alpha\}) .$$

Then $\mathcal{V}' = \{V(x) \mid x \in X\}$ satisfies the desired condition.

E) Let $\mathcal{U}$ be a locally finite open covering of a normal space X. Then there is a locally finite open covering $\mathcal{A}$ such that $\mathcal{A}^* < \mathcal{U}$

Proof. Assume $\mathcal{U} = \{ U_\alpha \mid \alpha \in A \}$. Then find an open covering $\{ V_\alpha \mid \alpha \in A \}$ satisfying $\bar{V}_\alpha \subset U_\alpha$. For each finite subset B of A we put

$$W(B) = (\cap \{ U_\alpha \mid \alpha \in B \}) \cap (\cap \{ X - \bar{V}_\alpha \mid \alpha \in A - B \}) \text{, and}$$
$$\mathcal{W} = \{ W(B) \mid B \text{ is a finite subset of } A \}.$$

Then $\mathcal{W}$ is a locally finite open covering such that $\mathcal{W}^\Delta < \mathcal{U}$. By repeating the same argument we construct a locally finite open covering $\mathcal{A}$ such that $\mathcal{A}^\Delta < \mathcal{W}$. Then $\mathcal{A}$ is the desired covering.

Theorem VII. 4. *Let φ be a closed continuous mapping from a normal space X onto a metric space Z such that* $\dim \varphi^{-1}(z) \leq 0$ *for all* $z \in Z$. *Then* $\dim X = \operatorname{Ind} X$.

Proof. Since $\dim X \leq \operatorname{Ind} X$ is proved in A), we shall prove $\operatorname{Ind} X \leq \dim X$. Assume $\dim X \leq n$. Let $\mathcal{W}_1, \mathcal{W}_2, \ldots$ be a sequence of locally finite open coverings of Z such that

$$\mathcal{W}_1 > \mathcal{W}_2^* > \mathcal{W}_2 > \mathcal{W}_3^* > \ldots \quad \text{and} \quad \operatorname{mesh} \mathcal{W}_i \to 0 \quad \text{as} \quad i \to \infty .$$

Put

$$\mathcal{U}_i = f^{-1}(\mathcal{W}_i) = \{ f^{-1}(W) \mid W \in \mathcal{W}_i \} ;$$

then each $\mathcal{U}_i$ is a locally finite open covering of X. Since $\dim X \leq n$, by use of D) we can shrink $\mathcal{U}_1$ to an open covering $\mathcal{V}_1$ and construct a locally finite open covering $\mathcal{V}_1'$ each of whose members meets at most $n+1$ members of $\mathcal{V}_1$. By E) there is a locally finite open covering $\mathcal{A}$ of X such that

$$\mathcal{A}^* < \mathcal{V}_1 \wedge \mathcal{V}_1' \wedge \mathcal{U}_2 .$$

Shrink $\mathcal{A}$ to an open covering $\mathcal{V}_2$ and construct a locally finite open covering $\mathcal{V}_2'$ each of whose members meets at most $n+1$ members of $\mathcal{V}_2$. Then $\mathcal{V}_2^* < \mathcal{V}_1 \wedge \mathcal{U}_2$, and each member of $\mathcal{V}_2$ intersects at most $n+1$ members of $\mathcal{V}_1$. Repeating the same process we get a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of locally finite open coverings of X such that

$$V_1 > V_2^* > V_2 > V_3^* \ldots, \quad V_i < U_i \quad \text{for} \quad i = 1, 2, \ldots ,$$

and each member of V_{i+1} meets at most $n+1$ members of V_i .

Next, let us define the equivalence relation $\sim$ among the points of X by

$$x \sim y \quad \text{if and only if} \quad y \in \bigcap_{i=1}^{\infty} S(x, V_i) .$$

Then each equivalence class is a subset of $\varphi^{-1}(z)$ for some $z \in Z$. Thus we can talk about the natural mapping f from X onto Y , the set of all equivalence classes of X and the natural mapping g from Y onto Z . (Namely f maps $x \in X$ to the class containing x .) Introduce a topology into Y by defining that $\{ S(y, f(V_i)) \mid i = 1, 2, \ldots \}$ is a neighbourhood base at each point y of Y . Then it is easy to see that f and g are continuous mappings such that $\varphi = g \cdot f$ and that Y is a metric space with $\dim Y \leqq n$ (by virtue of Theorem V.3). Thus $\operatorname{Ind} X \leqq n$ follows by use of C). Hence $\dim X = \operatorname{Ind} X$.[8]

Negative results in this fields are due to A. L. Lunc [1] and O. V. Lokucievskiǐ [1], who constructed a compact T_2-space R such that $\dim R = 1$ and $\operatorname{ind} R = \operatorname{Ind} R = 2$.[9]

More generally, P. Vopenka [1] showed that for every m, n with $1 \leqq m \leqq n \leqq \infty$, there exist compact T_2-spaces R and S such that $\dim R = m$, $\operatorname{ind} R = n$, $\dim S = m$, and $\operatorname{Ind} S = n$.

On the other hand, C. H. Dowker [3] constructed a normal space R such that $\operatorname{ind} R = 0$ and $\dim R = \operatorname{Ind} R = 1$.

VII. 3. Sum theorem and subspace theorem for Ind

O. V. Lokucievskiǐ's compact T_2-space R which we quoted in the preceding section contains two closed subsets F_1 and F_2 such that

$$R = F_1 \cup F_2, \quad \operatorname{ind} F_1 = \operatorname{Ind} F_1 = 1 , \quad \text{and} \quad \operatorname{ind} F_2 = \operatorname{Ind} F_2 = 1 .$$

[8] In this proof factorization of the mapping φ plays an important role. See S. Mardešić [1] and B. Pasynkov [8] for factorization theorems in dimension theory.

[9] V. Fedorčuk [1] gave a first countable compact T_2-space X with $\dim X < \operatorname{ind} X$.

Thus the finite sum theorem for inductive dimension does not hold even if the space is compact T_2 . In the present section we shall prove a sum theorem and a subspace theorem for Ind of totally normal spaces, which are mainly due to C. H. Dowker [2]. In the rest of this section we shall be concerned only with large (strong) inductive dimension.

A) To prove the subspace theorem for every totally normal space, it suffices to prove the same for all open subsets.

Proof. Assume that $\operatorname{Ind} U \leqq \operatorname{Ind} X$ holds for every open set U of every totally normal space X. To prove the theorem in general, we use induction on $n = \operatorname{Ind} X$. If $n = -1$, then it is obviously true. Assume that the theorem has been proved for every $(n-1)$-dimensional space. Now, let $\operatorname{Ind} X = n$, and Y be an arbitrary subset of X. Suppose F and G are disjoint closed sets of Y. Then $U = X - \bar{F} \cap \bar{G}$ is an open set of X, and hence $\operatorname{Ind} U \leqq n$. Since $F \cap U$ and $G \cap U$ are disjoint closed sets of U, there is an open set V of U such that

$$F \cap U \subset V \subset U - G \cap U \quad \text{and} \quad \operatorname{Ind} B_U(V) \leqq n-1 \ .$$ [10]

Put $W = V \cap Y$. Then W is an open set of Y such that

$$F \subset W \subset Y - G \quad \text{and} \quad B_Y(W) \subset B_U(V) \ .$$

Hence by the induction hypothesis we get $\operatorname{Ind} B_Y(W) \leqq n-1$, which proves $\operatorname{Ind} Y \leqq n$.

B) Let X be a hereditarily normal space. If F is a closed subset of X such that $\operatorname{Ind} F \leqq n$ and $\operatorname{Ind} (X-F) \leqq n$, then $\operatorname{Ind} X \leqq n$.

Proof. For $n = -1$ the assertion is clearly true. Assume its validity for every F satisfying $\operatorname{Ind} F \leqq n-1$ and $\operatorname{Ind} (X-F) \leqq n-1$. Let G and H be disjoint closed sets of X. Since $\operatorname{Ind} F \leqq n$, there exists an open set U of F such that $F \cap G \subset U \subset \bar{U}^F \subset F - H$ [11] and

(1) $\qquad \operatorname{Ind} (\bar{U}^F - U) \leqq n-1$.

[10] $B_U(.)$ denotes the boundary in the subspace U.

[11] By $\bar{U}^F$ we denote the closure of U in the subspace F.

Then $G \cup U$ and $H \cup (F - \bar{U}^F)$ satisfy

$$\overline{G \cup U} \cap (H \cup (F - \bar{U}^F)) = \emptyset \quad \text{and} \quad (G \cup U) \cap \overline{H \cup (F - \bar{U}^F)} = \emptyset .$$

Hence, by virtue of the hereditary normality of X, and I.1 B), there exist open sets V and W such that

$$G \cup U \subset V , \ H \cup (F - \bar{U}^F) \subset W , \ V \cap W = \emptyset , \text{ and}$$
$$\bar{V} \cap \bar{W} \subset \overline{G \cup U} \cap \overline{H \cup (F - \bar{U}^F)} = \bar{U} \cap \overline{F - \bar{U}^F} \subset \bar{U}^F - U .$$

Thus $\bar{V} \cap (X - F)$ and $\bar{W} \cap (X - F)$ are disjoint closed sets of $X - F$. Since $\operatorname{Ind} (X - F) \leqq n$, there exists an open set P of $X - F$ such that $\bar{V} \cap (X - F) \subset P \subset (X - F) - \bar{W}$ and

(2) $$\operatorname{Ind} (\bar{P}^{X - F} - P) \leqq n - 1 .$$

Now, put $Q = V \cup P$; then Q is an open set of X which satisfies $G \subset Q \subset X - H$ and

(3) $$B(Q) \subset (\bar{P}^{X - F} - P) \cup (\bar{U}^F - U) = S ;$$

$\bar{U}^F - U$ is a closed subset of S such that

$$\bar{P}^{X - F} - P = S - (\bar{U}^F - U) .$$

Using the induction hypothesis we deduce from (1) and (2) that $\operatorname{Ind} S \leqq n - 1$. Therefore, by (3) and I D) we conclude that $\operatorname{Ind} B(Q) \leqq n - 1$. Hence $\operatorname{Ind} X \leqq n$.

C) Let X be a hereditarily normal space and

$$X = Y_0 \supset Y_1 \supset Y_2 \supset \dots$$

be a sequence of open sets such that $\bigcap_{i=1}^{\infty} Y_i = \emptyset$. If $\operatorname{Ind} (Y_{i-1} - Y_i) \leqq n$ for $i = 1, 2, \dots$, then $\operatorname{Ind} X \leqq n$.

Proof. This proposition can be proved by arguing similarly as in the proof of B).
The proposition is obviously true if $n = -1$. Assume that the proposition is true for $(n-1)$-dimensional sets. Let G and H be disjoint closed sets of X. Since $\operatorname{Ind} (Y_0 - Y_1) \leqq n$, there is an open set U_1 of $Y_0 - Y_1$ such that

$$G \cap (Y_0 - Y_1) \subset U_1 \subset (Y_0 - Y_1) - H \quad \text{and} \quad \text{Ind } B_{Y_0 - Y_1}(U_1) \leq n-1 .$$

Find open sets V_1 and W_1 in X such that

$$G \cup U_1 \subset V_1 , H \cup [(Y_0 - Y_1) - \bar{U}_1^{Y_0 - Y_1}] \subset W_1 ,$$
$$V_1 \cap W_1 = \emptyset , \text{ and } \bar{V}_1 \cap \bar{W}_1 \subset B_{Y_0 - Y_1}(U_1) .$$

Since $\text{Ind } (Y_1 - Y_2) \leq n$, there is an open set U_2 of $Y_1 - Y_2$ such that

$$\bar{V}_1 \cap (Y_1 - Y_2) \subset U_2 \subset (Y_1 - Y_2) - \bar{W}_1 \quad \text{and} \quad \text{Ind } B_{Y_1 - Y_2}(U_2) \leq n-1 .$$

Find open sets V_2 and W_2 in X such that

$$V_1 \cup U_2 \subset V_2 , W_1 \cup [(Y_1 - Y_2) - \bar{U}_2^{Y_1 - Y_2}] \subset W_2 ,$$
$$V_2 \cap W_2 = \emptyset , \text{ and } \bar{V}_2 \cap \bar{W}_2 \subset B_{Y_0 - Y_1}(U_1) \cup B_{Y_1 - Y_2}(U_2) .$$

Continuing the same process we get sequences $\{U_i\}$, $\{V_i\}$, $\{W_i\}$, where U_i is open in $Y_{i-1} - Y_i$, and V_i and W_i are open in X satisfying

$$\bar{V}_{i-1} \cap (Y_{i-1} - Y_i) \subset U_i \subset (Y_{i-1} - Y_i) - \bar{W}_{i-1} ,$$
$$\text{Ind } B_{Y_{i-1} - Y_i}(U_i) \leq n-1 ,$$
$$V_{i-1} \cup U_i \subset V_i , W_{i-1} \cup [(Y_{i-1} - Y_i) - \bar{U}_i^{Y_{i-1} - Y_i}] \subset W_i ,$$
$$V_i \cap W_i = \emptyset , \text{ and } \bar{V}_i \cap \bar{W}_i \subset \bigcup_{j=1}^{i} B_{Y_{j-1} - Y_j}(U_j) .$$

Then put $U = \bigcup_{i=1}^{\infty} U_i$. Now it is easy to verify that U is an open set such that

$$G \subset U \subset X - H \quad \text{and} \quad B(U) = \bigcup_{i=1}^{\infty} B_{Y_{i-1} - Y_i}(U_i) .$$

Thus by use of the induction hypothesis we obtain $\text{Ind } B(U) \leq n-1$. Hence $\text{Ind } X \leq n$.

D) Let U be an open set of a totally normal space X. Then U has coverings P and Q which are both σ-discrete in U and consist of closed sets of X such that $P^\bullet$ shrinks to Q, where $P^\bullet = \{P^\bullet \mid P \in P\}$. ($P^\bullet$ denotes the interior of P.)

Proof. By the definition of totally normal space U has an open covering $U = \{U_\alpha \mid \alpha \in A\}$ which is locally finite in U, and each U_α is an F_σ-set of X. Thus each U_α can be expressed as

$$U_\alpha = \bigcup_{i=1}^{\infty} W_{\alpha i}, \ \bar{W}_{\alpha i} \subset W_{\alpha i+1},$$

where $W_{\alpha i}$ are open sets of X. We may assume that the members of U are well-ordered, i.e. $U = \{U_\alpha \mid 0 \leqq \alpha < \tau\}$.

Now define

$$P_{\alpha i} = \bar{W}_{\alpha i} - \bigcup_{\beta < \alpha} W_{\beta i+1}, \ Q_{\alpha i} = \bar{W}_{\alpha i-1} - \bigcup_{\beta < \alpha} W_{\beta i+2},$$

$$P_i = \{P_{\alpha i} \mid 0 \leqq \alpha < \tau\}, \ Q_i = \{Q_{\alpha i} \mid 0 \leqq \alpha < \tau\}.$$

Then each P_i is discrete in U and consists of closed sets of X, and the same is true for Q_i, too, while $Q_{\alpha i} \subset P^\bullet_{\alpha i}$. Thus $P = \bigcup_{i=1}^{\infty} P_i$ and $Q = \bigcup_{i=1}^{\infty} Q_i$ are coverings as desired.

E) Assume that $\{F_\alpha \mid \alpha \in A\}$ is a discrete covering of X. (Thus each F_α is closed and open in X.) If $\operatorname{Ind} F_\alpha \leqq n$ for all $\alpha \in A$, then $\operatorname{Ind} \bigcup \{F_\alpha \mid \alpha \in A\} \leqq n$.

Proof. Obvious. (Use induction on n.)

F)$_n$ Let X be a totally normal space and $\{F_i \mid i = 1, 2, \ldots\}$ a closed covering of X such that $\operatorname{Ind} F_i \leqq n$, $i = 1, 2, \ldots$. Then $\operatorname{Ind} X \leqq n$.

G)$_n$ Let U be an open subset of a totally normal space X with $\operatorname{Ind} X \leqq n$. Then $\operatorname{Ind} U \leqq n$.

Proof of F)$_n$ *and* G)$_n$. We will prove the two propositions simultaneously by induction on n.

i) F)$_{-1}$ and G)$_{-1}$ are obviously true.

ii) F)$_{n-1}$ implies G)$_n$.

First observe that F)$_{n-1}$ implies the following assertion

(1) Let $\{F_i \mid i = 1, 2, \ldots\}$ be a closed covering of a totally normal space X and $\{D_i \mid i = 1, 2, \ldots\}$ a closed covering of X such that $F_i \subset D^\bullet_i$, $i = 1, 2, \ldots$. If $\operatorname{Ind} D_i \leqq n$, $i = 1, 2, \ldots$, then $\operatorname{Ind} X \leqq n$.

Proof of the assertion (1) is as follows. Let G and H be disjoint closed sets of X. For each i we can find open sets V_{ik}, W_{ik}, $k=1,2,\dots$ such that

$$G \cap F_i \subset V_{ik} \subset \bar{V}_{i\,k+1} \subset D_i^\bullet - H ,$$
$$H \cap F_i \subset W_{ik} \subset \bar{W}_{i\,k+1} \subset D_i^\bullet - G ,$$
$$\text{Ind } B(V_{ik}) \leqq n-1 \quad \text{and} \quad \text{Ind } B(W_{ik}) \leqq n-1 ,$$

because $\text{Ind } D_i \leqq n$. Put

$$V_k = X - \bar{V}_{1k} \cup \dots \cup \bar{V}_{kk} \quad \text{and} \quad W_k = X - \bar{W}_{1k} \cup \dots \cup \bar{W}_{kk} .$$

Then V_k and W_k are open sets such that

$$V_k \supset \bar{V}_{k+1} \supset H , \; G \cap \left(\bigcap_{k=1}^{\infty} \bar{V}_k \right) = \emptyset ,$$
$$W_k \supset \bar{W}_{k+1} \supset G , \; H \cap \left(\bigcap_{k=1}^{\infty} W_k \right) = \emptyset ,$$
$$B(V_k) \subset \bigcup_{i=1}^{k} B(V_{ik}) , \; B(W_k) \subset \bigcup_{i=1}^{k} B(W_{ik}) .$$

Hence by $F)_{n-1}$ we obtain $\text{Ind } B(V_k) \leqq n-1$ and $\text{Ind } B(W_k) \leqq n-1$.
Now define

$$P = (W_1 - \bar{V}_1) \cup (W_2 - \bar{V}_2) \cup \dots .$$

Then P is an open set such that

$$G \subset P \subset X - H \quad \text{and} \quad B(P) \subset \bigcup_{k=1}^{\infty} (B(V_k) \cup B(W_k)) .$$

(A somewhat similar technique was used to prove II.1 E).) Hence by $F)_{n-1}$ and 1 D) we get $\text{Ind } B(P) \leqq n-1$. Thus $\text{Ind } X \leqq n$, i.e. (1) is proved.

Now, to prove ii) we assume that U is an open set of X with $\text{Ind } X \leqq n$. Let $\mathcal{P} = \bigcup_{i=1}^{\infty} \mathcal{P}_i$ and $\mathcal{Q} = \bigcup_{i=1}^{\infty} \mathcal{Q}_i$ be the coverings of U obtained in D), where $\mathcal{P}_i = \{ P_\alpha \mid \alpha \in A_i \}$ and $\mathcal{Q}_i = \{ Q_\alpha \mid \alpha \in A_i \}$ are discrete in U and satisfy $Q_\alpha \subset P_\alpha^\bullet$. By 1 D) $\text{Ind } P_\alpha \leqq n$, because P_α is closed in X. Thus by E) $\text{Ind } P_i \leqq n$, where $P_i = \bigcup \{ P_\alpha \mid \alpha \in A_i \}$. If we write $Q_i = \bigcup \{ Q_\alpha \mid \alpha \in A_i \}$, then P_i and Q_i are closed sets in U such that $Q_i \subset P_i^\bullet$. Thus from (1) it follows that $\text{Ind } U \leqq n$, which proves ii).

iii) $G)_n$ implies $F)_n$.

Put

$$Y_0 = X \ , \ Y_i = X - (F_1 \cup \dots \cup F_i) \ , \ i = 1, 2, \dots \ .$$

Then Y_i are open sets with

$$Y_0 \supset Y_1 \supset Y_2 \supset \dots \quad \text{and} \quad \bigcap_{i=1}^{\infty} Y_i = \emptyset \ .$$

Now $Y_{i-1} - Y_i = F_i - (F_1 \cup \dots \cup F_{i-1})$ is an open subset of F_i. Since Ind $F_i \leqq n$, it follows from G)$_n$ that Ind $(Y_{i-1} - Y_i) \leqq n$. Thus by C) we get Ind $X \leqq n$, which proves F)$_n$.

iv) Now combine i), ii), iii) to establish F)$_n$ and G)$_n$ for all integers $n \geqq -1$.

From A), F)$_n$ and G)$_n$ we obtain the following theorems.

Theorem VII. 5. *Let* $\{ F_i \mid i = 1, 2, \dots \}$ *be a closed covering of a totally normal space* X *such that* Ind $F_i \leqq n$, $i = 1, 2, \dots$. *Then* Ind $X \leqq n$.

Theorem VII. 6. *Let* Y *be a subset of a totally normal space* X . *Then* Ind $Y \leqq$ Ind X .

The following statement is another sum theorem whose proof is similar to that of C).

Theorem VII. 7. *Let* $\{ F_\alpha \mid \alpha \in A \}$ *be a locally finite closed covering of a totally normal space* X *such that* Ind $F_\alpha \leqq n$ *for all* $\alpha \in A$. *Then* Ind $X \leqq n$.

Proof. Well-order the given covering as $\{ F_\alpha \mid 0 \leqq \alpha \leqq \tau \}$. We shall prove the theorem by induction on n .

It is obviously true if $n = -1$.

Assume its validity in the $(n-1)$-dimensional case. Let G and H be given disjoint closed sets of X . Put

$$F(\alpha) = \cup \{ F_\beta \mid 0 \leqq \beta \leqq \alpha \} , \ 0 \leqq \alpha \leqq \tau \ .$$

Then $F(\alpha)$ is a closed set. We shall construct sets U_α for $0 \leqq \alpha \leqq \tau$ by use of induction on α such that

$(1)_\alpha$ U_α is an open set in $F(\alpha)$,

$(2)_\alpha$ $G\cap F(\alpha)\subset U_\alpha$, $H\cap F(\alpha)\subset F(\alpha) - \bar{U}_\alpha^{F(\alpha)}$,

$(3)_\alpha$ $\operatorname{Ind} B_{F(\alpha)}(U_\alpha)\leqq n-1$,

$(4)_\alpha$ for every $\beta<\alpha$, $U_\beta = U_\alpha\cap F(\beta)$, $B_{F(\beta)}(U_\beta) = B_{F(\alpha)}(U_\alpha)\cap F(\beta)$.

Since $\operatorname{Ind} F_0\leqq n$, we can choose U_0 satisfying these conditions for $\alpha=0$. Assume that U_β has been defined for all $\beta<\alpha$. Define $F = \bigcup\{F(\beta)\mid\beta<\alpha\}$; then F is a closed set. Put

(5) $C = G\cup(\bigcup_{\beta<\alpha} U_\beta)$,

(6) $B = H\cup(\bigcup\{F(\beta)-\bar{U}_\beta^{F(\beta)}\mid\beta<\alpha\})$.

Then $(\bar{C}\cap B)\cup(C\cap\bar{B})=\emptyset$, because $\{F_\alpha\}$ is a locally finite collection of closed sets, and $(1)_\beta$, $(2)_\beta$ and $(4)_\beta$ hold for all $\beta<\alpha$. Since X is hereditarily normal, there are open sets P and Q of X such that

(7) $C\subset P$, $B\subset Q$,

(8) $P\cap Q=\emptyset$, $\bar{P}\cap\bar{Q}\subset\bar{C}\cap\bar{B}$.

Since F is closed, we have $\bar{C}\cap\bar{B}\subset F$, which implies $\bar{P}\cap\bar{Q}\subset F$.

On the other hand it follows from Theorem VII.6 that $\operatorname{Ind}(F_\alpha-F)\leqq n$, because $\operatorname{Ind} F_\alpha\leqq n$. Thus we can find out an open set V of $F_\alpha-F$ such that

(9) $\bar{P}\cap(F_\alpha-F)\subset V$,

(10) $\bar{V}^{F_\alpha-F}\subset(F_\alpha-F)-\bar{Q}$,

(11) $\operatorname{Ind} B_{F_\alpha-F}(V)\leqq n-1$.

Then put

(12) $U_\alpha = (\bigcup_{\beta<\alpha} U_\beta)\cup V$.

Now, $(1)_\alpha$ follows from $(1)_\beta$ $(4)_\beta$ for $\beta<\alpha$, (5), (7), (9) and (12). $(2)_\alpha$ follows from $(2)_\beta$ for $\beta<\alpha$, (5), (6), (7), (9), (10) and (12). $(4)_\alpha$ follows from (12), $(4)_\beta$ for $\beta<\alpha$, (6), (7) and (10).

To prove $(3)_\alpha$, we express the boundary of U_α in $F(\alpha)$ as

$$B_{F(\alpha)}(U_\alpha) = B_1\cup B_2 ,$$

where

$$B_1 = B_{F(\alpha)}(U_\alpha) \cap (F(\alpha) - F) \quad \text{and} \quad B_2 = B_{F(\alpha)}(U_\alpha) \cap F .$$

Note that by $(4)_\beta$ for $\beta < \alpha$,

$$B_2 = \cup \{ B_{F(\beta)}(U_\beta) \mid \beta < \alpha \} = \cup \{ B_{F(\beta)}(U_\beta) \cap F_\beta \mid \beta < \alpha \} .$$

Thus by use of the induction hypothesis and $(3)_\beta$ for $\beta < \alpha$, we obtain

(13) $\quad \operatorname{Ind} B_2 \leq n - 1$.

Further note that $B_1 = B_{F_\alpha - F}(V)$. Thus from (11) it follows that $\operatorname{Ind} B_1 \leq n - 1$, which combined with (13) implies that

$$\operatorname{Ind} B_{F(\alpha)}(U_\alpha) \leq n - 1 ,$$

i.e. $(3)_\alpha$ is proved. Thus the induction process is complete, and so we can eventually construct U_τ . Since U_τ satisfies $(1)_\tau - (3)_\tau$ and since $F(\tau) = X$, U_τ is an open set of X such that

$$G \subset U_\tau \subset X - H \quad \text{and} \quad \operatorname{Ind} B(U_\tau) \leq n - 1 .$$

Thus we conclude that $\operatorname{Ind} X \leq n$.

Corollary. *Let* $\{ F_\alpha \mid \alpha \in A \}$ *be a locally countable closed covering of a hereditarily paracompact* T_2*-space* X . *If* $\operatorname{Ind} F_\alpha \leq n$ *for all* $\alpha \in A$ *, then* $\operatorname{Ind} X \leq n$.

Proof. Combine Theorems VII.5 and VII.7.[12]

[12] We have proved sum theorems for dim and Ind under considerably general conditions. On the other hand another dimension, ind, behaves very badly in this respect. E. van Douwen [1] and T. Przymusiński [1] proved that the finite sum theorem does not hold for ind in the class of complete metric spaces. In fact the former showed that adding a single point could raise ind from 0 to 1 in the class of metric spaces.
As for Katětov-Smirnov's dimension (Definition I.4'), E. Pol [1] proved that there is a Tychonoff space X such that Katětov-Smirnov $\dim X > 0$, and $X = X_1 \cup X_2$ for functionally closed sets X_1 and X_2 with Katětov-Smirnov $\dim X_i = 0$, $i = 1,2$; this fact rather sharply contrasts Theorem VII.1. However, this dimension is useful to some extent when one tries to extend dimension theory beyond normal spaces. See K. Morita [8] for dimension theory in general spaces.

VII. 4. Characterisation of dim by partitions

In the present section we will discuss a generalization of Theorem II.8 to normal spaces (due to E. Hemmingsen and K. Morita).

A) Let $\mathcal{U} = \{U_a \mid a = 1,\dots,k\}$ be a finite open collection in a topological space X and $\mathrm{F} = \{F_a \mid a = 1,\dots,k\}$ a closed collection such that $F_a \subset U_a$. We define binary coverings $\mathcal{V}_a$, $a = 1,\dots,k$ by $\mathcal{V}_a = \{U_a, X - F_a\}$. Then $\mathcal{V} = \wedge\{\mathcal{V}_a \mid a = 1,\dots,k\}$ is a finite open covering of R satisfying $S(F_a, \mathcal{V}) \subset U_a$.

Proof. It is clear that $\mathcal{V}$ is a finite open covering. Let

$$V = [\cap\{U_a \mid a \in \gamma\}] \cap [\cap\{X - F_a \mid a \notin \gamma\}]$$

be a given member of $\mathcal{V}$, where γ denotes a subset of $\{1,\dots,k\}$. If $V \cap F_a \neq \emptyset$, then $a \in \gamma$, which implies $V \subset U_a$. Hence we obtain $S(F_a, \mathcal{V}) \subset U_a$.

B) Let $\mathcal{U} = \{U_a \mid a = 1,\dots,k\}$ be a finite open collection in a normal space X and $\mathrm{F} = \{F_a \mid a = 1,\dots,k\}$ a closed collection such that $F_a \subset U_a$. If F is a closed subset of X of dimension $\leq n$, then there exist open sets V_a and W_a such that

$$F_a \subset V_a \subset \bar{V}_a \subset W_a \subset U_a \quad \text{and} \quad \operatorname{ord}\{F \cap (\bar{W}_a - V_a) \mid a = 1,\dots,k\} \leq n \;.$$[13]

Proof. Denoting by Γ the family of all finite subsets of $\{1,\dots,k\}$, we let $\mathcal{L} = \{L_\gamma \mid \gamma \in \Gamma\}$, where

$$(1) \qquad L_\gamma = [\cap\{U_a \mid a \in \gamma\}] \cap [\cap\{X - F_a \mid a \notin \gamma\}] \;.$$

Since $\dim F \leq n$, we obtain open coverings

$$\mathcal{M} = \{M_\gamma \mid \gamma \in \Gamma\} \quad \text{and} \quad \mathcal{N} = \{N_\gamma \mid \gamma \in \Gamma\}$$

such that

$$(2) \qquad \bar{N}_\gamma \subset M_\gamma \subset L_\gamma \quad \text{and} \quad \operatorname{ord}\{F \cap M_\gamma \mid \gamma \in \Gamma\} \leq n + 1 \;.$$

[13] We can prove this theorem for every locally finite open covering $\mathcal{U}$ in an analogous way, by applying Theorem II.6 to the normal space X. (Theorem II.6 is valid for every normal space R.)

Define $\delta(\gamma) = \{a \mid F_a \cap M_\gamma \neq \emptyset\}$; then $\delta(\gamma) \subset \gamma$. In order to show this, suppose a is an element of $\delta(\gamma)$. Then $F_a \cap M_\gamma \neq \emptyset$, which implies by (2) $F_a \cap L_\gamma \neq \emptyset$. Hence from (1) it follows that $a \in \gamma$, which proves $\delta(\gamma) \subset \gamma$.

By virtue of the normality of X we can define open sets $P_{\gamma a}$, $Q_{\gamma a}$ for each $\gamma \in \Gamma$ and $a \in \delta(\gamma)$ such that

(3) $$\bar{N}_\gamma \subset P_{\gamma a} \subset \bar{P}_{\gamma a} \subset Q_{\gamma a} \subset \bar{Q}_{\gamma a} \subset P_{\gamma a'} \subset M_\gamma \quad \text{if } a, a' \in \delta(\gamma) \text{ and } a < a' .$$

Now, for a fixed a with $1 \leqq a \leqq k$ we put

(4) $$V_a = \bigcup \{P_{\gamma a} \mid \gamma \in \Gamma , \ a \in \delta(\gamma)\} ,$$

(5) $$W_a = \bigcup \{Q_{\gamma a} \mid \gamma \in \Gamma , \ a \in \delta(\gamma)\} .$$

It follows from (3) and (4) that $F_a \subset S(F_a, N) \subset V_a$, because $F_a \cap N_\gamma \neq \emptyset$ implies $F_a \cap M_\gamma \neq \emptyset$, and hence $N_\gamma \subset P_{\gamma a}$ and $a \in \delta(\gamma)$. From (3) and (5) we obtain $W_a \subset S(F_a, M)$, because $a \in \delta(\gamma)$ implies $F_a \cap M_\gamma \neq \emptyset$, and hence $Q_{\gamma a} \subset M_\gamma \subset S(F_a, M)$. On the other hand, from (2) and A), it follows that

$$S(F_a, M) \subset S(F_a, L) \subset U_a .$$

Therefore $W_a \subset U_a$. Moreoever, from (3), (4) and (5), we obtain

$$\bar{V}_a = \bigcup \{\bar{P}_{\gamma a} \mid \gamma \in \Gamma , \ a \in \delta(\gamma)\} \subset W_a .$$

Thus it remains only to prove

$$\operatorname{ord} \{F \cap (\bar{W}_a - V_a) \mid a = 1, \ldots, k\} \leqq n .$$

We suppose a_i , $i = 1, \ldots, n+1$ are given distinct numbers between 1 and k . From (5) we obtain

$$\bar{W}_{a_i} = \bigcup \{\bar{Q}_{\gamma a_i} \mid \gamma \in \Gamma , \ a_i \in \delta(\gamma)\} ,$$

and hence

$$\bar{W}_{a_i} - V_{a_i} = \bigcup \{\bar{Q}_{\gamma a_i} \mid \gamma \in \Gamma , \ a_i \in \delta(\gamma)\} - \bigcup \{P_{\gamma a_i} \mid \gamma \in \Gamma , \ a_i \in \delta(\gamma)\}$$
$$\subset \bigcup \{\bar{Q}_{\gamma a_i} - P_{\gamma a_i} \mid \gamma \in \Gamma , a_i \in \delta(\gamma)\} .$$

Therefore to prove

$$F \cap \left[\bigcap_{i=1}^{n+1} (\bar{W}_{a_i} - V_{a_i}) \right] = \emptyset$$

it suffices to show

$$(6) \qquad F \cap \left[\bigcap_{i=1}^{n+1} (\bar{Q}_{\gamma_i a_i} - P_{\gamma_i a_i}) \right] = \emptyset$$

for every choice of $n+1$ indices $\gamma_1, \ldots, \gamma_{n+1}$. If $\gamma_i = \gamma_j$ for some distinct i, j, in view of (3), we may assume

$$\bar{Q}_{\gamma_i a_i} \subset P_{\gamma_j a_j} .$$

Hence

$$F \cap (\bar{Q}_{\gamma_i a_i} - P_{\gamma_i a_i}) \cap (\bar{Q}_{\gamma_j a_j} - P_{\gamma_j a_j}) = \emptyset$$

which implies (6). If $\gamma_1, \ldots, \gamma_{n+1}$ are all different, we assume the contrary of (6). Then we select a point

$$(7) \qquad p \in F \cap \left[\bigcap_{i=1}^{n+1} (\bar{Q}_{\gamma_i a_i} - P_{\gamma_i a_i}) \right] ,$$

which implies

$$p \notin P_{\gamma_i a_i} \supset N_{\gamma_i} , \quad i = 1, \ldots, n+1 .$$

Since $N = \{ N_\gamma \mid \gamma \in \Gamma \}$ covers X, there exists N_γ which contains p, and in consequence $\gamma \neq \gamma_i$, $i = 1, \ldots, n+1$. Thus by (2) we obtain

$$(8) \qquad p \in F \cap M_\gamma .$$

On the other hand, from (7) combined with (3), we obtain

$$(9) \qquad p \in F \cap \left[\bigcap_{i=1}^{n+1} M_{\gamma_i} \right] .$$

But (9) combined with (8) contradicts ord $\{F \cap M_\gamma \mid \gamma \in \Gamma\} \leqq n+1$. Thus in either case we conclude the validity of (6). Therefore

$$\text{ord}\,\{F \cap (\bar{W}_a - V_a) \mid a = 1,\dots,k\} \leqq n ,$$

which completes the proof of this assertion.

Theorem VII. 8. *A normal space* X *has dimension* $\leqq n$ *if and only if for every finite open collection* $\{U_i \mid i=1,\dots,k\}$ *and closed collection* $\{F_i \mid i=1,\dots,k\}$ *satisfying* $F_i \subset U_i$, $i=1,\dots,k$, *there exists an open collection* $\{V_i \mid i=1,\dots,k\}$ *such that* $F_i \subset V_i \subset U_i$, $i=1,\dots,k$, *and* ord $\{B(V_i) \mid i=1,\dots,k\} \leqq n$.

Proof. The "only if" part is a direct consequence of B). As for the proof of the "if" part, we refer to the proof of Theorem II.8. Although in that theorem the metrizability was assumed, only the normality was used in the proof.

The following corollary can be deduced from Theorem VII.8 in the same way as in the case of metric spaces.

Corollary. *A normal space* X *has dimension* $\leqq n$ *if and only if for every open collection* $\{U_i \mid i=1,\dots,n+1\}$ *and closed collection* $\{F_i \mid i=1,\dots,n+1\}$ *satisfying* $F_i \subset U_i$, *there exists an open collection* $\{V_i \mid i=1,\dots,n+1\}$ *such that* $F_i \subset V_i \subset U_i$, $i=1,\dots,n+1$, $\bigcap_{i=1}^{n+1} B(V_i) = \emptyset$.[14]

VII. 5. Dimension and mappings

Many theorems on dimension and mappings can be extended to considerably general non-metrizable spaces. In this section we shall prove for non-metrizable spaces only a few of the relevant theorems in Chapter III and leave the reexamination of the remaining ones to the reader.

We take Definition III.1 as the definition of the unstable value of a mapping defined on non-metrizable spaces. Then we obtain the following theorem which runs exactly as Theorem III.1.

[14] Generally, let F and G be disjoint sets and C a closed set such that $X-C=U \cup V$ for disjoint open sets U and V satisfying $U \supset F$, $V \supset G$. Then C is called a *partition* between F and G . If W is an open set such that $F \subset W \subset \bar{W} \subset X-G$, then $B(W)$ is a partition between F and G . Thus Theorem VII.8 and its corollary can be stated in terms of partitions in place of $B(V_i)$.

Theorem VII. 9. *A normal space X has $\dim \leqq n$ if and only if all values of every continuous mapping of X into I^{n+1} are unstable.*

Proof. As for the "if" part, we can prove its validity using the corollary to Theorem VII.8 by the same argument as in the proof of III.1 B). In that proof we in fact did not use the metrizability of R but only its normality.

As for the "only if" part, we define closed sets F_i and G_i as in the proof of III.1 A). Then we define by use of 4 B) two open sets V_i and W_i, $i = 1, \ldots, n+1$ satisfying

$$F_i \subset V_i \subset \bar{V}_i \subset W_i \subset X - G_i \quad \text{and} \quad \bigcap_{i=1}^{n+1} (\bar{W}_i - V_i) = \emptyset .$$

We can construct a continuous function φ_i over X such that $|\varphi_i| \leqq \varepsilon$, $\{x \mid \varphi_i(x) = 0\} \subset \bar{W}_i - V_i$, $\{x \mid \varphi_i(x) = \varepsilon\} = F_i$, and $\{x \mid \varphi_i(x) = -\varepsilon\} = G_i$.
The remainder of the proof runs parallel to the metric case.

From Theorem VII.9 we can easily deduce the following theorem which corresponds to Theorem III.2 in the metric case.

Theorem VII. 10. *A normal space X has $\dim \leqq n$ if and only if for every closed set C of X and every continuous mapping f of C into S^n there exists a continuous extension of f over X.*

Proof. All we have to do is to check in the non-metrizable case the proof of Theorem III.2; this will be left to the reader.

Theorem VII. 11. *A normal space X has $\dim \leqq n$ if and only if every mapping of X into I^{n+1} is inessential, where we take the same Definition III.5 for non-metrizable spaces.*

Proof. In fact the proof of Theorem III.5 only uses the normality of R, so we can apply it to the non-metrizable case.

A) Let $\{U_i \mid i = 1, \ldots, k\}$ be an open covering of a normal space X. If there exists an open covering $V < \{U_i\}$ with $\operatorname{ord} V \leqq n+1$, then there exists a closed covering $\{F_i \mid i = 1, \ldots, k\}$ such that $F_i \subset U_i$ and $\operatorname{ord} \{F_i \mid i = 1, \ldots, k\} \leqq n+1$.

Proof. Without loss of generality, we may suppose

$$V = \{V_i \mid i = 1, \ldots, k\}, \quad V_i \subset U_i, \quad i = 1, \ldots, k .$$

Since X is normal, we can find an open covering $\{W_i \mid i = 1, \ldots, k\}$ for which $\bar{W}_i \subset V_i$, $i = 1, \ldots, k$. Thus $\{F_i \mid i = 1, \ldots, k\}$, $F_i = \bar{W}_i$ is the desired closed covering.

B) Let $\{F_\alpha \mid \alpha < \tau\}$ be a locally finite closed covering of a normal space X and $\mathcal{U} = \{U_i \mid i = 1,\dots,k\}$ a finite open covering of X. If

i) there exists an open covering $\mathcal{U}_\alpha$ of F_α such that $\operatorname{ord} \mathcal{U}_\alpha \leqq n+1$ and $\mathcal{U}_\alpha < \mathcal{U}$,

ii) $\dim F_\alpha \cap F_\beta \leqq n-1$ for $\alpha \neq \beta$,

then there exists an open covering $\mathcal{W}$ of X satisfying $\operatorname{ord} \mathcal{W} \leqq n+1$, $\mathcal{W} < \mathcal{U}$.

Proof. We shall define by induction closed sets $G_{\alpha i}$ for $\alpha < \tau$, $i = 1,\dots,k$ such that

$$G_{\alpha i} \subset F_\alpha \cap U_i, \quad \bigcup_{i=1}^{k} \bigcup_{\beta \leqq \alpha} G_{\beta i} = \bigcup_{\beta \leqq \alpha} F_\beta, \text{ and}$$

$$\operatorname{ord} \{ \bigcup_{\beta \leqq \alpha} G_{\beta i} \mid i = 1,\dots,k \} \leqq n+1 .$$

For $\alpha = 0$, we can define G_{0i}, $i = 1,\dots,k$ by use of A) combined with condition i).

Let us assume that $G_{\beta i}$, $i = 1,\dots,k$ have been constructed for every $\beta < \alpha$. Let

$$G_i = \bigcup_{\beta < \alpha} G_{\beta i} ;$$

then G_i is closed because by virtue of the assumption $\{F_\beta \mid \beta < \alpha\}$ and accordingly $\{G_{\beta i} \mid \beta < \alpha\}$ are locally finite. Hence by the induction hypothesis $\{G_i \mid i = 1,\dots,k\}$ is a closed collection of order $\leqq n+1$. By I B) there are open sets M_i, $i = 1,\dots,k$, for which

$$G_i \subset M_i \subset U_i \quad \text{and} \quad \operatorname{ord} \{M_i \mid i = 1,\dots,k\} \leqq n+1 .$$

It follows from condition i) combined with A) and I B) that there exist open sets N_i and closed sets H_i, $i = 1,\dots,k$, for which

$$H_i \subset N_i \subset U_i, \quad \operatorname{ord} \{N_i \mid i = 1,\dots,k\} \leqq n+1, \text{ and}$$

$$(1) \qquad \bigcup_{i=1}^{k} H_i = F_\alpha .$$

It follows from condition ii) combined with Theorem VII.1 that

$$\dim (F_\alpha \cap (\bigcup_{\beta < \alpha} F_\beta)) \leqq n-1 .$$

Hence we can construct, by use of A) and I B), a finite open collection $\mathcal{W} = \{ W_j \mid j = 1,\ldots,l \}$ such that

(2) $$F_\alpha \cap (\bigcup_{\beta<\alpha} F_\beta) \subset \bigcup_{j=1}^{l} W_j ,$$

(3) $$\bar{\mathcal{W}} < \mathcal{U} \wedge [\bigwedge_{i=1}^{k} (\{ M_i , X - G_i \} \wedge \{ N_i , X - H_i \})] ,$$

(4) $$\operatorname{ord} \bar{\mathcal{W}} \leqq n .$$

We define closed sets J_i , J and F by

(5) $$J_i = H_i \cap (X - \bigcup_{j=1}^{l} W_j) ,$$

(6) $$J = \bigcup_{i=1}^{k} J_i ,$$

(7) $$F = \bigcup_{\beta<\alpha} F_\beta = \bigcup_{i=1}^{k} G_i .$$

Now, we shall construct closed sets K_j , L_j , $j = 1,\ldots,l$, such that

(8) $$\bar{W}_j = K_j \cup L_j , \ K_j \subset X - \bigcup_{h=1}^{j-1} (K_h \cap L_h) \cup J , \ L_j \subset X - F .$$

Since by (2), (5), (6) and (7) $F \cap J = \emptyset$, we can find closed sets K_1 and L_1 for which

$$\bar{W}_1 = K_1 \cup L_1, \ K_1 \subset X - J , \ L_1 \subset X - F .$$

Suppose we have defined $K_1,\ldots,K_{j-1}$; $L_1,\ldots,L_{j-1}$. Then $\bigcup_{h=1}^{j-1} (K_h \cap L_h) \cup J$ and F are disjoint closed sets. Therefore, we can construct closed sets K_j and L_j satisfying (8).

We note that the second formula of (8) implies that

(9) $$(K_h \cap L_h) \cap (K_j \cap L_j) = \emptyset \quad \text{if} \quad h \neq j .$$

We shall prove

(10) $$\operatorname{ord} \{ K_j , L_j \mid j = 1,\ldots,l \} \leqq n + 1 .$$

Let

$$K = (\bigcap_{p=1}^{r} K_{i_p}) \cap (\bigcap_{q=1}^{s} L_{j_q}) \neq \emptyset .$$

If $\{ i_p \mid p=1,\dots,r\} \cap \{ j_q \mid q=1,\dots,s\}$ contains two distinct numbers i_p and $i_{p'}$, then

$$K \subset (K_{i_p} \cap L_{i_p}) \cap (K_{i_{p'}} \cap L_{i_{p'}})$$

which contradicts (9). Hence $\{ i_p \mid p=1,\dots,r\} \cap \{ j_q \mid q=1,\dots,s\}$ contains at most one number. Thus it follows from (4) and

$$K \subset (\bigcap_{p=1}^{r} \bar{W}_{i_p}) \cap (\bigcap_{q=1}^{s} \bar{W}_{j_q}) ,$$

which is implied by (8), that $r+s-1 \leqq n$. So we have shown (10).

Now, for $i=1,\dots,k$, let

$$K_i' = \cup \{ K_j \mid K_j \cap G_h = \emptyset \text{ for } h=1,\dots,i-1 \text{ ; } K_j \cap G_i \neq \emptyset \},$$

$$L_i' = \cup \{ L_j \mid L_j \cap J_h = \emptyset \text{ for } h=1,\dots,i-1 \text{ ; } L_j \cap J_i \neq \emptyset \}.$$

Then it follows from (3), (5) and from $G_i \subset M_i$, $H_i \subset N_i$ that

(11) $\quad K_i' \cup G_i \subset M_i$, $L_i' \cup J_i \subset N_i$.

Furthermore, let us suppose that

$$K_{j_p} \cap F = \emptyset \text{ for } p=1,\dots,r ,$$

$$K_j \cap F \neq \emptyset \text{ for } j \neq j_p ,$$

$$L_{j'_q} \cap J = \emptyset \text{ for } q=1,\dots,s ,$$

$$L_{j'} \cap J \neq \emptyset \text{ for } j' \neq j'_q .$$

Then we have from (8)

$$(12)\qquad \begin{cases} F\cap J=\emptyset\ , \\ K_{j_p}\cap(F\cup J)=\emptyset\ ,\ L_{j'_q}\cap(F\cup J)=\emptyset\ . \end{cases}$$

We note that by (7) $j\neq j_p$ implies

$$K_j\cap(\bigcup_{i=1}^{k}G_i)\neq\emptyset\ ,$$

i.e.

$$(13)\qquad K_j\subset K'_i \text{ for some } i$$

and that by (6) $j'\neq j'_q$ implies

$$L_{j'}\cap(\bigcup_{i=1}^{k}J_i)\neq\emptyset\ ,$$

i.e.

$$(14)\qquad L_{j'}\subset L'_{i'} \text{ for some } i'\ .$$

We put

$$\mathsf{F}=\{(K'_i\cap F_\alpha)\cup G_i\ ,\ (L'_i\cap F_\alpha)\cup J_i\ ,\ K_{j_p}\cap F_\alpha\ ,\ L_{j'_q}\cap F_\alpha\ |\ i=1,\dots,k\ ;\ p=1,\dots,r\ ;\ q=1,\dots,s\}\ .$$

We shall prove that F is a closed covering of $\bigcup_{\beta\leq\alpha}F_\beta$ with $\operatorname{ord}\mathsf{F}\leq n+1$. From (5) and (1) it follows that

$$\bigcup_{i=1}^{k}J_i=(\bigcup_{i=1}^{k}H_i)\cap(X-\bigcup_{j=1}^{l}W_j)=F_\alpha\cap(X-\bigcup_{j=1}^{l}W_j)\ .$$

From (13), (14) and (8) we deduce that

$$(\bigcup_{i=1}^{k}K'_i)\cup(\bigcup_{p=1}^{r}K_{j_p})\cup(\bigcup_{i=1}^{k}L'_i)\cup(\bigcup_{q=1}^{s}L_{j'_q})\supset\bigcup_{j=1}^{l}(K_j\cup L_j)=\bigcup_{j=1}^{l}\bar{W}_j\ .$$

Thus a given point x of F_α satisfies at least one of the following conditions,

a) $x \in J_i \subset (L'_i \cap F_\alpha) \cup J_i$ for some i ,

b) $x \in K'_i$ for some i ,

which implies $x \in (K'_i \cap F_\alpha) \cup G_i$ and

c) $x \in K_{j_p}$ for some p ,

which implies $x \in K_{j_p} \cap F_\alpha$ and

d) $x \in L'_i$ for some i ,

which implies $x \in (L'_i \cap F_\alpha) \cup J_i$ and

e) $x \in L_{j'_q}$ for some q ,

which implies $x \in L_{j'_q} \cap F_\alpha$.

If x is a given point of $\bigcup_{\beta<\alpha} F_\beta$, then by (7) $x \in G_i \subset (K'_i \cap F_\alpha) \cup G_i$ for some i . Hence $\mathcal{F}$ is a closed covering of $\bigcup_{\beta\leq\alpha} F_\beta$.

To verify ord $\mathcal{F} \leq n+1$ it suffices to prove it on F_α , because the same obviously holds outside of F_α by virtue of the induction hypothesis. Let

$$L = [\bigcap_{m=1}^{m_0} ((K'_{i_m} \cap F_\alpha) \cup G_{i_m})] \cap [\bigcap_{n=1}^{n_0} ((L'_{i'_n} \cap F_\alpha) \cup J_{i'_n})] \cap$$
$$\cap [\bigcap_{t=1}^{t_0} (K_{j_{p(t)}} \cap F_\alpha)] \cap [\bigcap_{u=1}^{u_0} (L_{j'_{q(u)}} \cap F_\alpha)] \cap F_\alpha \neq \emptyset .$$

Unless $t_0 = u_0 = 0$, $m_0 n_0 = 0$, from (6), (7) and (12) we obtain

$$L \subset (\bigcap_{m=1}^{m_0} K'_{i_m}) \cap (\bigcap_{n=1}^{n_0} L'_{i'_n}) \cap (\bigcap_{t=1}^{t_0} K_{j_{p(t)}}) \cap (\bigcap_{u=1}^{u_0} L_{j'_{q(u)}}) \cap F_\alpha ,$$

where generally the intersection of A_m, m ranging through an empty set of indices, means X. Hence, in view of the definitions of K_i', K_{j_p}, L_i', $L_{j'_q}$, we obtain

$$L \subset [\bigcap_{m=1}^{m_0} \bigcap_{n=1}^{n_0} (K_{i''_m} \cap L_{i'''_n})] \cap [\bigcap_{t=1}^{t_0} \bigcap_{u=1}^{u_0} (K_{j_{p(t)}} \cap L_{j'_{q(u)}})] \cap F_\alpha$$

for some i''_m and i'''_n for which

$$K_{i''_m} \subset K'_{i_m} , \quad L_{i'''_n} \subset L'_{i'_n} , \quad i''_m \neq j_{p(t)} , \quad i'''_n \neq j'_{q(u)} .$$

Thus from (10) it follows that

$$m_0 + n_0 + t_0 + u_0 \leqq n+1 .$$

If $t_0 = u_0 = 0$, $m_0 n_0 = 0$, then assume, for example, $m_0 \geqq 1$, $n_0 = 0$. From (11) we obtain

$$L = \bigcap_{m=1}^{m_0} [(K'_{i_m} \cap F_\alpha) \cup G_{i_m}] \cap F_\alpha \subset \bigcap_{m=1}^{m_0} M_{i_m} ,$$

which implies $m_0 \leqq n+1$, because $\operatorname{ord} \{M_i \mid i = 1, \ldots, k\} \leqq n+1$.

Similarly we can prove $n_0 \leqq n+1$ in case $t_0 = u_0 = 0$, $m_0 = 0$, $n_0 \geqq 1$. Thus we can conclude $\operatorname{ord} \mathrm{F} \leqq n+1$.

Now, we define $G_{\alpha i}$, $i = 1, \ldots, k$, by

$$\begin{aligned} G_{\alpha i} = (K_i' \cap F_\alpha) \cup (L_i' \cap F_\alpha) \cup J_i \cup \\ \cup [\cup \{K_{j_p} \mid K_{j_p} \subset U_i ;\ K_{j_p} \not\subset U_{i'}, \text{ for } i' < i\} \cap F_\alpha] \cup \\ \cup [\cup \{L_{j'_q} \mid L_{j'_q} \subset U_i ;\ L_{j'_q} \not\subset U_{i'}, \text{ for } i' < i\} \cap F_\alpha] . \end{aligned}$$

Then it follows from the above property of F, (1), (11), (7), the induction hypothesis and $G_i = \bigcup_{\beta < \alpha} G_{\beta i}$, that $G_{\alpha i}$, $i = 1, \ldots, k$, are the closed sets which were wanted at the beginning of the proof. We shall leave the details of the proof to the reader.

To complete our proof, let $G_i = \bigcup_{\alpha < \tau} G_{\alpha i}$. Since $\{F_\alpha \mid \alpha < \tau\}$ is locally finite, and $G_{\alpha i} \subset F_\alpha$, it follows that $\{G_{\alpha i} \mid \alpha < \tau\}$ is locally finite, and hence G_i is closed. Thus $G = \{G_i \mid i = 1,\dots,k\}$ is a closed covering of $\cup_{\alpha<\tau} F_\alpha = X$ such that $G < U$ and $\operatorname{ord} G \leq n+1$. Hence by use of 1 B) we obtain the desired open covering W.

Theorem VII. 12. *Let f be a closed continuous mapping of a normal space X onto a paracompact space Y such that $\dim f^{-1}(q) \leq k$ for each point q of Y. Then* $\dim X \leq \operatorname{Ind} Y + k$.[15]

Proof. Since the theorem is clearly true if $\operatorname{Ind} Y = -1$, we shall assume it for Y with $\operatorname{Ind} Y < n$, and proceed to prove it for Y with $\operatorname{Ind} Y = n$.

Let $U = \{U_i \mid i = 1,\dots,s\}$ be a given open covering of X. Since $\dim f^{-1}(q) \leq k$ for each point $q \in Y$, by A) and 1. B) there exist open sets V_i, $i = 1,\dots,s$ such that

$$f^{-1}(q) \subset \bigcup_{i=1}^{s} V_i, \quad V_i \subset U_i, \text{ and}$$

(1) $$\operatorname{ord}\{V_i \mid i = 1,\dots,s\} \leq k+1 .$$

Let

(2) $$V'(q) = \bigcup_{i=1}^{s} V_i ;$$

then $\{V'(q) \mid q \in X\}$ is an open covering of X which satisfies $V'(q) \supset f^{-1}(q)$. We take for each q an open set $V(q)$ for which

(3) $$V'(q) \supset \overline{V(q)} \supset V(q) \supset f^{-1}(q) .$$

It follows from $q \in Y - f(X - V(q))$ and the closedness of f, that $V = \{Y - f(X - V(q)) \mid q \in Y\}$ is an open covering of Y. Since Y is a paracompact space with $\operatorname{Ind} Y \leq n$, there exists a locally finite open covering $W = \{W_\alpha \mid \alpha < \tau\}$ such that $W < V$ and $\operatorname{Ind} B(W_\alpha) \leq n-1$.

[15] Due to K. Morita [6].

Since $ff^{-1}(B(W_\alpha)) = B(W_\alpha)$, f is a closed continuous mapping of $f^{-1}(B(W_\alpha))$ onto $B(W_\alpha)$. Hence it follows from the induction hypothesis that

(4) $\dim f^{-1}(B(W_\alpha)) \leqq n+k-1$.

Since by the continuity of f , $B(f^{-1}(W_\alpha)) \subset f^{-1}(B(W_\alpha))$, we obtain from (4)

(5) $\dim B(f^{-1}(W_\alpha)) \leqq n+k-1$.

Now, let

(6) $K_\alpha = \overline{f^{-1}(W_\alpha)}$;

then

(7) $\{ K_\alpha \mid \alpha < \tau \}$ is a locally finite open covering of X .

We put

(8) $F_\alpha = K_\alpha - \bigcup_{\beta<\alpha} f^{-1}(W_\beta)$.

For a given α there is q for which $\bar{W}_\alpha \subset Y - f(X - V(q))$. Then one can easily see that

$$F_\alpha \subset K_\alpha \subset \overline{V(q)} \subset V'(q) ,$$

and hence by (1) and (2) there exists an open covering $\mathcal{U}_\alpha$ of F_α such that

(9) $\begin{cases} \mathcal{U}_\alpha < \mathcal{U} \\ \operatorname{ord} \mathcal{U}_\alpha \leqq k+1 \leqq n+k+1 . \end{cases}$

On the other hand, from (6) and (8) we can easily see that

$$F_\alpha \cap F_\beta \subset B(f^{-1}(W_\beta)) \quad \text{if} \quad \beta < \alpha .$$

Therefore, by virtue of (5),

(10) $\dim (F_\alpha \cap F_\beta) \leqq n+k-1$.

Thus $\{F_\alpha \mid \alpha < \tau\}$ satisfies (7), (9) and (10), and hence by B) there exists an open covering P with $P < U$ and ord $P \leq n + k + 1$. Therefore $\dim X \leq n + k$.

Theorem VII. 13. *Let f be a closed continuous mapping of a totally normal space X onto a normal space Y such that for each point q of Y, $B(f^{-1}(q))$ contains at most $m+1$ points $(m \geq 0)$. Then* $\operatorname{Ind} Y \leq \operatorname{Ind} X + m$.

Proof. The proof of this theorem is similar to that of Theorem III.7. All we have to do is to modify it as follows. Throughout the proof of Theorem III.7 we change 'dim' into 'Ind'. In iii) of that proof we consider the open subset $J = B(V) - H$ of $B(V)$ instead of H_k, $k = 1,2,\dots$. Define

$$J' = f^{-1}(J) \cap (X - \bar{U}) = \cup\{ f^{-1}(q)_2 \mid q \in J\} .$$

As is easily seen, f restricted to J' is a closed continuous mapping of J' onto J such that each point of J has an inverse image of at most m points. Hence by the induction hypothesis and Theorem VII.6 $\operatorname{Ind} J \leq n + m - 1$. Thus by 3 B) combined with $\operatorname{Ind} H \leq n + m - 1$ we conclude that $\operatorname{Ind} B(V) = \operatorname{Ind} (H \cup J) \leq n + m - 1$. (Note that Y is actually a hereditarily normal space.) This proves $\operatorname{Ind} Y \leq n + m$.

E. Sklyarenko [4] proved $\dim X \leq \dim Y + n$ if there is a closed continuous mapping f from a paracompact T_2-space X onto a paracompact T_2-space Y such that $\dim f^{-1}(q) \leq n$ for every $q \in Y$.

K. Morita [6] proved $\dim Y \leq \operatorname{Ind} X + m$ if there is a closed continuous mapping f from a normal space X onto a normal space Y such that each $f^{-1}(q)$ contains at most $m + 1$ points.

A. Zarelua [4] proved $\dim Y \leq \dim X + m$ under the same assumption.

A. V. Arhangelskii [3] obtained a different type of theorem as follows: Let f be a closed continuous mapping from a normal space X onto a normal space Y. Put $T = \{ y \in Y \mid f^{-1}(y)$ is not a singleton$\}$, $\operatorname{rd} T = \sup\{\dim G \mid G \subset T$, and G is closed in $Y\}$. Then $\dim Y \leq \dim X + \operatorname{rd} T + 1$.

VII. 6. Product theorem

A) Let X be a compact T_2-space and Y a metric space. Suppose $X \neq \emptyset$ or $Y \neq \emptyset$. Then $\dim X \times Y \leq \dim X + \dim Y$.

Proof. Assume $X \neq \emptyset$ and $Y \neq \emptyset$. Let $\dim X = n$, $\dim Y = m$. Then it is easy to see that the projection $\pi: X \times Y \to Y$ is a closed continuous mapping because X is compact. ($\pi(x,y) = y$.) It is also easy to see that $X \times Y$ is normal. (See e.g.

J. Nagata [8].) Since $\dim \pi^{-1}(y) = \dim X = n$ and $\dim Y = \operatorname{Ind} Y = m$, by Theorem VII.12 we obtain $\dim X \times Y \leqq n + m$.

We owe the following theorem to K. Morita [7].

Theorem VII. 14. *Let X be a compact T_2-space and $X \times Y$ a normal space. Suppose $X \neq \emptyset$ or $Y \neq \emptyset$. Then* $\dim X \times Y \leqq \dim X + \dim Y$.

Proof. Suppose $X \neq \emptyset$, $Y \neq \emptyset$, $\dim X = n$ and $\dim Y = m$. Let A be a given finite open covering of $X \times Y$. Then by I.1 C) there is a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of finite open coverings of $X \times Y$ such that

$$A > \mathcal{W}_1^* > \mathcal{W}_1 > \mathcal{W}_2^* > \dots .$$

Define for $y_1, y_2 \in Y$

$$y_1 \underset{i}{\sim} y_2 \text{ if and only if } (x, y_1) \in S((x, y_2), \mathcal{W}_i) \text{ for every } x \in X .$$

(The binary relation $\underset{i}{\sim}$ is no equivalence relation.) Define

$$V_i(y) = \{ y' \in Y \mid y \underset{i}{\sim} y' \} \quad \text{for } y \in Y, \text{ and } \mathcal{V}_i = \{ V_i(y) \mid y \in Y \} .$$

Then we can prove that $\mathcal{V}_i$ is an open covering of Y.

To prove that $V_i(y)$ is open, let $y' \in V_i(y)$ be given. Then for each $x \in X$ there is $W_x \in \mathcal{W}_i$ such that (x,y), $(x,y') \in W_x$. Choose open neighbourhoods $U(x)$, $V_x(y)$, $V_x(y')$ of x, y, y', respectively such that

$$U(x) \times V_x(y) \subset W_x \quad \text{and} \quad U(x) \times V_x(y') \subset W_x .$$

Since X is compact, $X = \bigcup_{j=1}^{k} U(x_j)$ for some finitely many x_j's. Then

$$V(y') = \bigcap_{j=1}^{k} V_{x_j}(y') \subset V_i(y) .$$

Because if $y'' \in V(y')$, then for any $x \in X$ there is j for which $x \in U(x_j)$. Thus

$$(x,y) \in U(x_j) \times V_{x_j}(y) \subset W_{x_j} ,$$
$$(x,y'') \in U(x_j) \times V_{x_j}(y') \subset W_{x_j} ,$$

which implies $y \underset{i}{\sim} y''$. Hence $V(y') \subset V_i(y)$ which proves that $V_i(y)$ is open.

It is also easy to see that $V^*_{i+1} < V_i$ follows from $W^*_{i+1} < W_i$, because $S(V_{i+1}(y), V_{i+1}) \subset V_i(y)$. Thus by Theorem I.1' each V_i has a locally finite open refinement V'_i. Since $\dim Y \leqq m$, we can construct, as we did in the proof of Theorem VII.4, a sequence $P_1, P_2, \ldots$ of locally finite open coverings of Y such that

$$P_1 > P^*_2 > P_2 > P^*_3 > \ldots, \quad P_i < V'_i < V_i ,$$

and each member of P_{i+1} meets at most $(m+1)$ members of P_i. We continue to use the method of the above mentioned proof to define a metric space Z with $\dim Z \leqq m$ and a continuous mapping f from Y onto Z such that $\{ S(z, f(P_i)) \mid i = 1,2,\ldots \}$ is a neighbourhood base at each $z \in Z$. Note that $f^{-1}(z)$ for each $z \in Z$ is an equivalence class in Y with respect to $\sim$ defined by

$$y \sim y' \text{ if and only if } y' \in \bigcap_{i=1}^{\infty} S(y, P_i) .$$

Further observe that $y \sim y'$ implies

$$y \in \bigcap_{i=1}^{\infty} S(y', P_i) \subset \bigcap_{i=1}^{\infty} S(y', V_i)$$

and accordingly

(1) $$y \underset{i}{\sim} y' , \quad i = 1,2,\ldots .$$

Now we define a continuous mapping φ from $X \times Y$ onto $X \times Z$ by $\varphi((x,y)) = (i(x), f(y))$, where i is the identity mapping of X.

Let

$$A = \{ A_l \mid l = 1,\ldots,k \} ,$$
$$B = \{ (X \times Z - \varphi(X \times Y - A_l))^{\circ} \mid l = 1,\ldots,k \} .$$

Then B is an open covering of $X \times Z$. To prove this let $q = (x,z) \in X \times Z$ be given. Then $\varphi^{-1}(q) \cap W \neq \emptyset$ for some $W \in \mathcal{W}_2$. Select $p = (x,y) \in \varphi^{-1}(q) \cap W$. Then x has an open neighbourhood U in X such that

(2) $$U \times \{ y \} \subset W .$$

Since $\mathcal{W}_2^* < \mathcal{W}_1 < \mathcal{W}_1^* < \mathcal{A}$, there is ℓ for which

(3) $$S^4(W,\mathcal{W}_2) \subset A_\ell \ .$$

Now, put $V = S(z, f(\mathcal{P}_2))$. Then $U \times V$ is a neighbourhood of q in $X \times Z$. We claim that $U \times V \subset X \times Z - \varphi(X \times Y - A_\ell)$.

To prove it, let $q' = (x', z') \in U \times V$ be given. Then $x' \in U$, and there is $V(y_0) \in \mathcal{V}_2$ such that $z, z' \in f(V(y_0))$, because $\mathcal{P}_2 < \mathcal{V}_2$. Hence we can choose y_1 and y_2 such that

(4) $$y_1 \in f^{-1}(z) \cap V(y_0) \ , \ y_2 \in f^{-1}(z') \cap V(y_0) \ .$$

Note that

(5) $$y \in f^{-1}(z)$$

follows from $p \in \varphi^{-1}(q)$. Define $p' = (x', y') \in \varphi^{-1}(q')$; then

(6) $$y' \in f^{-1}(z') \ .$$

Thus it follows from (4), (5), (6) that

$$y \underset{2}{\sim} y_1 \underset{2}{\sim} y_0 \underset{2}{\sim} y_2 \underset{2}{\sim} y' \ .$$

Thus there are $W_i \in \mathcal{W}_2$, $i = 1, \ldots, 4$ such that every successive two points of the sequence (x', y), (x', y_1), (x', y_0), (x', y_2), (x', y') are contained in some W_i . On the other hand, since $x' \in U$, $(x', y) \in W$ follows from (2). Thus $(x', y') \in S^4(W, \mathcal{W}_2)$, which combined with (3) implies $(x', y') \in A_\ell$. Hence we have proved $\varphi^{-1}(q') \subset A_\ell$. Thus $q' \in X \times Z$ but $q' \notin \varphi(X \times Y - A_\ell)$.

Recall that q' is a given point of the neighbourhood $U \times V$ of q , and hence

$$U \times V \cap \varphi(X \times Y - A_\ell) = \emptyset \ ,$$

proving

$$q \in (X \times Y - \varphi(X \times Y - A_\ell))^{\cdot}$$

Therefore $\mathcal{B}$ covers $X \times Z$.

Now, since X is compact T_2 and Z metric, by A) we have

$$\dim X \times Z \leqq \dim X + \dim Z \leqq n + m .$$

Hence there is an open covering $\mathcal{D}$ of $X \times Z$ such that $\mathcal{D} < \mathcal{B}$ and ord $\mathcal{D} \leqq n+m+1$. Then $\mathcal{D}' = \{\varphi^{-1}(D) \mid D \in \mathcal{D}\}$ is an open covering of $X \times Y$ such that $\mathcal{D}' < \mathcal{A}$ and ord $\mathcal{D}' \leqq n+m+1$. This proves that $\dim X \times Y \leqq n+m$.

Corollary. *Suppose that X is a paracompact T_2-space, and $X = \bigcup_{i=1}^{\infty} F_i$ for locally compact closed sets F_i, $i = 1, 2, \ldots$. Further suppose that $X \times Y$ is normal, and $X \neq \emptyset$ or $Y \neq \emptyset$. Then $\dim X \times Y \leqq \dim X + \dim Y$.*[16]

Proof. Assume $\dim X = n$, $\dim Y = m$. Each F_i is a locally finite sum of compact sets. Thus by Theorem VII.14 and the sum theorem (for locally finite sums) we obtain $\dim F_i \times Y \leqq n+m$. By use of the countable sum theorem we conclude that $\dim X \times Y \leqq n+m$.

Let us review here some simple facts about the Stone-Čech compactification.

B) Let X be a Tychonoff space; then there is a unique compact T_2-space βX (called Stone-Čech compactification of X) satisfying

(i) X is a dense subset of βX,

(ii) $\{\bar{Z} \mid Z \in \mathcal{Z}\}$ is a closed base for βX, where $\mathcal{Z}$ denotes the collection of all zero sets of X,

(iii) if $Z_i \in \mathcal{Z}$, $i = 1, \ldots, k$, then $\bigcap_{i=1}^{k} \bar{Z}_i = \overline{\bigcap_{i=1}^{k} Z_i}$.

(In (ii) and (iii) the closure is taken in βX.) Every real-valued bounded continuous function on X can be continuously extended to βX, and thus for every cozero set D of X there is a cozero set D' of βX such that $D' \cap X = D$.[17]

C) Let X be a normal space; then $\dim X = \dim \beta X$, $\operatorname{Ind} X = \operatorname{Ind} \beta X$.

Proof. Let $\dim X = n$, $\dim \beta X = m$. Suppose $\mathcal{U}$ is a given finite open covering of X. Then there is a finite covering $\mathcal{V}$ of X by cozero sets such that $\mathcal{V} < \mathcal{U}$. (Use I.1 A) and Urysohn's lemma.) Now, by B), $\beta\mathcal{V} = \{\beta X - \overline{(X - V)} \mid V \in \mathcal{V}\}$ is an open

---

[16] K. Morita [7] proved this theorem for the Katětov-Smirnov dimension without assuming the normality of $X \times Y$. K. Morita [3] proved $\dim X \times Y \leqq \dim X + \dim Y$ in case that X and Y are T_2 and $X \times Y$ has the star-finite property.

[17] See e.g. J. Nagata [8] for a proof.

covering of βX. Hence there is an open covering $\mathcal{V}'$ of βX such that $\mathcal{V}' < \beta\mathcal{V}$, ord $\mathcal{V}' \leq m+1$. The restriction of $\mathcal{V}'$ to X is an open refinement of $\mathcal{U}$ with ord $\leq m+1$. Thus $n \leq m$.

Let $\mathcal{W}$ be a finite open covering of βX. Then by B) there is a finite covering $\mathcal{W}'$ of βX such that $\mathcal{W}' < \mathcal{W}$, and each member of $\mathcal{W}'$ is of the form $\beta X - \overline{(X-Z)}$ for a zero set Z of X. Then select a finite covering A of X by cozero sets of X such that $\mathsf{A} < \mathcal{W}'$, ord $\mathsf{A} \leq n+1$. Now it is easy to check $\beta\mathsf{A} < \mathcal{W}$ and ord $\beta\mathsf{A} \leq n+1$, where

$$\beta\mathsf{A} = \{\beta X - \overline{(X-A)} \mid A \in \mathsf{A}\}.$$

Thus $m \leq n$, i.e. $n = m$. The proof of the second equality is omitted.

D) Let $\mathcal{U}$ be a σ-locally finite open covering of a countably paracompact space X. Then there is a locally finite open covering $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$.

Proof. Let $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$, where each $\mathcal{U}_i$ is locally finite. Put $U_i = \bigcup\{U \mid U \in \mathcal{U}_i\}$. Then, since X is countably paracompact, there is a locally finite open covering $\{V_i \mid i = 1,2,\dots\}$ such that $V_i \subset U_i$. Put $\mathcal{V}_i = \{V_i \cap U \mid U \in \mathcal{U}_i\}$; then $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a locally finite open refinement of $\mathcal{U}$.

E) Let Y be a metric space and $X \times Y$ a countably paracompact normal space. Suppose that $\mathcal{B} = \{B_h \mid h = 1,\dots,k\}$ is an open covering of $X \times Y$. Then there is a cozero set D of $\beta X \times Y$ and an open covering $\mathcal{D} = \{D_h \mid h = 1,\dots,k\}$ of D such that $X \times Y \subset D$ and $D_h \cap (X \times Y) \subset B_h$, $h = 1,\dots,k$.

Proof. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be a σ-locally finite base for Y, where $\mathcal{V}_i = \{V_\alpha \mid \alpha \in A_i\}$, $i = 1,2,\dots$ are locally finite open coverings of Y. We may assume $\mathcal{V}_i \subset \mathcal{V}_{i+1}$, and thus $A_i \subset A_{i+1}$. Define

$$A_i' = \{(\alpha_1,\dots,\alpha_i) \mid \alpha_1 \in A_1,\dots,\alpha_i \in A_i,\ \bigcap_{\ell=1}^{i} V_{\alpha_\ell} \neq \emptyset\}$$

and

$$V(\alpha_1,\dots,\alpha_i) = \bigcap_{\ell=1}^{i} V_{\alpha_\ell}, \quad (\alpha_1,\dots,\alpha_i) \in A_i'.$$

Then $V(\alpha_1,\dots,\alpha_i)$ is a non-empty open set of Y. Further we define open sets of X by

$$U(\alpha_1,\dots,\alpha_i\ ;\ h) = \bigcup\left\{ U \;\middle|\; \begin{array}{l} U \text{ is an open set of } X \text{ such that} \\ U \times V(\alpha_1,\dots,\alpha_i) \subset B_h \end{array} \right\},$$
$(\alpha_1,\dots,\alpha_i) \in A_i'$, $1 \leq h \leq k$.

Then we obtain

(1) $$U(\alpha_1,\dots,\alpha_i\ ;\ h) \times V(\alpha_1,\dots,\alpha_i) \subset B_h .$$

Then we put

$$U(\alpha_1,\dots,\alpha_i) = \bigcup_{h=1}^{k} U(\alpha_1,\dots,\alpha_i\ ;\ h) .$$

Now it is obvious that

$$\mathcal{W} = \{ U(\alpha_1,\dots,\alpha_i) \times V(\alpha_1,\dots,\alpha_i) \mid (\alpha_1,\dots,\alpha_i) \in A_i'\ ;\ i = 1,2,\dots \}$$

is an open covering of $X \times Y$. For each fixed i , $\{ V(\alpha_1,\dots,\alpha_i) \mid (\alpha_1,\dots,\alpha_i) \in A_i' \}$ is a locally finite open covering of Y . Hence $\mathcal{W}$ is a σ-locally finite open covering of $X \times Y$. Since $X \times Y$ is countably paracompact and normal, by use of D) we can find a locally finite open covering $\mathcal{N} = \{ N_\gamma \mid \gamma \in \Gamma \}$ of $X \times Y$ such that $\mathcal{N} < \mathcal{W}$. Shrink $\mathcal{N}$ to an open covering $\mathcal{L} = \{ L_\gamma \mid \gamma \in \Gamma \}$ satisfying $\bar{L}_\gamma \subset N_\gamma$. Then put

(2) $$L(\alpha_1,\dots,\alpha_i\ ;\ \gamma) = \bigcup\left\{ U \;\middle|\; \begin{array}{l} U \text{ is an open set of } X \text{ such that} \\ U \times V(\alpha_1,\dots,\alpha_i) \subset L_\gamma \end{array} \right\}$$
$(\alpha_1,\dots,\alpha_i) \in A_i'$, $\gamma \in \Gamma$.

Each $L(\alpha_1,\dots,\alpha_i\ ;\ \gamma)$ is an open set of X satisfying

(3) $$L(\alpha_1,\dots,\alpha\ \ ;\ \gamma) \times V(\alpha_1,\dots,\alpha_i) \subset L_\gamma \subset \bar{L}_\gamma \subset N_\gamma .$$

For $(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j) \in A_{i+j}'$, we define

(4) $$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j) = \bigcup \{ \overline{L(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j\ ;\ \gamma)} \mid \gamma \text{ satisfies } N_\gamma \subset U(\alpha_1,\dots,\alpha_i) \times V(\alpha_1,\dots,\alpha_i) \} .$$

Then $H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)$ is a closed set of X because of (3) and the local finiteness of $\mathcal{L}$. Furthermore, observe that

(5) $$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\times\overline{V(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)} \subset U(\alpha_1,\dots,\alpha_i)\times V(\alpha_1,\dots,\alpha_i)$$

follows from (3). Hence

$$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset U(\alpha_1,\dots,\alpha_i)$$

holds in X, because $V(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j)\neq\emptyset$.

Since X is a normal space, for each $(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j)\in A'_{i+j}$ we can select a cozero set $M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)$ of X such that

$$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset U(\alpha_1,\dots,\alpha_i)\ .$$

Since

$$U(\alpha_1,\dots,\alpha_i)=\bigcup_{h=1}^{k} U(\alpha_1,\dots,\alpha_i\ ;\ h)\ ,$$

there are cozero sets $M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j\ ;\ h)$, $h=1,\dots,k$, of X such that

(6) $$\bigcup_{h=1}^{k} M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j\ ;\ h)=M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\ ,$$
$$M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j\ ;\ h)\subset U(\alpha_1,\dots,\alpha_i\ ;\ h)\ .$$

Now, let $p=(x,y)$ be a given point of $X\times Y$. Then $p\in L_\gamma$ for some $\gamma\in\Gamma$, and $N_\gamma\subset U(\alpha_1,\dots,\alpha_i)\times V(\alpha_1,\dots,\alpha_i)$ for some i and $(\alpha_1,\dots,\alpha_i)\in A'_i$, because $N<\mathcal{W}$. Since $\mathcal{V}$ is a base for Y and $A_i\subset A_{i+1}$, there are $\beta_1\in A_{i+1},\dots,\beta_j\in A_{i+j}$ such that

$$p=(x,y)\in U\times V(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j)\subset L_\gamma$$

for some open neighbourhood U of x. Hence by (2)

$$x\in L(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j\ ;\ \gamma)\ .$$

By (4)

$$x\in H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\ .$$

and hence

$$p \in M(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) .$$

This proves that $\{ M(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \in A'_{i+j} \,;\, i,j = 1,2,\dots\}$ covers $X \times Y$.

Therefore $\{ M(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \in A'_{i+j} \,;\, i,j = 1,2,\dots \,;\, h = 1,\dots,k \}$ covers $X \times Y$.

By B) there is a cozero set $D(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h)$ of βX whose intersection with X is $M(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h)$. Then $\{ D(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \in A'_{i+j} \,;\, i,j = 1,2,\dots \,;\, h = 1,\dots,k \}$ is a collection of cozero sets in $\beta X \times Y$ which covers $X \times Y$.

Define for $h = 1,\dots,k$

$$D_h = \bigcup \{ D(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \in A'_{i+j} \,;\, i,j = 1,2,\dots \} .$$

Then $D_h \cap (X \times Y) \subset B_h$ can be proved as follows. $D(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \cap (X \times Y) = M(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j)$. On the other hand (6) and (1) imply

$$M(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \subset U(\alpha_1,\dots,\alpha_i \,;\, h) \times V(\alpha_1,\dots,\alpha_i) \subset B_h .$$

Thus it is proved that $D_h \cap (X \times Y) \subset B_h$.

Finally we shall prove that D_h is a cozero set of $\beta X \times Y$. For fixed i, j $\{ V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \in A'_{i+j} \}$ is locally finite in Y. Thus $\{ D(\alpha_1,\dots,\alpha_i \,;\, \beta_1,\dots,\beta_j \,;\, h) \times V(\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i \,,\, \beta_1,\dots,\beta_j) \in A'_{i+j} \,;\, i,j = 1,2,\dots\}$ is a collection of cozero sets of $\beta X \times Y$, which is σ-locally finite in $\beta X \times Y$. Thus D_h is a cozero set of $\beta X \times Y$. Put $D = \bigcup_{h=1}^{k} D_h$ and $\mathcal{D} = \{ D_1,\dots,D_k \}$. We obtained the desired result.

We owe the following theorem to Y. Kodana [3].

Theorem VII. 15. *Let Y be a metric space and $X \times Y$ a countably paracompact normal space. Suppose $X \neq \emptyset$ or $Y \neq \emptyset$. Then* $\dim X \times Y \leqq \dim X + \dim Y$.[18]

Proof. Assume $\dim X = n$, $\dim Y = m$. Then by C) $\dim \beta X = n$. Thus by Theorem VII.14 $\dim \beta X \times Y \leqq n + m$. (Note that the product of a compact T_2-space and a paracompact T_2-space is normal. See J. Nagata [8] for a proof.) Let $\mathcal{B}$ be a given finite open covering of $X \times Y$. Then we consider a cozero set D of $\beta X \times Y$ and its open covering $\mathcal{D}$ mentioned in E). Since D is an F_σ-set in the normal space $\beta X \times Y$, it is normal as a subspace, and thus by the countable sum theorem $\dim D \leqq \dim \beta X \times Y \leqq n + m$. Hence there is a finite open covering $\mathcal{D}'$ of D such that $\mathcal{D}' < \mathcal{D}$, $\text{ord}\, \mathcal{D}' \leqq n + m + 1$. Then the restriction of $\mathcal{D}'$ to $X \times Y$ is an open refinement of $\mathcal{B}$ with $\text{ord} \leqq n + m + 1$. Hence $\dim X \times Y \leqq n + m$.

V. Filippov [3] generalized Theorem VII.15 as follows: If $X \times Y$ is countably paracompact and normal, and if there is a perfect mapping from Y onto a metric space, then $\dim X \times Y \leqq \dim X + \dim Y$ $(X \neq \emptyset$ or $Y \neq \emptyset)$.

This theorem implies practically Theorem VII.14.

B. Pasynkov [6] further extended this theorem as follows: $\dim X \times Y \leqq \dim X + \dim Y$ whenever $X \times Y$ is a rectangular product and X, Y Tychonoff $(X \neq \emptyset$ or $Y \neq \emptyset)$, where dim is the Katětov-Smirnov dimension, and where it is defined that a product $X \times Y$ is called *rectangular* if every finite covering of $X \times Y$ by cozero sets has a σ-locally finite open refinement by sets of the form $U \times V$ for cozero sets U and V of X and Y, respectively.[19]

T. Hoshina - K. Morita [1] studied general conditions under which $X \times Y$ is rectangular and extended Pasynkov's theorem further to non-Tychonoff spaces.

As for large induction dimension, $\text{Ind}\, X \times Y \leqq \text{Ind}\, X + \text{Ind}\, Y$ (where $X \neq \emptyset$ or $Y \neq \emptyset$) was proved under various conditions, for examples in case that $X \times Y$ is hereditarily paracompact T_2 and has the star-finite property (Y. Katsuta [1]), each of X and Y is the inverse image of a metric space by a perfect mapping and

[18] M. E. Rudin - M. Starbird [1] proved that if the product $X \times Y$ of a countably paracompact space X and a metric space Y is normal, then $X \times Y$ is countably paracompact. Thus Theorem VII.15 can be improved as follows:
Let Y be a metric space, X countably paracompact and $X \times Y$ normal. Suppose $X \neq \emptyset$ or $Y \neq \emptyset$. Then $\dim X \times Y \leqq \dim X + \dim Y$.

[19] The product $X \times Y$ of Tychonoff spaces is rectangular, e.g., in the following cases: (1) Y is locally compact and paracompact, (2) the projection $\pi: X \times Y \to X$ is a closed mapping, (3) there is a perfect mapping from Y onto a metric space, and $X \times Y$ is countably paracompact and normal, (4) (implied by (3)) there is a perfect mapping from each of X and Y onto a metric space.

$X \times Y$ totally normal (J. Nagata [12]), $X \times Y$ is normal, X locally compact and paracompact, and the finite sum theorem holds for Ind in X and Y (V. Filippov [3]), and more generally in case that $X \times Y$ is a rectangular product and normal, and the finite sum theorem holds for Ind in X and Y (B. Pasynkov [6]).[20]

On the other hand the equality $\dim X \times Y = \dim X + \dim Y$ holds under rather restrictive conditions. K. Morita [7] proved it in case that $\dim X = 1$, and Y is a paracompact T_2-space which is a countable sum of locally compact closed subsets, where dim is Katětov-Smirnov dimension.

It is obvious that $\dim X \times Y \leqq \dim X + \dim Y$ does not hold in general. Because $\dim Z = 0$ implies that Z is a normal space, and hence if X, Y are 0-dimensional spaces such that $X \times Y$ is not normal, then the product theorem does not hold. For example, let $X = Y =$ the Sorgenfrey line (= the real numbers with the open base $\{ [x, x + \varepsilon) \mid -\infty < x < +\infty,\ \varepsilon > 0 \}$.)

What can be said if $X \times Y$ is normal? This had been a long standing question until it was negatively answered by M. Wage [1] under the assumption of the continuum hypothesis and by T. Przymusiński [2] without the hypothesis.[21] The former gave a space X such that $\dim X = 0$, $\dim X \times X \neq 0$, and $X \times X$ is locally compact and perfectly normal. The latter gave, for every natural number n, a separable first-countable space X such that X^n is Lindelöf and $\dim X^n = 0$, X^{n+1} is normal and $\dim X^{n+1} > 0$ (X^n denotes the n-ple product of X).

Since $\dim X = 0$ if and only if $\operatorname{Ind} X = 0$, the above examples show that the product theorem for Ind does not hold either even if $X \times Y$ is normal.[22]

V. Filippov [2] constructed compact T_2-spaces X and Y such that $\operatorname{ind} X = \operatorname{Ind} X = 1$, $\operatorname{ind} Y = \operatorname{Ind} Y = 2$, $\operatorname{ind} X \times Y \geqq 4$. Thus the product theorem for the inductive dimension does not hold even for compact T_2-spaces.

---

[20] Also note that $\operatorname{Ind} X \times Y \leqq \operatorname{Ind} X + \operatorname{Ind} Y$ holds for regular spaces X and Y ($X \neq \emptyset$ or $Y \neq \emptyset$) if the characterization theorem by means of a σ-closure-preserving base (footnote 5) holds for X and Y. Because it is known that if a regular space has a σ-closure-preserving base, then it is hereditarily paracompact and thus the product theorem can be proved in a similar way as done for metric spaces by use of the finite sum theorem and induction. Thus the product theorem is true especially if X and Y are the images of metric spaces by closed continuous mappings.

[21] The problem still remains open if $X \times Y$ is paracompact T_2.

[22] S. Mrowka [1] and P. Nyikos [1] studied relations between 0-dimensional spaces and N-compact spaces (= closed sets in products of countable discrete spaces).

VII. 7. Characterization of dim by $\Delta_k(X)$

Pontrjagin-Schnirelmann's theorem (Theorem IV.6) was generalized to non-metrizable spaces by J. Bruijning [1] by use of totally bounded pseudo-metrics. However, in the present section we are going to define a new function $\Delta_k(X)$, which has a remote resemblance to Pontrjagin-Schnirelmann's function $N(\varepsilon,R,\rho)$ in its basic idea, and we shall see that $\Delta_k(X)$ is a more appropriate function to characterize dim of non-metrizable spaces. The main results and method of this section are due to J. Bruijning - J. Nagata [1]. In the following we denote by C^k_m the set of all m-element subsets of $\{1,2,\dots,k\}$ and by $\binom{k}{m}$ its cardinality, i.e. $\binom{k}{m} = k!/m!(k-m)!$.

A) A normal space X has $\dim \leqq n$ if and only if every open covering $\{U_1,\dots,U_{n+2}\}$ can be shrunk to an open covering $\{V_1,\dots,V_{n+2}\}$ such that $\bigcap_{i=1}^{n+2} V_i = \emptyset$.

Proof. The 'only if' part follows directly from I A).

To prove the 'if' part, assume the condition and let $\mathcal{B} = \{B_1,\dots,B_k\}$ be an open covering of X. Assume $k \geqq n+2$ and put $C^k_{n+2} = \{A_1,\dots,A_p\}$. Suppose $A_1 = \{i_1,\dots,i_{n+2}\}$.

Then, let $\{F_1,\dots,F_k\}$ be a closed covering of X such that $F_i \subset B_i$, $i=1,\dots,k$. Put

$$K_1 = F_{i_1} \cup \dots \cup F_{i_{n+2}} \quad \text{and} \quad U_j = (X-K_1) \cup B_{i_j} \quad \text{for } j=1,\dots,n+2 .$$

Then $\{U_1,\dots,U_{n+2}\}$ is an open covering of X, and hence it can be shrunk to an open covering $\{V_1,\dots,V_{n+2}\}$ such that $\bigcap_{j=1}^{n+2} V_j = \emptyset$. Now put

$$U^1_{i_j} = V_j \cap B_{i_j}, \quad j=1,\dots,n+2,$$
$$U^1_i = B_i \quad \text{for } i \text{ satisfying } 1 \leqq i \leqq k,\ i \neq i_1,\dots,i_{n+2} .$$

Then $\mathcal{U}^1 = \{U^1_1,\dots,U^1_k\}$ is an open covering of X because $V_j \cap K_1 \subset B_{i_j} \cap K_1$. Observe that $\mathcal{U}^1$ shrinks $\mathcal{B}$ and satisfies $\bigcap\{U^1_i \mid i \in A_1\} = \emptyset$.

By use of the same argument we can shrink $\mathcal{U}^1$ to an open covering $\mathcal{U}^2 = \{U^2_1,\dots,U^2_k\}$ such that $\bigcap\{U^2_i \mid i \in A_2\} = \emptyset$, and accordingly $\bigcap\{U^2_i \mid i \in A_1\} = \emptyset$.

Repeating the same argument we obtain an open covering $\mathcal{U}^p = \{U^p_1,\dots,U^p_k\}$ which shrinks $\mathcal{B}$ and satisfies $\bigcap\{U^p_i \mid i \in A_l\} = \emptyset$ for $l=1,\dots,p$. Thus $\operatorname{ord} \mathcal{U}^p \leq n+1$, proving $\dim X \leqq n$.

Definition VII. 2. *Let X be a topological space and k a natural number. Then we define $\Delta_k(X)$ as the least natural number m such that for every open covering $\mathcal{U}$ of X with $|\mathcal{U}| \leq k$, there is an open covering $\mathcal{V}$ of X such that $|\mathcal{V}| \leq m$, $\mathcal{V} \overset{\Delta}{<} \mathcal{U}$.*

B) Let X be a normal space with $\dim X \leq n$ $(n \geq 0)$, and k a natural number. Then

$$\Delta_k(X) \leq 2^k - 1 \quad \text{if} \quad k \leq n+1 ,$$
$$\Delta_k(X) \leq \binom{k}{1} + \dots + \binom{k}{n+1} \quad \text{if} \quad k \geq n+1 .$$

Proof. We shall prove the inequality in case of $k \geq n+1$. Let $\mathcal{U}$ be an open covering of X with $|\mathcal{U}| \leq k$. Then we may put $\mathcal{U} = \{U_1, \dots, U_k\}$ by counting a same element repeatedly if necessary. Now, shrink $\mathcal{U}$ to an open covering $\mathcal{V} = \{V_1, \dots, V_k\}$ with $\operatorname{ord} \mathcal{V} \leq n+1$. Then select an open covering $\mathcal{W} = \{W_1, \dots, W_k\}$ such that $\bar{W}_i \subset V_i$. For each $A \in \bigcup_{m=1}^{n+1} C_m^k$, we put

$$P(A) = (\cap \{ V_i \mid i \in A \}) \cap (\cap \{ X - \bar{W}_i \mid i \notin A ,\ 1 \leq i \leq k \}) .$$

Then it is obvious that

$$\mathcal{P} = \{ P(A) \mid A \in \bigcup_{m=1}^{n+1} C_m^k \}$$

is an open covering of X such that

$$\mathcal{P} \overset{\Delta}{<} \mathcal{V} < \mathcal{U} , \quad \text{and} \quad |\mathcal{P}| \leq \binom{k}{1} + \dots + \binom{k}{n+1} .$$

Thus

$$\Delta_k(X) \leq \binom{k}{1} + \dots + \binom{k}{n+1} .$$

In case of $k \leq n+1$, we can use the above argument to get

$$\Delta_k(X) \leq \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k} = 2^k - 1 .$$

C) Let X be an infinite normal space with $\dim X \geq n \geq 0$. Let k be a natural number. Then there are k closed subsets of X with $\dim \geq n$ which are disjoint with each other.

Proof. If $n=0$, then, since X is infinite (contains infinitely many points), just choose k distinct points.

If $n>0$, then it suffices to prove that X contains two disjoint closed sets with $\dim \geqq n$. Since $\dim X \geqq n$, there is an open covering $\mathcal{U} = \{U_1, \ldots, U_j\}$ of X which cannot be shrunk to an open covering of $\text{ord} \leqq n$. Let $\mathcal{F} = \{F_1, \ldots, F_j\}$ be a closed covering of X such that $F_i \subset U_i$. Then $\dim F_i \geqq n$ for some i because of the sum theorem. Let V be an open set such that $F_i \subset V \subset \bar{V} \subset U_i$.

Then $\dim (X-V) \geqq n$: Assume the contrary, $\dim (X-V) < n$. Since $\{U_\ell - \bar{V}, U_i \mid \ell \neq i\}$ covers $X-V$, there is a collection $\mathcal{W} = \{W_1, \ldots, W_j\}$ of open sets of X such that $W_i \subset U_i$, $W_\ell \subset U_\ell - \bar{V}$ for $\ell \neq i$, $\text{ord}\, \mathcal{W} \leqq n$, and $\bigcup_{k=1}^{j} W_k \supset X-V$. Now, put $P_i = W_i \cup V$ and $P_\ell = W_\ell$ for $\ell \neq i$. Then $\mathcal{P} = \{P_1, \ldots, P_j\}$ is an open covering of X such that $\text{ord}\, \mathcal{P} \leqq n$, and $P_k \subset W_k$ for $k=1,\ldots,j$, which is a contradiction. Thus we obtain two disjoint closed sets F_i and $X-V$ each of which has $\dim \geqq n$.

D) Let X be an infinite normal space with $\dim X \geqq n$. Let k be a natural number. Then

$$\Delta_k(X) \geqq 2^k - 1 \text{ if } k \leqq n+1 \text{ , and}$$
$$\Delta_k(X) \geqq \binom{k}{1} + \ldots + \binom{k}{n+1} \text{ if } k \geqq n+1 \text{ .}$$

Proof. Assume $k \geqq n+1$. By C) there is a collection $\{C(\alpha) \mid \alpha \in C^k_{n+1}\}$ of disjoint closed sets of X with $\dim C(\alpha) \geqq n$. By A) there is an open covering $\mathcal{U}(\alpha) = \{U^\alpha_i \mid i \in \alpha\}$ of $C(\alpha)$ which cannot be shrunk to an open covering of $C(\alpha)$ with an empty intersection. Define open sets of X by

$$(1) \qquad U_i = (X - \cup\{C(\alpha) \mid \alpha \in C^k_{n+1}\}) \cup (\cup\{U^\alpha_i \mid i \in \alpha\}) , \; i=1,\ldots,k \text{ .}$$

Then $\mathcal{U} = \{U_i \mid i=1,\ldots,k\}$ is obviously an open covering of X . Let $\mathcal{V}$ be an open covering of X satisfying $\mathcal{V}^\Delta < \mathcal{U}$. Then we claim that

$$(2) \qquad |\mathcal{V}| \geqq \binom{k}{1} + \ldots + \binom{k}{n+1} \text{ .}$$

To prove it, we shall show that for every $\beta \in C^k_1 \cup C^k_2 \cup \ldots \cup C^k_{n+1}$, there is a member $V(\beta)$ of $\mathcal{V}$ such that

$$(3) \qquad V(\beta) \subset U_i \text{ if and only if } i \in \beta \text{ .}$$

If this is shown, we know that, for distinct β's , the $V(\beta)$'s are distinct sets, which implies that $\mathcal{V}$ contains at least $\binom{k}{1} + \ldots + \binom{k}{n+1}$ distinct members.

Now, select a subset γ of $\{1,\dots,k\}$ such that $\beta\cap\gamma=\emptyset$, and $\alpha=\beta\cup\gamma\in C^k_{n+1}$. Put

(4) $K=C(\alpha)-\cup\{U^\alpha_i \mid i\in\gamma\}$.

Then observe that $\{U^\alpha_j \mid j\in\beta\}$ is a collection of open sets of $C(\alpha)$ which covers K . Further we note that

(5) $\{U^\alpha_j \mid j\in\beta\}$ is a covering of K which cannot be shrunk to a closed covering of K with an empty intersection.

We assume the contrary. Then there is a closed covering $\{K_j \mid j\in\beta\}$ of K such that

$$K_j\subset U^\alpha_j \quad \text{and} \quad \bigcap_{j\in\beta} K_j=\emptyset \ .$$

Use 1 B) to get open sets H_j , $j\in\beta$, of $C(\alpha)$ such that

$$K_j\subset H_j\subset U^\alpha_j \quad \text{and} \quad \bigcap_{j\in\beta} H_j=\emptyset \ .$$

Then $\{H_j \mid j\in\beta\}\cup\{U^\alpha_i \mid i\in\gamma\}$ is an open covering of $C(\alpha)$ with an empty intersection, and this covering shrinks $U(\alpha)$, which contradicts the definition of $U(\alpha)$.

Now, we define closed sets G_j , $j\in\beta$, by

(6) $G_j=\{x\in K \mid S(x,V)\subset U_j\}$, $j\in\beta$.

Then $G_j\subset U^\alpha_j$ follows from (1) and (4). It follows from (4) and $V^\Delta<U$ that $\cup\{G_j \mid j\in\beta\}=K$. Hence by (5) we obtain $\cap\{G_j \mid j\in\beta\}\neq\emptyset$.

Select a point $x\in\cap\{G_j \mid j\in\beta\}$. Let $V(\beta)$ be an arbitrary element of V such that $x\in V(\beta)$. Then $V(\beta)\subset U_j$ for each $j\in\beta$, because $x\in G_j$. (Recall (6).) On the other hand $x\notin U_i$ for each $i\notin\beta$ follows from (1) and (4). Hence $x\in V(\beta)-U_i$, proving $V(\beta)\not\subset U_i$ for each $i\notin\beta$.

Thus (3) and accordingly our claim (2) is proved. This implies that

$$\Delta_k(X)\geqq\binom{k}{1}+\dots+\binom{k}{n+1} \ .$$

In case of $k \leq n+1$, $\dim X \geq k-1$ holds. Thus by use of the above results we obtain

$$\Delta_k(X) \geq \binom{k}{1} + \ldots + \binom{k}{k} = 2^k - 1 .$$

Theorem VII. 16. *Let* X *be an infinite normal space with* $\dim X = n$ $(0 \leq n \leq \infty)$. *Let* k *be a natural number. Then*

$$\Delta_k(X) = 2^k - 1 \quad \text{if} \quad k \leq n+1 ,$$
$$\Delta_k(X) = \binom{k}{1} + \ldots + \binom{k}{n+1} \quad \text{if} \quad k \geq n+1 .$$[23]

Proof. Combine B) and D).

Corollary. *Let* X *be an infinite normal space. Then*

$$\dim X = \lim_{k \to \infty} \frac{\log \Delta_k(X)}{\log k} - 1 .$$

Proof. This follows directly from Theorem VII.16.

VII. 8. Characterizations in terms of $C(X)$

In IV.8 we characterized dim of a compact metric space R by the ring $C(R)$ of all real-valued continuous functions on R . In the present section we shall discuss two characterizations of $\dim X$ of a normal space X in terms of $C(X)$. These characterizations are based on quite different ideas as compared to those of IV.8 and are due to M. Canfell [1] and J. Hejcman [2], respectively.

Definition VII. 3. *Let us denote by* C *the ring* $C(X)$ *of continuous real-valued functions of a topological space* X. *A subset* $fC = \{ f\varphi \mid \varphi \in C \}$ *of* C *is called a principal ideal, where* f *is a fixed element of* C . *A finite set* $\{ f_i C \mid i = 1, \ldots, n \}$ *of principal ideals is called uniquely generated if the following holds:*

[23] Theorem VII.16 and its corollary can be extended to a general topological space if we adopt the Katětov-Smirnov dimension. (See J. Bruijning - J. Nagata [1].)

Whenever $f_iC = g_iC$, $i = 1,\ldots,n$, *there are* $\theta_i \in C$, $i = 1,\ldots,n$ *such that* $f_i = g_i\theta_i$, *and* $\theta_1 C + \ldots + \theta_n C = \{\theta_1\varphi_1 + \ldots + \theta_n\varphi_n \mid \varphi_1,\ldots,\varphi_n \in C\} = C$.

Further we define $\text{p-dim}\, C$ *as the least integer* n *such that every set of* $(n+1)$ *principal ideals is uniquely generated, and* $\text{p-dim}\, C = \infty$ *if there is no such integer.*

Theorem VII. 17. *Let* X *be a non-empty normal space. Then* $\dim X = \text{p-dim}\, C(X)$.

Proof. Let $\dim X \leqq n < \infty$. Then suppose $\{f_iC \mid i = 1,\ldots,n+1\}$ is a given set of principal ideals. Assume $f_iC(X) = g_iC(X)$ for $i = 1,\ldots,n+1$. Then

$$(1)\qquad \begin{aligned} f_i &= g_i\varphi_i \ , \quad i = 1,\ldots,n+1 \ , \\ g_i &= f_i\psi_i \ , \quad i = 1,\ldots,n+1 \ , \end{aligned}$$

for some φ_i , $\psi_i \in C(X)$. Hence $g_i = g_i\varphi_i\psi_i$, i.e.

$$(2)\qquad g_i(1 - \varphi_i\psi_i) = 0 \ .$$

Let

$$(3)\qquad \begin{aligned} Z_i &= Z(\varphi_i) \ , \quad i = 1,\ldots,n+1 \ , \\ Z_i' &= Z(\varphi_i + 1 - \varphi_i\psi_i) \ , \quad i = 1,\ldots,n+1 \ , \end{aligned}$$

where we denote by $Z(\varphi)$ the zero set $\{x \in X \mid \varphi(x) = 0\}$ for $\varphi \in C(X)$. Then Z_i and Z_i' are disjoint zero sets, and hence there is $u_i \in C(X)$ such that

$$(4)\qquad u_i(Z_i) = 1 \ , \quad u_i(Z_i') = 0 \ , \quad 0 \leqq u_i \leqq 1 \ .$$

Define a continuous mapping u from X to I^{n+1} by $u(x) = (u_1(x),\ldots,u_{n+1}(x))$, $x \in X$. Then $u(Z_i) \subset S^n$, $u(Z_i') \subset S^n$, where we denote by S^n the boundary of I^{n+1} in E^{n+1} . Since $\dim X \leqq n$, by Theorem VII.11, the mapping u is inessential. Thus there is a continuous mapping v from X into S^n such that $v(x) = u(x)$ for $x \in u^{-1}(S^n)$.

Let us express v as $v = (v_1,\ldots,v_{n+1})$, where $v_i \in C(X)$, $i = 1,\ldots,n+1$. Then put $\theta_i = \varphi_i + v_i\,(1 - \varphi_i\psi_i)$, $i = 1,\ldots,n+1$. Let us prove

$$(5)\qquad Z(\theta_1) \cap \ldots \cap Z(\theta_{n+1}) = \emptyset \ .$$

To do so, let x be an arbitrary point of X . Since $v(x) \in S^n$, either $v_i(x) = 0$ or $v_i(x) = 1$ holds for some i . Suppose $v_i(x) = 0$; then $u_i(x) \neq 1$, because otherwise we reach the contradiction $v_i(x) = u_i(x) = 1$.

Hence by (4) $x \notin Z_i$. By (3) this implies $\varphi_i(x) \neq 0$. Hence $\theta_i(x) = \varphi_i(x) \neq 0$, i.e. $x \notin Z(\theta_i)$.

Suppose $v_i(x) = 1$; then $u_i(x) \neq 0$, and thus $x \notin Z_i'$ follows from (4). Hence by (3)

$$\theta_i(x) = \varphi_i(x) + 1 - \varphi_i(x)\psi_i(x) \neq 0 .$$

Therefore $x \notin Z(\theta_i)$. Thus (5) is proved. This implies that

$$\frac{\theta_i}{\theta_1^2 + \ldots + \theta_{n+1}^2} \in C(X) \ , \ i = 1,\ldots,n+1 .$$

Hence

$$\theta_1 C(X) + \ldots + \theta_{n+1} C(X) = C(X) \ ,$$

and it follows from (1) and (2) that

$$g_i\theta_i = g_i\varphi_i + g_i v_i(1 - \varphi_i\psi_i) = g_i\varphi_i = f_i .$$

Namely $\{ f_1 C(X),\ldots,f_{n+1}C(X) \}$ is uniquely generated, and hence $\text{p-dim}\ C(X) \leq n$.

Conversely, assume that $\text{p-dim}\ C(X) \leq n < \infty$. Let Z_i , Z_i' , $i = 1,\ldots,n+1$, be given pairs of disjoint closed sets of X . Find, by use of Urysohn's lemma, $p_i \in C(X)$ such that $p_i(Z_i) = 1$ and $p_i(Z_i') = -1$. Put

$$F_i = \{ x \in X \mid p_i(x) \geq 1/2 \},\ F_i' = \{ x \in X \mid p_i(x) \leq -1/2 \} ,$$
$$G_i = \{ x \in X \mid p_i(x) \leq 1/2 \},\ G_i' = \{ x \in X \mid p_i(x) \geq -1/2 \} .$$

Now find $q_i \in C(X)$ such that $q_i(F_i) = 1$, $q_i(F_i') = -1$, $-1 \leq q_i \leq 1$. Furthermore, select s_i , $t_i \in C(X)$ such that

(1) $$s_i(Z_i) = 1 \ , \ s_i(G_i) = 0 \ , \ 0 \leq s_i \leq 1 \ ,$$
$$t_i(Z_i') = -1 \ , \ t_i(G_i') = 0 \ , \ -1 \leq t_i \leq 0 .$$

Put $f_i = s_i + t_i$. Then we claim that

(2) $$f_i = q_i \mid f_i \mid .$$

If $x \in F_i$, then $q_i(x) = 1$, and $f_i(x) = s_i(x) + t_i(x) = s_i(x) \geq 0$.
If $x \in F_i'$, then $q_i(x) = -1$, and $f_i(x) = s_i(x) + t_i(x) = t_i(x) \leq 0$.
If $x \in X - (F_i \cup F_i')$, then $x \in G_i \cap G_i'$, and hence $f_i(x) = s_i(x) + t_i(x) = 0$.

Thus the equality (2) holds in every case. Observe that (2) implies

(3) $|f_i| = q_i f_i$.

Therefore

$$f_i C(X) = |f_i| C(X) \ , \ i = 1, \dots, n+1 \ .$$

Since p-dim $C(X) \leq n$, there are $\theta_1, \dots, \theta_{n+1} \in C(X)$ such that $f_i = \theta_i |f_i|$, $i = 1, \dots, n+1$, and $\theta_1 C(X) + \dots + \theta_{n+1} C(X) = C(X)$. The last equality implies that

$$Z(\theta_1) \cap \dots \cap Z(\theta_{n+1}) = \emptyset \ .$$

Put $U_i = \{ x \in X | \ \theta_i(x) > 0 \}$. Then, since for each $x \in Z_i$ $f_i(x) = s_i(x) + t_i(x) = 1$ follows from (1), we obtain $Z_i \subset U_i$. On the other hand, for each $x \in Z_i'$, $f_i(x) = -1$ follows from (1), and hence $Z_i' \cap U_i = \emptyset$.

It is obvious that

$$\bigcap_{i=1}^{n+1} B(U_i) \subset \bigcap_{i=1}^{n+1} Z(\theta_i) = \emptyset \ .$$

Thus by Theorem VII.8, dim $X \leq n$. Hence the theorem is proved. [24]

We now turn to another method of characterization. It is interesting that dim can be characterized in terms of the concept "partition of unity" which is important in topology and analysis, though the characterization is not purely of algebraic nature.

Definition VII. 4. *A subset* D *of* $C(X)$ *is called a partition of unity if every element of* D *is non-negative and* $\sum_{f \in D} f = 1$ *(meaning* $\sum_{f \in D} f(x) = 1$ *for all* $x \in X$*) . If* D *contains only finitely many elements, the partition is called finite. A partition of unity* D *is said to be subordinated to a covering* $\mathcal{U}$ *of* X *if* $\{ P(f) \mid f \in D \} < \mathcal{U}$ *, where* $P(f) = \{ x \in X | \ f(x) \neq 0 \}$.

To every partition of unity D and covering $\mathcal{U}$ the following real numbers can be associated:

$$\lambda D = \inf \{ \sup \{ f(x) \mid f \in D \} \mid x \in X \} \quad \text{and}$$

[24] M. Canfell [1] proved the theorem in a more general form.

$$c(\mathcal{U}) = \sup \left\{ \lambda D \,\middle|\, \begin{array}{l} D \text{ is a finite partition of} \\ \text{unity subordinated to } \mathcal{U} \end{array} \right\} .$$

Theorem VII. 18. *Let X be a non-empty normal space and Ω the collection of all finite open coverings of X. Then* $\inf\{ c(\mathcal{U}) \mid \mathcal{U} \in \Omega \} = 1/(\dim X + 1)$.

Proof. Assume $\dim X \leqq n < \infty$. Then for each $\mathcal{U} \in \Omega$ there is $\mathcal{V} = \{ V_i \mid i = 1,\dots,k \}$ such that $\mathcal{V} < \mathcal{U}$, ord $\mathcal{V} \leqq n+1$. Let $\mathcal{W} = \{ W_i \mid i = 1,\dots,k \} \in \Omega$ satisfy $\overline{W}_i \subset V_i$. Then select $f_i \in C(X)$, $i = 1,\dots,k$, such that $f_i(\overline{W}_i) = 1$, $f_i(X - V_i) = 0$, $0 \leqq f_i \leqq 1$.

Put

$$g_i = f_i \Big/ \sum_{j=1}^{k} f_j \ , \quad i = 1,\dots,k \ .$$

Then $D = \{ g_1,\dots,g_k \}$ is a partition of unity subordinated to $\mathcal{U}$. Let x be an arbitrary point of X ; then $g_i(x) \neq 0$ for at most $n+1$ distinct elements of D , because ord $\mathcal{V} \leqq n+1$, and $g_i = f_i = 0$ holds outside of V_i . Since $\sum_{i=1}^{k} g_i(x) = 1$ there exists i with $g_i(x) \geqq 1/(n+1)$. Thus $\lambda D \geqq 1/(n+1)$, which implies $c(\mathcal{U}) \geqq 1/(n+1)$, and accordingly

$$\inf \{ c(\mathcal{U}) \mid \mathcal{U} \in \Omega \} \geqq 1/(n+1) \ .$$

Thus

$$\inf \{ c(\mathcal{U}) \mid \mathcal{U} \in \Omega \} \geqq 1/(\dim X + 1) \ .$$

Conversely, assume $\dim X \geqq n$. Then there is $\mathcal{U} \in \Omega$ such that every open refinement $\mathcal{V}$ of $\mathcal{U}$ satisfies ord $\mathcal{V} \geqq n+1$. Assume that $\lambda D > 1/(n+1)$ holds for some finite partition of unity $D = \{ f_i \mid i = 1,\dots,k \}$ subordinated to $\mathcal{U}$. Then $\lambda D > 1/(n+1) + \varepsilon$ for some $\varepsilon > 0$, and hence $\sup \{ f_i(x) \mid i = 1,\dots,k \} > 1/(n+1)$ for every $x \in X$, i.e. $f_i(x) > 1/(n+1)$ holds for every $x \in X$ and some i .

Now, put $V_i = \{ x \mid f_i(x) > 1/(n+1) \}$; then $\mathcal{V} = \{ V_1,\dots,V_k \} \in \Omega$. Since $\mathcal{V} < \mathcal{U}$ follows from the fact that D is subordinated to $\mathcal{U}$, we have ord $\mathcal{V} \geqq n+1$.

Assume that $\text{ord}_x \mathcal{V} \geqq n+1$ holds at $x \in X$. Thus there are $n+1$ distinct elements, say $f_1,\dots,f_{n+1}$ of D such that $f_i(x) > 1/(n+1)$ for $i = 1,\dots,n+1$. Hence

$$\sum_{i=1}^{k} f_i(x) \geqq \sum_{i=1}^{n+1} f_i(x) > 1 \ ,$$

which is a contradiction.

Therefore $\lambda D \leq 1/(n+1)$ holds for all finite partitions of unity subordinated to $\mathcal{U}$. Hence $c(\mathcal{U}) \leq 1/(n+1)$, which implies

$$\inf\{c(\mathcal{U}) \mid \mathcal{U} \in \Omega\} \leq 1/(\dim X + 1) ,$$

proving the theorem.[25]

[25] Let Ω' be the set of all locally finite open coverings of X. Define $c'(\mathcal{U}) = \sup\{D \mid D$ is a (not necessarily finite) partition of unity subordinated to $\mathcal{U}\}$. Then $\inf\{c'(\mathcal{U}) \mid \mathcal{U} \in \Omega'\} = \inf\{c'(\mathcal{U}) \mid \mathcal{U} \in \Omega\} = \inf\{c(\mathcal{U}) \mid \mathcal{U} \in \Omega\} = 1/(\dim X + 1)$ can be easily proved for a normal space X in a similar way. J. Hejcman [2] proved the theorem in a more general form.