CHAPTER VI

INFINITE-DIMENSIONAL SPACES

We shall first introduce notions which distinguish some types of infinite-dimensional spaces from general infinite-dimensional spaces. It will be a natural idea to extend definitions of Ind and ind by transfinite induction.

Definition VI. 1. *i) A space* R *has strong (weak) inductive dimension* -1, Ind $R = -1$ (ind $R = -1$) *if* $R = \emptyset$.

ii) Let α *be a transfinite ordinal number. If for every disjoint closed sets* F *and* G *(for any neighbourhood* $U(p)$ *of any point* p*) of* R *there exists an open set* V *such that*

$$F \subset V \subset R - G \ (p \in V \subset U(p)) \text{ and } \operatorname{Ind} B(V) < \alpha \ (\operatorname{ind} B(V) < \alpha),$$

then R *has strong (weak) transfinite inductive dimension* $\leq \alpha$,

$$\operatorname{Ind} R \leq \alpha \ (\operatorname{ind} R \leq \alpha).$$

In the above definition the word 'large (small)' may be used in place of 'strong (weak)'.

Definition VI. 2. *If for every countable number of pairs* $\{F_i, G_i\}$, $i = 1, 2, \ldots$ *of disjoint closed sets* F_i *and* G_i *of a space* R, *there exist open sets* U_i, $i = 1, 2, \ldots$ *such that*

$$F_i \subset U_i \subset R - G_i \text{ and } \bigcap_{i=1}^{k} B(U_i) = \emptyset \text{ for some number } k,$$

then we call R a weakly infinite-dimensional space. Otherwise R is called strongly infinite-dimensional.

The decomposition theorem states that a space R has dimension $\leq n$ if and only if R can be decomposed into $n+1$ 0-dimensional spaces. Therefore it is natural to define countable-dimensional spaces as follows:

Definition VI. 3. *A space R is called countable-dimensional if* $R = \bigcup_{i=1}^{\infty} R_i$ *for some subspaces* R_i *of dimension* ≤ 0 [1]. *By the decomposition theorem this is equivalent to the statement that R is the countable sum of finite-dimensional spaces.*

Definition VI. 4. *If a space R is the countable sum of finite-dimensional closed sets, then we call R a countable-dimensional space in the strong sense or strongly countable-dimensional.*

A countable dimensional space in the strong sense is countable-dimensional, but the converse is not true as shown by the counter example R^{ω}, the set of points in I^{ω} at most finitely many of whose coordinates are rational [2].

We shall discuss some relation between weak infinite-dimensionality and countable-dimensionality later in VI.1. A famous problem of P. Alexandroff in this aspect has recently been answered by R. Pol [1] in the negative [3]. Namely he gave an example of a compact metric space which is weakly infinite-dimensional but not countable-dimensional. Thus for compact metric spaces the latter is a stronger condition than the former.

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[1] L. Tumarkin [5] proved that if R is a countable-dimensional separable space, then R can be decomposed into countably many 0-dimensional subspaces R_i, $i = 1,2,\ldots$ so that for every n, $\bigcup_{i=1}^{n} R_i$ is 0-dimensional. K. Nagami [6] generalized this result to general metric spaces. (See also A. Arhangelskii [1]).

[2] See Yu. Smirnov [6] or J. Nagata [3].

[3] In the construction of R. Pol the following two propositions play important roles: A. Lelek's [1] proposition that for every separable completely metrizable space X there is a metric compactification $\tilde{X}$ such that $\tilde{X} - X$ is strongly countable-dimensional, and L. Rubin - R. Schori - J. Walsh's [1] proposition that there is a separable completely metrizable space which is totally disconnected but not countable-dimensional.

As for relationships among compactification, weak-dimensionality, and countable-dimensionality, see (besides the above-mentioned paper of Lelek) E. Sklyarenko [1], A. W. Schurle [1], Z. Shmuely [1], and K. Nagami - J. H. Robers [1] in which it is proved that the set K^{ω} of points in I^{ω}, whose coordinates are all zero except for at most finitely many, has no countable-dimensional (metric) completion.

It is also known that a normal space X is weakly infinite-dimensional if and only if βX is weakly infinite-dimensional and that $\operatorname{Ind} X \leq \alpha$ if and only if $\operatorname{Ind} \beta X \leq \alpha$, where βX denotes the Stone-Čech compactification of X.

Another famous problem was Tumarkin's: Does every infinite-dimensional compact metric space contain closed sets of arbitrary high finite dimension? (Note that every $n(<\infty)$-dimensional space has an m-dimensional closed set for every $m \leq n$ and also that every countable-dimensional compact metric space has an m-dimensional closed set for every natural number m because of 3 C).) L. Tumarkin [4] himself proved that every infinite-dimensional compact metric space contains either compact sets of arbitrary high finite dimension or an *infinite-dimensional Cantor manifold*, i.e. an infinite-dimensional compact set F such that for every finite dimensional subset A of F the complement $F-A$ is connected. This problem has remained unsolved for a long time until D. Henderson [1] constructed an infinite-dimensional compact metric space which has no n-dimensional closed subset for every n with $1 \leq n < \infty$ in 1965. His complicated original construction was improved meanwhile by several authors, e.g. R. H. Bing [2] and A. V. Zarelua [3]. Further J. J. Walsh [1] constructed an infinite-dimensional compact metric space containing no n-dimensional $(1 \leq n < \infty)$ subsets.

Up to the present we do not know as much about infinite-dimensional spaces as about finite-dimensional spaces because of some difficult circumstances which are peculiar to infinite-dimensional cases. In this chapter we shall give a brief account of infinite-dimensional spaces and especially of countable-dimensional spaces, and strong transfinite inductive dimension [4].

VI. 1. Countable-dimensional spaces

The purpose of this section is to extend some results of the theory of finite-dimensional spaces to countable-dimensional spaces. As a matter of fact, the foundation of the theory of countable-dimensional spaces is established by III.4 A), III.7 A) and III.7 B).

To begin with, let us show that not every space is countable-dimensional [5].

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[4] P. Fletcher - R. McCoy - R. Solver [1] defined another kind of infinite-dimensionality as follows and studied its properties: a space X is called *boundedly paracompact* if for each open cover $\mathcal{U}$ of X there is a natural number n (dependent on $\mathcal{U}$) such that $\mathcal{U}$ has a locally finite open refinement of order $\leq n$. Another class of infinite-dimensional spaces was considered by D. Addis - J. Gresham [1].

[5] In contrast to this fact every metric space is homeomorphic to a subset of the topological product of countably many one-dimensional metric spaces. H. J. Kowalsky [1] proved that every metric space is homeomorphic to a subset of the topological product of countably many star spaces. A star space is a 1-dimensional metric space defined by Definition VI.6. As for finite-dimensional spaces J. Nagata [2] proved that every n-dimensional metric space can be imbedded in the product of $n+1$ 1-dimensional metric spaces.

On the other hand K. Borsuk [4] proved that S^2 cannot be imbedded as a subset of the product of two 1-dimensional spaces. Thus it is generally impossible to represent an n-dimensional space as a subset of the product of n 1-dimensional spaces.

A) Consider the n-dimensional cube

$$I_1^n = \{x \mid |x_i| \leq 1,\ i = 1,\dots,n\}$$

where we denote by x_i the i-th coordinate of the point $x \in E^n$. Put

$$F_i = \{x \mid x \in I_1^n,\ x_i = 1\},\ F_i' = \{x \mid x \in I_1^n,\ x_i = -1\}\ ,$$

for $i = 1,\dots,n$. If V_i, $i = 1,\dots,n$ are open sets of I_1^n satisfying $F_i \subset V_i \subset I_1^n - F_i'$, then $\cap_{i=1}^n B(V_i) \neq \emptyset$.

Proof. Assume the contrary, i.e. $\cap_{i=1}^n B(V_i) = \emptyset$.

Then by use of the corollary to Theorem I.6, we can construct real-valued continuous functions f_i, $i = 1,\dots,n$ such that

$$f_i(F_i) = 1\ ,\ f_i(F_i') = -1\ ,\ B(V_i) = \{x \mid f_i(x) = 0\}\ ,\ |f_i| \leq 1\ .$$

We define by $f(x) = (f_1(x),\dots,f_n(x))$ a continuous mapping of I_1^n into itself. It is clear that

$$f(F_i) \subset F_i\ ,\ f(F_i') \subset F_i'\ ,\ p_0 \notin f(I_1^n)\ ,$$

where p_0 denotes the origin of E^n. Letting $I_2^n = \{x \mid |x_i| \leq 2\ ,\ i = 1,\dots,n\}$ we consider a given point x of $I_2^n - (I_1^n \cup B(I_2^n))$. By a,b we denote the projections of x from p_0 to $B(I_1^n)$ and $B(I_2^n)$ respectively. Then we denote by $g(x)$ the point which divides the segment joining $f(a)$ and b, in the ratio $\rho(a,x)/\rho(x,b)$. Finally, we define a continuous mapping h of I_2^n into itself by

$$h(x) = \begin{cases} f(x) & \text{for } x \in I_1^n\ , \\ g(x) & \text{for } x \in I_2^n - (I_1^n \cup B(I_2^n))\ , \\ x & \text{for } x \in B(I_2^n)\ . \end{cases}$$

Then h leaves every point of $B(I_2^n)$ fixed while $p_0 \notin h(I_2^n)$. But this contradicts IV.3 A). Thus our assertion is proved.

B) The Hilbert cube I^ω is neither countable-dimensional nor weakly infinite-dimensional.

Proof. To proof the first assertion let us assume the contrary. Let

$$I^\omega = \prod_{i=1}^{\infty} [-\tfrac{1}{i}, \tfrac{1}{i}]\ ,$$

$$F_i = \{ x \mid x \in I^\omega,\ x_i = 1/i \},\ F'_i = \{ x \mid x \in I^\omega,\ x_i = -1/i \},$$

where we denote by x_i the i-th coordinate of x. Since by assumption I^ω can be decomposed into countably many 0-dimensional subsets, III.4 A) implies the existence of a sequence V_i, $i = 1,2,\ldots$ of open sets such that

$$F_i \subset V_i \subset I^\omega - F'_i \text{, and}$$
$$\operatorname{ord}_p \{ B(V_i) \mid i = 1,2,\ldots \} < +\infty \quad \text{at every} \quad p \in I^\omega .$$

Define

$$I^n = \{ x \in I^\omega \mid |x_i| \leqq 1/i \quad \text{for} \quad i = 1,\ldots,n \ ;\ x_j = 0 \quad \text{for} \quad j \geqq n+1 \}.$$

Then I^n is topologically equivalent to the n-cube. It is clear that $V_i \cap I^n$ is an open set of I^n such that

$$F_i \cap I^n \subset V_i \cap I^n \subset I^n - F'_i \cap I^n .$$

Hence by A) we obtain

$$\bigcap_{i=1}^{n} B(V_i) \supset \bigcap_{i=1}^{n} B_{I^n}(V_i \cap I^n) \neq \emptyset .$$

Since I^ω is compact, this implies

$$\bigcap_{i=1}^{\infty} B(V_i) \neq \emptyset ,$$

which is a contradiction. Thus I^ω is not countable-dimensional. Note that the second assertion is also implicitely proved in the above.

Theorem VI. 1 [6]. *A space R is countable-dimensional if and only if there exists a σ-locally finite open basis V such that* $\operatorname{ord}_p B(V) < +\infty$ *at every point p of R.*

[6] Theorems in Sections 1 and 2 are mainly due to J. Nagata [3]. Characterizations of strongly countable-dimensional spaces are discussed in J. Nagata [2] and A. Arhangelskii [1]. The latter proved that a metric space X is strongly countable-dimensional if and only if it has a base U such that $\operatorname{rank}_x U < +\infty$ at every point x of X.

Proof. Let R be countable-dimensional; then R is decomposable into countably many 0-dimensional spaces A_i, $i = 1,2,\dots$. By the corollary to Theorem I.1 there exists a sequence of locally finite open coverings $\mathcal{U}_i$ of R such that $\{ S(p,\mathcal{U}_i) \mid i = 1,2,\dots \}$ is a neighbourhood basis of each point p. Assume

$$\mathcal{U}_i = \{ U_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \},\ \tau_0 = 0 .$$

Then by virtue of the local finiteness of $\mathcal{U}_i$ there is a closed covering $\{ F_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \}$ such that $F_\alpha \subset U_\alpha$. Defining

$$\tau = \sup \{ \tau_i \mid i = 1,2,\dots \}$$

we get an open collection $\{ U_\alpha \mid \alpha < \tau \}$ and a closed collection $\{ F_\alpha \mid \alpha < \tau \}$ which satisfy the condition of III.4 A). Hence we can construct an open collection $\mathcal{V} = \{ V_\alpha \mid \alpha < \tau \}$ according to that proposition. Now, letting

$$\mathcal{V}_i = \{ V_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \} ,$$

we know that $\mathcal{V}$ is the desired σ-locally finite open basis of R.

Conversely, if there exists an open basis $\mathcal{V}$ which satisfies the condition in the statement, then we let

$$A_n = \{ p \mid \operatorname{ord}_p \mathcal{B}(\mathcal{V}) = n - 1 \},\ n = 1,2,\dots .$$

Since $\mathcal{U} = \{ V \cap A_n \mid V \in \mathcal{V} \}$ is a σ-locally finite open basis of A_n such that

$$\operatorname{ord} \mathcal{B}_{A_n}(\mathcal{U}) \leqq n - 1$$

from Theorem II.9 we can deduce that $\dim A_n \leqq n - 1$. Thus from the decomposition theorem it follows that $R = \bigcup_{n=1}^{\infty} A_n$ is countable-dimensional.

C) A space R is countable-dimensional if and only if for every open collection $\{ U_\alpha \mid \alpha < \tau \}$ and closed collection $\{ F_\alpha \mid \alpha < \tau \}$ such that $F_\alpha \subset U_\alpha$ and such that $\{ U_\beta \mid \beta < \alpha \}$ is locally finite for every $\alpha < \tau$, there exists an open collection $\{ V_\alpha \mid \alpha < \tau \}$ such that

$$F_\alpha \subset V_\alpha \subset U_\alpha ,$$

$$\operatorname{ord}_p \{ \mathcal{B}(V_\alpha) \mid \alpha < \tau \} < +\infty \text{ at every point } p \in R .$$

Proof. The "only if" part is a direct consequence of III.4 A).

To prove the "if" part we consider a σ-locally finite open basis $\mathcal{U}$ of R which is decomposed into locally finite open coverings

$$\mathcal{U}_i = \{ U_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \}, \quad i = 1,2,\dots .$$

Then there exists a closed covering $\{ F_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \}$ of R such that $F_\alpha \subset U_\alpha$. If we put

$$\tau = \sup \{ \tau_i \mid i = 1,2,\dots \} ,$$

then

$$\mathcal{U} = \{ U_\alpha \mid \alpha < \tau \} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$$

and $\{ F_\alpha \mid \alpha < \tau \}$ satisfy the condition of the assertion. Hence we can construct the open collection $\mathcal{V} = \{ V_\alpha \mid \alpha < \tau \}$ in the assertion. Now, it is easy to see that $\mathcal{V}$ is an open basis satisfying the condition of Theorem VI.1, and hence R is countable-dimensional.

Theorem VI. 2. *A space R is countable-dimensional if and only if for all sequences $\{ U_i \mid i = 1,2,\dots \}$ of open sets and $\{ F_i \mid i = 1,2,\dots \}$ of closed sets satisfying $F_i \subset U_i$, $i = 1,2,\dots$ there exists a sequence $\mathcal{V} = \{ V_i \mid i = 1,2,\dots \}$ of open sets such that*

$$F_i \subset V_i \subset U_i , \quad i = 1,2,\dots \quad \text{and}$$
$$\operatorname{ord}_p B(\mathcal{V}) < +\infty \quad \text{at every point } p \text{ of } R .$$

Proof. Since the "only if" part is an immediate consequence of C), we show only the "if" part. By use of Theorem I.4 we can find a σ-discrete open basis

$$\mathcal{U}' = \bigcup_{i=1}^{\infty} \mathcal{U}'_i$$

of the space R. Suppose $\mathcal{U}'_i = \{ U'_\gamma \mid \gamma \in \Gamma_i \}$ is a discrete open collection; then we decompose each U'_γ as

$$U'_\gamma = \bigcup_{j=1}^{\infty} F'_{\gamma j}$$

for closed sets $F'_{\gamma j}$. Furthermore, we let

$$U_i = \cup \{ U'_\gamma \mid \gamma \in \Gamma_i \} \text{ , and } F_i^j = \cup \{ F'_{\gamma j} \mid \gamma \in \Gamma_i \} \text{ .}$$

Note that F_i^j is closed by virtue of the discreteness of U'_i. Then, since $F_i^j \subset U_i$, $i,j = 1,2,\ldots$, by use of the condition of the theorem we construct an open collection $V = \{ V_i^j \mid i,j = 1,2,\ldots \}$ such that

$$F_i^j \subset V_i^j \subset U_i \text{ , and}$$
$$\operatorname{ord}_p B(V) < +\infty \quad \text{at every } p \in R \text{ .}$$

Defining

$$W_\gamma^j = V_i^j \cap U'_\gamma \quad \text{for} \quad \gamma \in \Gamma_i \text{ ,}$$

we get a locally finite open collection

$$\mathcal{W}_i^j = \{ W_\gamma^j \mid \gamma \in \Gamma_i \} \text{ .}$$

Here we note that

$$B(W_\gamma^j) \cap B(W_{\gamma'}^j) = \emptyset$$

if $\gamma, \gamma' \in \Gamma_i$, $\gamma \neq \gamma'$, because in this case it follows from the discreteness of U'_i that

$$\bar{W}_\gamma^j \cap \bar{W}_{\gamma'}^j \subset \bar{U}'_\gamma \cap \bar{U}'_{\gamma'} = \emptyset \text{ .}$$

Hence

$$\mathcal{W} = \cup \{ \mathcal{W}_i^j \mid i,j = 1,2,\ldots \}$$

is a σ-locally finite open basis of R such that $\operatorname{ord}_p B(\mathcal{W}) < +\infty$ at every $p \in R$. Hence by Theorem VI.1, we conclude that R is countable-dimensional.

Corollary. *Every countable-dimensional compact space* R *is weakly infinite-dimensional* [7].

Theorem VI. 3. *A space* R *is countable-dimensional if and only if there exists a sequence* $\{\mathcal{F}_i \mid i = 1,2,\dots\}$ *of locally finite closed coverings of* R *satisfying*

i) *for every neighbourhood* $U(p)$ *of every point* p *of* R *there exists some* i *with* $S(p,\mathcal{F}_i) \subset U(p)$,

ii) $\mathcal{F}_i = \{F(\alpha_1,\dots,\alpha_i) \mid \alpha_k \in \Omega\ ,\ k = 1,\dots,i\}$, *where* $F(\alpha_1,\dots,\alpha_i)$ *may be empty*,

iii) $F(\alpha_1,\dots,\alpha_{i-1}) = \cup\ \{F(\alpha_1,\dots,\alpha_{i-1},\beta) \mid \beta \in \Omega\}$,

iv) $\sup\ \{\operatorname{ord}_p \mathcal{F}_i \mid i = 1,2,\dots\} < +\infty$ *at every point* p *of* R .

Proof. Let R be a countable-dimensional space such that $R = \bigcup_{n=1}^{\infty} A_n$ for 0-dimensional spaces A_n . Then III.7 B) assures us that there exists a required sequence $\{\mathcal{F}_i \mid i = 1,2,\dots\}$ of closed coverings.

Conversely, if there exists a sequence $\{\mathcal{F}_i \mid i = 1,2,\dots\}$ satisfying i)-iv), then we define

$$A_n = \{p \mid \sup\{\operatorname{ord}_p \mathcal{F}_i \mid i = 1,2,\dots\} = n+1\}\ ,\ n = 0,1,\dots\ .$$

Since $\operatorname{ord}_p \mathcal{F}_i \leqq n+1$ at every point $p \in A_n$, by Theorem III.9 we obtain that $\dim A_n \leqq n$. Since the condition iv) of $\mathcal{F}_i$ implies that $R = \bigcup_{n=0}^{\infty} A_n$, R is countable-dimensional.

Theorem VI. 4. *A space* R *is countable-dimensional if and only if there exists a subset* P *of a generalized Baire's 0-dimensional space* $N(\Omega)$ *for suitable* Ω *and a closed continuous mapping* f *of* P *onto* R *such that for each point* q *of* R *the inverse image* $f^{-1}(q)$ *consists of finitely many points* [8] .

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[7] This assertion is not true unless R is compact, because we can easily see that K^{ω} , the set of points in I^{ω} at most finitely many of whose coordinates are different from zero, is not weakly infinite-dimensional though it is countable-dimensional in the strong sense.
There is another definition of *weak infinite-dimensionality* which is obtained by replacing "$\bigcap_{i=1}^{k} B(U_i) = \emptyset$ for some k" in Definition VI.2 with "$\bigcap_{i=1}^{\infty} B(U_i) = \emptyset$". This definition is due to P. S. Alexandroff while the previous one is due to Yu. Smirnov. They are equivalent in compact spaces. The discrete sum $\bigcup_{n=1}^{\infty} I^n$ of n-dimensional cubes is strongly countable-dimensional and weakly infinite-dimensional in the sense of Alexandroff, but not in the sense of Smirnov. Every countably-dimensional space is weakly infinite-dimensional in the sense of Alexandroff. (In fact the same is true for every hereditarily normal space.)

[8] E. Sklyarenko [2] proved that a compact space R is countable-dimensional if and only if there exists a continuous mapping f of the Cantor's perfect set D^{∞} on R such that for each point q of R , $f^{-1}(q)$ consists of countably many points. J. Walker - B. Wenner [1] proved a similar theorem for strongly countable-dimensional spaces.

Proof. The "only if" part is an immediate consequence of Theorem VI.3. In fact, let R be a countable-dimensional space and $\{F_i \mid i = 1,2,\dots\}$ a sequence of closed coverings satisfying i)-iv) of Theorem VI.3. Then we define a subset P of $N(\Omega)$ by

$$P = \{(\alpha_1,\alpha_2,\dots) \mid \bigcap_{i=1}^{\infty} F(\alpha_1,\dots,\alpha_i) \neq \emptyset\}$$

and a mapping f of P onto R by

$$f(\alpha) = \bigcap_{i=1}^{\infty} F(\alpha_1,\dots,\alpha_i) \quad \text{for} \quad \alpha = (\alpha_1,\alpha_2,\dots) .$$

We can show, quite analogously to the proof of Theorem III.8, that f is a closed continuous mapping such that for every $q \in R$, $f^{-1}(q)$ consists of finitely many points.

Conversely, if there exist such $N(\Omega)$, P and f, then we let

$$A_n = \{q \mid f^{-1}(q) \text{ consists of } n+1 \text{ points}\} .$$

Since f restricted to $f^{-1}(A_n)$ is obviously a closed continuous mapping of $f^{-1}(A_n)$ onto A_n, by Theorem III.8 A_n is an at most n-dimensional subset of R. Thus from the fact that $R = \bigcup_{n=0}^{\infty} A_n$ we deduce that R is countable-dimensional.

Theorems VI.1, VI.2, VI.3 and Theorem VI.4 are extensions of Theorems II.9, II.8, III.9 and Theorem III.8 respectively to the countable-dimensional case.

VI. 2. Imbedding of countable-dimensional spaces

In this section we shall give a universal space for countable-dimensional spaces.

A) Let R be a countable-dimensional space for which $R = \bigcup_{n=1}^{\infty} A_n$, $\dim A_n = 0$. Let $\{U_m \mid m = 1,2,\dots\}$ be an open collection and $\{F_m \mid m = 1,2,\dots\}$ a closed collection such that $F_m \subset U_m$. Then there exists an open collection $V = \{V_{mr} \mid m = 1,2,\dots, |r| < \sqrt{2}/2m,\ r \text{ rational}\}$ satisfying

i) $F_m \subset V_{mr} \subset \bar{V}_{mr} \subset V_{mr'} \subset \bar{V}_{mr'} \subset U_m$ if $r < r'$.

ii) $\bar{V}_{mr} = \bigcap\{V_{mr'} \mid r' > r\}$, $V_{mr} = \bigcup\{\bar{V}_{mr'} \mid r' < r\}$,

iii) $\operatorname{ord}_p B(V) \leq n-1$ at each point p of A_n.

Proof. First we consider m fixed and index all the rational numbers satisfying $|r| < \sqrt{2}/2m$ so that

$$
\begin{array}{c}
r_{m1} , \\
r_{m2} < r_{m1} < r_{m3} , \\
r_{m4} < r_{m2} < r_{m5} < r_{m1} < r_{m6} < r_{m3} < r_{m7} , \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

Note that these rational numbers are naturally ordered as

$$
\begin{array}{l}
\frac{0}{1} , \\
\frac{-k_2}{2} , \frac{-k_2+1}{2} , \ldots , \frac{k_2}{2} , \\
\frac{-k_3}{3} , \frac{-k_3+1}{3} , \ldots , \frac{k_3}{3} , \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

for $k_i = [\sqrt{2}/2m]i$; then to suit our requirement this order must only be modified slightly.

Now, we put

$$
\begin{array}{l}
N_{m1} = \{ r_{m1} \} , \\
N_{m2} = \{ r_{m2}, r_{m3} \} , \\
N_{m3} = \{ r_{m4}, r_{m5}, r_{m6}, r_{m7} \} , \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

We shall define open sets V_{mr} satisfying i) iii) and

iv) if r_{mk} succeeds r_{mi} in $\cup_{h=1}^{s-1} N_{mh}$, i.e. $r_{mi} < r_{mk}$ and there exists no number r in $\cup_{h=1}^{s-1} N_{mh}$ satisfying $r_{mi} < r < r_{mk}$, and if $r_{mj} \in N_{ms}$, $r_{mi} < r_{mj} < r_{mk}$, then

$V_{r_{mj}} \subset S_{1/s}(\bar{V}_{r_{mi}})$ for odd s ,

$R - \bar{V}_{r_{mj}} \subset S_{1/s}(R - V_{r_{mk}})$ for even s ,

where we denote, for brevity, $V_{mr_{mi}}$ by $V_{r_{mi}}$.

Then it will be easily seen that

$$\{ V_{r_{mi}} \mid r_{mi} \in \bigcup_{h=1}^{\infty} N_{mh} \} = \{ V_{mr} \mid |r| < \sqrt{2}/2m \}$$

satisfies also ii). In fact, assume that $\{ V_{r_{mi}} \mid r_{mi} \in \cup_{h=1}^{\infty} N_{mh} \}$ satisfies iv). We consider a number $r_{mi} \in N_{m\,s-1}$ and a point p for which $p \notin \bar{V}_{r_{mi}}$. Then we select an odd t satisfying

$$t \geqq \max \left[s, \frac{1}{\rho(p, \bar{V}_{r_{mi}})} \right]$$

and the number $r_{mj} \in N_{mt}$ which succeeds r_{mi} in $\cup_{k=1}^{t} N_{mk}$. It follows from iv) that

$$V_{r_{mj}} \subset S_{1/t}(\bar{V}_{r_{mi}}) \not\ni p .$$

Hence

$$\bar{V}_{mr} = \cap \{ V_{mr'} \mid r' > r \} .$$

Subsequently we consider a number $r_{mk} \in N_{m\,s-1}$ and a point p for which $p \in V_{r_{mk}}$. Then we choose an even t satisfying

$$t \geqq \max \left[s, \frac{1}{\rho(p, R - V_{r_{mk}})} \right]$$

and the number $r_{mj} \in N_{mt}$ which r_{mk} succeeds in $\cup_{k=1}^{t} N_{mk}$. It follows from iv) that

$$p \in R - S_{1/t}(R - V_{r_{mk}}) \subset \bar{V}_{r_{mj}} .$$

Hence

$$V_{mr} = \cup \{ \bar{V}_{mr'} \mid r' < r \} ,$$

proving ii).

Now, we define V_{mr} for all m and r by induction in the following order

$$V_{r_{11}}, V_{r_{12}}, V_{r_{21}}, V_{r_{13}}, V_{r_{22}}, V_{r_{31}}, V_{r_{14}}, \dots .$$

(The point is that the order of $\{ r_{m1}, r_{m2}, \dots \}$ for a fixed m is kept in the sequence $\{ r_{11}, r_{12}, r_{21}, \dots \}$. As long as this condition is satisfied we may choose any other ordering.)

First, by use of $\dim A_1 = 0$ we construct an open set $V_{r_{11}}$ such that

$$F_1 \subset V_{r_{11}} \subset \bar{V}_{r_{11}} \subset U_1 \quad \text{and} \quad B(V_{r_{11}}) \cap A_1 = \emptyset .$$

Since the construction procedure is quite analogous to that for V_0 in III.4 A) it will be omitted here.

Suppose that we have defined all $V_{r_{nh}}$ before $V_{r_{mj}}$ and that $r_{mj} \in N_{ms}$. Then to define $V_{r_{mj}}$ we choose numbers

$$r_{mi}, r_{mk} \in \bigcup_{h=1}^{s-1} N_{mh}$$

such that $r_{mi} < r_{mj} < r_{mk}$ and such that r_{mk} succeeds r_{mi} in $\bigcup_{k=1}^{s-1} N_{mk}$. By use of $\dim A_n = 0$, we can construct $V_{r_{mj}}$ such that

$$\bar{V}_{r_{mi}} \subset V_{r_{mj}} \subset \bar{V}_{r_{mj}} \subset V_{r_{mk}} \cap S_{1/s}(\bar{V}_{r_{mi}}) \quad \text{if } s \text{ is odd,}$$

$$[R - S_{1/s}(R - V_{r_{mk}})] \cup \bar{V}_{r_{mi}} \subset V_{r_{mj}} \subset \bar{V}_{r_{mj}} \subset V_{r_{mk}} \quad \text{if } s \text{ is even}$$

and such that

$$\operatorname{ord}_p B(\mathcal{V}_{mj}) \leqq n-1 , \text{ at each point } p \in A_n ,$$

where

$$\mathcal{V}_{mj} = \{ V_{r_{11}}, V_{r_{12}}, V_{r_{21}}, \ldots, V_{r_{mj}} \} .$$

The construction procedure for $V_{r_{mj}}$ is quite parallel to that for V_α in III.4 A) and hence its details are left to the reader. Thus the proof of the assertion is complete.

Recall that we defined $I^\omega = \prod_{i=1}^{\infty} [-\frac{1}{i}, \frac{1}{i}]$.

Theorem VI. 5. *Let R be a space with a σ-star-finite open base. Denote by R^ω the set of points in I^ω at most finitely many of whose coordinates are rational.*

Then R is countable-dimensional if and only if R is homeomorphic to a subset of $N(\Omega)\times R^{\omega}$ *for suitable* Ω [9] .

Proof. Let R be a space with a σ-star-finite open basis. Then, since by I.2 F) R is homeomorphic to a subset of $N(\Omega)\times I^{\omega}$, we obtain a sequence

$$N_1 > N_2 > \dots$$

of star-finite open coverings such that $\{ S(p,N_i) \mid i=1,2,\dots \}$ is a neighbourhood basis of each point p of R . We put

$$S_i = \{ S^{\infty}(N,N_i) \mid N\in N_i \} ,$$

recalling that

$$S^{\infty}(N,N_i) = \bigcup_{n=1}^{\infty} S^{n}(N,N_i) .$$

Then, for brevity, we define

$$S_i = \{ S_{\alpha} \mid \alpha\in\Omega_i \} , \text{ where } S_{\alpha}\cap S_{\beta}=\emptyset \text{ for } \alpha\neq\beta .$$

Since N_i is star-finite, each set S_{α} is a sum of countably many elements of N_i . Hence we can decompose S_{α} as

$$S_{\alpha} = \bigcup \{ N_{\alpha j} \mid j=1,2,\dots \} \text{ for } N_{\alpha j}\in N_i .$$

For each i we construct an open covering P_i of R such that

$$P_i = \{ P_{\alpha j} \mid \alpha\in\Omega_j , j=1,2,\dots \} \text{ with } \bar{P}_{\alpha j}\subset N_{\alpha j} .$$

Defining

$$U_{ij} = \bigcup \{ N_{\alpha j} \mid \alpha\in\Omega_i \} \text{ and } F_{ij} = \bigcup \{ \bar{P}_{\alpha j} \mid \alpha\in\Omega_i \}$$

[9] Concerning the imbedding of countable-dimensional spaces in the strong sense Yu. Smirnov [6] and J. Nagata [3] proved that a space R with a σ-star-finite open basis is countable-dimensional in the strong sense if and only if it is homeomorphic to a subset of $N(\Omega)\times K^{\omega}$ for suitable Ω , where K^{ω} denotes the set of points in I^{ω} at most finitely many of whose coordinates are different from zero. It is easy to see that in Theorem VI.5 Ω may be taken as the set whose cardinality is equal to the weight of R (namely the minimal cardinality of its base). The same applies in the strongly countable-dimensional case as well.

we obtain an open set U_{ij} and a closed set F_{ij} satisfying $F_{ij} \subset U_{ij}$. Then we put, for brevity,

$$\{U_{ij} \mid i,j = 1,2,\dots\} = \{U_m \mid m = 1,2,\dots\}$$
$$\{F_{ij} \mid i,j = 1,2,\dots\} = \{F_m \mid m = 1,2,\dots\} .$$

Now, suppose R is countable-dimensional. Then for these U_m and F_m we define V_{mr} by use of A) and construct a real-valued continuous function f_m over R by

$$f_m(p) = \begin{cases} \inf\{r \mid p \in V_{mr}\} & \text{if } p \in V_{mr} \text{ for some } r , \\ \sqrt{2}/2m & \text{if } p \notin V_{mr} \text{ for all } r . \end{cases}$$

Then by i) of A) it is clear that f_m is a continuous function such that

$$f_m(F_m) = -\sqrt{2}/2m , \quad f_m(R - U_m) = \sqrt{2}/2m , \quad |f_m| \leqq \sqrt{2}/2m .$$

Suppose

$$p \notin \bigcup\{B(V_{mr}) \mid |r| < \sqrt{2}/2m, \ r \text{ rational}\} ;$$

then we shall show that $f_m(p)$ is irrational. To this end we consider a given rational number r satisfying $|r| < \sqrt{2}/2m$. If $p \in V_{mr}$, then by ii) and i) of A) there exists r' for which $r' < r$, $p \in V_{mr'}$. Hence $f_m(p) \leqq r' < r$. If $p \notin \bar{V}_{mr}$, then by ii) of A) there exists r' for which $r' > r$, $p \notin V_{mr'}$. Hence $f_m(p) \geqq r' > r$. Thus $f_m(p) \neq r$ in either case. So from iii) of A) it follows that at most finitely many of $\{f_1(p), f_2(p), \dots\}$ are rational, and hence

$$f(p) = (f_1(p), f_2(p), \dots)$$

is a continuous mapping of R into R^ω.

Now, we put $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ and consider the generalized Baire's 0-dimensional space $N(\Omega)$. Then we define a continuous mapping c of R into $N(\Omega)$ by

$$c(p) = \alpha = (\alpha_1, \alpha_2, \dots) \text{ if } p \in S_{\alpha_i} , \ \alpha_i \in \Omega_i , \ i = 1,2,\dots .$$

Finally, we define a continuous mapping φ of R into $N(\Omega) \times R^\omega$ by

$$\varphi(p) = (c(p), f(p)), \ p \in R .$$

Note that φ is a one-to-one continuous mapping. To see that φ is homeomorphic, we suppose $U(p)$ is a given neighbourhood of a point p of R. Let us take i for which $S(p,N_i)\subset U(p)$. Since $\{F_{ij} \mid j=1,2,\dots\}$ is a covering of R, there exists F_{ij} satisfying $p\in F_{ij}$. We suppose $F_{ij}=F_m$.

Since $S_k=\{S_\alpha \mid \alpha\in\Omega_k\}$ is a covering of R, for each k, $1\leqq k\leqq i$, there exists $\alpha_k\in\Omega_k$ satisfying $p\in\bigcap_{k=1}^{i} S_{\alpha k}$. Then we define a neighbourhood $V(\varphi(p))$ of $\varphi(p)$ by

$$V(\varphi(p)) = N(\alpha_1,\dots,\alpha_i)\times N(f(p)) ,$$

where

$$N(\alpha_1,\dots,\alpha_i) = \{(\alpha_1',\alpha_2',\dots) \mid \alpha_k'=\alpha_k,\ k=1,\dots,i\}\subset N(\Omega) ,$$
$$N(f(p)) = \{(a_1,a_2,\dots) \mid a_m<0\}\subset R^\omega .$$

One can easily see that

$$\varphi^{-1}(V(\varphi(p)))\subset U(p) ,$$

which proves that φ is a homeomorphism. Thus the "only if" part of this theorem is established.

Conversely, it is clear that

$$N(\Omega)\times R^\omega = \bigcup_{n=1}^{\infty} [N(\Omega)\times R^n]$$

for

$$R^n = \{(a_1,a_2,\dots) \mid a_j, j>n \text{ , are irrational; } |a_i|\leqq 1/i \text{ for } i=1,2,\dots\} .$$

Since $\dim R^n=n$, the product theorem implies

$$\dim [N(\Omega)\times R^n] = n .$$

Hence $N(\Omega)\times R^\omega$ is countable-dimensional. This proves the "if" part of the theorem.

The following is an immediate consequence of this theorem.

Corollary. *Let R be a separable metric space. Then R is countable-dimensional if and only if R is homeomorphic to a subset of R^ω.*

VI. 3. Mappings and countable-dimensional spaces

Various good results were obtained on mappings and infinite-dimensional spaces. We have already seen such an example in Theorem VI.4 and shall show another example in the following. In particular we will discuss mappings which preserve countable-dimensionality.

A) Let A be a non-empty countable compact set in a space R. Then A contains an isolated point.

Proof. The easy proof is left to the reader.

B) Let $\mathcal{U}$ be an open covering of R. Then there is a σ-discrete open covering $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$.

Proof. Let $\mathcal{F}$ be a locally finite closed covering such that $\mathcal{F} < \mathcal{U}$. Apply the method used in the proof of the corollary of Theorem II.1 to get a σ-discrete closed covering $\mathcal{G}$ such that $\mathcal{G} < \mathcal{F}$. By swelling the members of $\mathcal{G}$ slightly we obtain a desired open covering $\mathcal{V}$.

C) Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ be a σ-locally finite covering of R such that each member of $\mathcal{U}$ is countable-dimensional. Then R is countable-dimensional.

Proof. First observe that the proposition is true if $\mathcal{U}$ is σ-discrete. Because in that case $\mathcal{U}_n$ is discrete, and thus each member of $\mathcal{U}_n$ is closed in the subspace $U_n = \bigcup \{ U \mid U \in \mathcal{U}_n \}$, and hence U_n is countable-dimensional by the sum theorem.

Now, by use of B), we can find a σ-discrete open covering $\mathcal{V}_n = \bigcup_{m=1}^{\infty} \mathcal{V}_{nm}$ such that each member of $\mathcal{V}_n$ intersects at most finitely many members of $\mathcal{U}_n$. Then $\mathcal{V}_{nm} \cap \mathcal{U}_n$ is obviously σ-discrete. Hence $\bigcup_{n,m=1}^{\infty} \mathcal{V}_{nm} \wedge \mathcal{U}_n$ is a σ-discrete covering which refines $\mathcal{U}$. Thus by the previous observation we conclude that R is countable-dimensional.

D) Let f be a perfect mapping from R onto S. (Namely f is closed continuous, and $f^{-1}(y)$ is compact for each $y \in S$.) If $\mathcal{U}$ is a locally finite collection in R, then $f(\mathcal{U}) = \{ f(U) \mid U \in \mathcal{U} \}$ is locally finite in S.

Proof. Let y be a given point of S. Then each $x \in f^{-1}(y)$ has an open neighbourhood $W(x)$ which intersects at most finitely many members of $\mathcal{U}$. Since $f^{-1}(y)$ is compact, it is covered by finitely many of $W(x)$'s, say $W(x_i)$, $i = 1, \ldots, k$. Then $V = S - f(R - \bigcup_{i=1}^{k} W(x_i))$ is an open neighbourhood of y because f is closed. It is obvious that V intersects at most finitely many members of $f(\mathcal{U})$. Thus the proposition is proved.

E) Let f be a perfect mapping from R onto S such that $f^{-1}(y)$ has an isolated point for each $y \in S$. If R is countable-dimensional, then so is S.

Proof. Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ be a σ-discrete base, where each $\mathcal{U}_n = \{ U_\alpha \mid \alpha \in A_n \}$ is a discrete open collection. Pick an isolated point $x(y)$ from each $f^{-1}(y)$. For each $\alpha \in A_n$ we put

$$F_{n\alpha} = \{ x(y) \mid U_\alpha \cap f^{-1}(y) = \{x(y)\}\}, \text{ and } G_{n\alpha} = f(F_{n\alpha}) .$$

It is almost obvious that f maps $F_{n\alpha}$ topologically onto $G_{n\alpha}$. Since R is countable-dimensional, so is $F_{n\alpha}$ and $G_{n\alpha}$. Note that $F_{n\alpha} \subset \bar{U}_\alpha$, and $\{ \bar{U}_\alpha | \alpha \in A_n \}$ is discrete. Therefore $\{ F_{n\alpha} \mid \alpha \in A_n \}$ is discrete. Hence by D) $\{ G_{n\alpha} \mid \alpha \in A_n, n = 1,2,\ldots \}$ is σ-locally finite. Hence S is countable-dimensional by C).

The following theorem is an infinite-dimensional counter-part of Theorem III.7.

Theorem VI. 6. *Let f be a closed continuous mapping from a countable-dimensional space R onto S such that $B(f^{-1}(y))$ is countable for each $y \in S$. Then S is countable-dimensional* [10].

Proof. Put

$$S_1 = \{ y \in S \mid B(f^{-1}(y)) = \emptyset \} \text{ and } S_2 = \{ y \in S \mid B(f^{-1}(y)) \neq \emptyset \} .$$

To each $y \in S_1$ we assign a point $x(y)$ of $f^{-1}(y)$. Put $R_1 = \{ x(y) \mid y \in S_1 \}$; then it is obvious that f maps R_1 topologically onto S_1 because $f^{-1}(y)$ is open for each $y \in S_1$. Thus S_1 is countable-dimensional.

Theorem I.9 implies that $B(f^{-1}(y))$ is compact for each $y \in S$. Thus it follows from A) that $B(f^{-1}(y))$ has an isolated point for each $y \in S_2$. Put

$$R_2 = \bigcup \{ B(f^{-1}(y)) \mid y \in S_2 \} .$$

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[10] Several people contributed to this theorem, especially E. G. Sklyarenko [2], A. V. Arhangelskii [2], K. Nagami [7] and A. Okuyama [2]; the last proved this theorem for spaces more general than metric spaces. E. G. Sklyarenko [3] and I. M. Leǐbo [1] proved similar types of theorems on weakly infinite-dimensional spaces.

The following theorem (a counter-part of Theorem III.6) is also due to Arhangelskii: Let f be a closed continuous mapping from R onto a countable-dimensional space S. If $\dim f^{-1}(q) < \infty$ for each $q \in S$, then R is countable-dimensional.

Then R_2 is countable-dimensional, and $f \mid R_2$ satisfies the condition of E). Thus $S_2 = f(R_2)$ is countable-dimensional. Since $S = S_1 \cup S_2$, S is also countable-dimensional.

Corollary. *Let f be a closed continuous mapping from a countable-dimensional space R onto S such that*

$$S' = \{ y \in S \mid |B(f^{-1}(y))| > \aleph_0 \}$$

is countable-dimensional. Then S is also countable-dimensional.

Proof. Let $R_0 = f^{-1}(S - S')$. Then $f \mid R_0$ satisfies the condition of the theorem. Hence $S - S'$ is countable-dimensional, and so is S.

Another aspect to the study of mappings (in relation to dimension) is the characterization of dimension in terms of mappings as shown in III.3. E. G. Sklyarenko characterized in [3] strongly infinite-dimensional spaces as follows:

A compact (T_2-) space X is strongly infinite-dimensional if and only if there is a continuous mapping f from X into the Hilbert cube I^{ω} satisfying the condition that for every finite dimensional face F of I^{ω} the restriction of f to $f^{-1}(F)$ is essential [11].

D. Henderson [3] defined a compact metric space J^{α} for every countable ordinal number α such that $J^{\alpha} = I^{\alpha}$ if $0 < \alpha < \infty$, and $\operatorname{Ind} J^{\alpha} = \alpha$, and also essential mappings onto J^{α}. Thus he proved that if there is an essential mapping from a normal space X onto J^{α}, then $\operatorname{Ind} X \geqq \alpha$ or $\operatorname{Ind} X$ does not exist.

VI. 4. Transfinite inductive dimension

In this section we shall study results obtained by Yu. Smirnov [7] and E. Sklyarenko [1] on the transfinite inductive dimension and related concepts [12].

A) If a space R has a strong transfinite inductive dimension, then it is weakly infinite-dimensional.

Proof. If $\operatorname{Ind} R = -1$, then this assertion is obviously true.

[11] See also B. Levšenko [1].

[12] "Weakly infinite-dimensional" is understood in the sense of Smirnov unless otherwise stated.

Given a transfinite ordinal number α, we assume the assertion for every R with $\operatorname{Ind} R < \alpha$. Then we suppose $\{F_i, G_i\}$, $i = 1,2,\dots$ are pairs of disjoint closed sets of a space R with $\operatorname{Ind} R = \alpha$. There exists an open set U_1 such that

$$F_1 \subset U_1 \subset R - G_1 \quad \text{and} \quad \operatorname{Ind} B(U_1) < \alpha .$$

Thus by the induction hypothesis $B(U_1)$ is weakly infinite-dimensional, and hence there exist open sets $V_2, V_3, \dots$ of $B(U_1)$ which satisfy

$$F_i \cap B(U_1) \subset V_i \subset B(U_1) - G_i \cap B(U_1) , \quad i = 2,3,\dots,$$

$$\bigcap_{i=2}^{k} B_{B(U_1)}(V_i) = \emptyset \quad \text{for some } k .$$

We can easily extend V_i to an open set U_i of R such that

$$F_i \subset U_i \subset R - G_i \quad \text{and} \quad B(U_i) \cap B(U_1) \subset B_{B(U_1)}(V_i) .$$

Hence $\bigcap_{i=1}^{k} B(U_i) = \emptyset$, which proves that R is weakly infinite-dimensional.

B) If a space R has a strong transfinite inductive dimension, then it is countable-dimensional.

Proof. If $\operatorname{Ind} R = -1$, then the assertion is clearly true. Assume B) for every space R of $\operatorname{Ind} < \alpha$. Given a space R of $\operatorname{Ind} = \alpha$, we can construct a σ-locally finite open basis $\mathcal{V}$ such that $\operatorname{Ind} B(V) < \alpha$ for every $V \in \mathcal{V}$. By the induction hypothesis each $B(V)$ is countable-dimensional. Therefore by use of the sum theorem we can easily verify that $\bigcup\{B(V) \mid V \in \mathcal{V}\}$ is countable-dimensional. Since it is obvious that

$$\dim (R - \bigcup\{B(V) \mid V \in \mathcal{V}\}) \leq 0 ,$$

R itself is countable-dimensional.

The converse of B) is generally not true, but in the compact case equivalence holds between the two conditions as follows.

C) If R is compact, then it has a strong transfinite inductive dimension if and only if it is countable-dimensional.

Proof. Suppose that R is countable-dimensional. Then $R = \bigcup_{i=1}^{\infty} A_i$ for 0-dimensional subsets A_i. Assume that R has no $\operatorname{Ind} R$. Then there are disjoint closed sets F_1 and G_1 such that every open set U_1 satisfying $F_1 \subset U_1 \subset R - G_1$ has a boundary with no Ind. We can select an open set U_1 such that $B(U_1) \cap A_1 = \emptyset$. Put $B_1 = B(U_1)$ and note that $B_1 \neq \emptyset$. Since B_1 has no Ind, we can repeat the above argument, this time for the subspace B_1 to obtain an open set U_2 of B_1 such that

$$B_{B_1}(U_2) \cap A_2 = \emptyset ,$$

and $B_{B_1}(U_2)$ has no Ind. Put $B_2 = B_{B_1}(U_2)$ and note that $B_2 \neq \emptyset$. Repeating this process we get a decreasing sequence $B_1 \supset B_2 \supset \dots$ of non-empty closed sets such that $B_i \cap A_i = \emptyset$. Since R is compact, $\cap_{i=1}^{\infty} B_i \neq \emptyset$ in contradiction to $\cup_{i=1}^{\infty} A_i = R$. Thus R has Ind R.[13]

Theorem VI. 7. *A space R is weakly infinite-dimensional if and only if it can be decomposed as*

$$R = K \cup (\bigcup_{n=1}^{\infty} P_n)$$

for a weakly infinite-dimensional compact set K and finite-dimensional open sets P_n such that if $\{ x_i \mid i = 1, 2, \dots \}$, $x_i \in \cup_{n=1}^{\infty} P_n$ is a sequence of points having no accumulation point, then it is residual in some P_n, i.e. for some k and for all $i \geqq k$ $x_i \in P_n$.

Proof. First we suppose R is weakly infinite-dimensional. Define

$$P_n = \cup \{ U \mid U \text{ is an open set of } R \text{ with } \dim U \leqq n \}, \text{ and}$$

$$K = R - \bigcup_{n=1}^{\infty} P_n .$$

Then by the sum theorem P_n is an open set of dimension $\leqq n$. (Note that every subset of R is paracompact.)

To prove that K is compact we assume the contrary. Then there exists a sequence $\{ y_i \mid i = 1, 2, \dots \}$ of points of K which has no accumulation point. Note that every open neighbourhood of each y_i has no finite dimension.

[13] As easily seen from this argument, a complete metric space R has a weak transfinite inductive dimension if and only if it is countable-dimensional. (Apply Baire's theorem.)

To prove simultaneously the property of P_n, $n = 1, 2, \ldots$ we assume that there exists a sequence $\{x_i \mid i = 1, 2, \ldots\}$ of points of $\bigcup_{n=1}^{\infty} P_n$ which has no accumulation point and is residual in no P_n. Then we can easily select a subsequence $\{x'_i\}$ of $\{x_i\}$ such that

$$x'_n \notin \bigcup_{i=1}^{n} P_i, \quad n = 1, 2, \ldots .$$

Note that by the definition of P_n every open neighbourhood of x'_n has dimension $> n$. Thus it follows from either of the preceding assumptions that there exists a sequence $\{z_i \mid i = 1, 2, \ldots\}$ of points, each z_i having an open neighbourhood $V(z_i)$ such that

(1) $\dim \overline{V(z_i)} > i$,

(2) $\{V(z_i) \mid i = 1, 2, \ldots\}$ is discrete.

Combining (1) with the corollary to Theorem II.8 for each $\overline{V(z_i)}$ we get closed sets F_i^j and G_i^j, $j = 1, \ldots, i+1$ such that

(3) $$\begin{cases} F_i^j \cup G_i^j \subset \overline{V(z_i)}, \\ F_i^j \cap G_i^j = \emptyset, \end{cases}$$

and such that if W^j, $j = 1, \ldots, i+1$ are open sets of R satisfying $F_i^j \subset W^j \subset R - G_i^j$, then

(4) $$\bigcap_{j=1}^{i+1} B(W^j) \neq \emptyset .$$

Further we define that

$$F_i^j = G_i^j = \emptyset \quad \text{for} \quad j > i+1 .$$

Letting

$$F^j = \bigcup_{i=1}^{\infty} F_i^j \quad \text{and} \quad G^j = \bigcup_{i=1}^{\infty} G_i^j ,$$

we get, by virtue of (2) and (3), disjoint closed sets F^j and G^j. Suppose W^j, $j=1,2,\dots$ are given open sets such that $F^j \subset W^j \subset R - G^j$; then for any fixed number i

$$F_i^j \subset W^j \subset R - G_i^j , \quad j=1,\dots,i+1 .$$

Therefore from (4) it follows that $\bigcap_{j=1}^{i+1} B(W^j) \neq \emptyset$ which contradicts the fact that R is weakly infinite-dimensional. Thus we have proved the compactness of K as well as the desired property of $\{P_n\}$. Since K is a closed subset of the weakly infinite-dimensional space R, it is easily seen to be also weakly infinite-dimensional.

Conversely, suppose R has a decomposition

$$R = K \cup \left(\bigcup_{n=1}^{\infty} P_n \right) ,$$

as described. Let $\{F_i, G_i\}$, $i=1,2,\dots$ be given pairs of disjoint closed sets of R. Since K is weakly infinite-dimensional, there exist open sets U_i, $i=1,2,\dots$ of K such that

$$F_i \subset U_i \subset \bar{U}_i \subset K - G_i , \quad \text{and} \quad \bigcap_{i=1}^{k} B_K(U_i) = \emptyset \quad \text{for some } k .$$

To each point p of $B_K(U_i)$ we assign $\varepsilon(p) > 0$ such that

$$S_{\varepsilon(p)}(p) \cap G_i = \emptyset \quad \text{and} \quad S_{\varepsilon(p)}(p) \cap B_K(U_j) = \emptyset$$

for every j with $1 \leqq j \leqq k$, $p \notin B_K(U_j)$. Then we put

$$V_i = K \cap \left(\cup \{ S_{\frac{1}{2}\varepsilon(p)}(p) \mid p \in B_K(U_i) \} \right) \quad \text{and} \quad W_i = V_i \cup U_i .$$

We can easily see that W_i is an open set of K which satisfies $\bar{U}_i \subset W_i \subset K - G_i$ and

$$(5) \qquad \bigcap_{i=1}^{k} (W_i - \bar{U}_i) = \emptyset .$$

Now, since $\dim P_n < +\infty$, using III.4 A) and the decomposition theorem we can construct open sets N_i, $i=1,2,\dots$ such that

(6) $\bar{U}_i \subset N_i \subset \bar{N}_i \subset R - (K - W_i)$, $i = 1,\dots,k$,

$F_i \subset N_i \subset R - G_i$, $i = k+1,\ k+2,\dots,$

(7) $\operatorname{ord}_p \{ B(N_i) \mid i = 1,2,\dots \} \leqq m_n < +\infty$

at each point p of P_n . Assume that $\cap_{i=1}^{\ell} B(N_i) \neq \emptyset$ for every integer ℓ . Then we take $x_\ell \in \cap_{i=1}^{\ell} B(N_i)$. From (7) it follows that $x_\ell \notin P_n$ for every ℓ with $\ell > m_n$, and hence $\{x_\ell\}$ is residual in no P_n and has no accumulation point in $\cup_{n=1}^{\infty} P_n$. (Because every accumulation point of $\{x_\ell\}$ belongs to $\cap_{i=1}^{\infty} B(N_i)$, and hence by (7) it cannot be in any P_n .) Therefore $\{x_\ell\}$ must have an accumulation point x in K . Since $x_k, x_{k+1}, \dots$ are contained in the closed set $\cap_{i=1}^{k} B(N_i)$, we obtain

(8) $$x \in K \cap \left(\bigcap_{i=1}^{k} B(N_i) \right) .$$

On the other hand, (6) implies

$$K \cap B(N_i) \subset W_i - \bar{U}_i ,$$

and hence (8) implies

$$\bigcap_{i=1}^{k} (W_i - \bar{U}_i) \neq \emptyset$$

which contradicts (5). Thus we can conclude that $\cap_{i=1}^{\ell} B(N_i) = \emptyset$ for some ℓ .
This proves that R is weakly infinite-dimensional.

D) We define a mapping $\beta(\alpha)$ of the ordinal numbers $\geqq -1$ in itself as follows:

$\beta(-1) = \omega_0$, where ω_0 denotes the first countable ordinal number,

$\beta(\alpha) = \sup \{ \beta(\alpha') \mid \alpha' < \alpha \} + 1$.

Then $\beta(\alpha)$ satisfies

i) if $\alpha' < \alpha$, then $\beta(\alpha') < \beta(\alpha)$,

ii) if $\alpha < \omega_1$, then $\beta(\alpha) < \omega_1$, where ω_1 denotes the first uncountable ordinal number

iii) $\alpha \leqq \beta(\alpha)$ for every α .

E) Let a space R have a decomposition as in Theorem VI.6. If K has a strong transfinite inductive dimension $\operatorname{Ind} K$, then $\operatorname{Ind} R \leqq \beta(\operatorname{Ind} K)$, where $\beta(\alpha)$ is the mapping defined in D).

Proof. We shall prove this assertion by induction on the dimension number of K. If $\operatorname{Ind} K = -1$, then $R = \bigcup_{n=1}^{\infty} P_n$. From the property of $\{P_n\}$ it easily follows that we can select a finite subcovering of $\{P_n \mid n = 1, 2, \dots\}$. Since $\operatorname{Ind} P_n < +\infty$, by the sum theorem we conclude that $\operatorname{Ind} R < +\infty$, i.e. $\operatorname{Ind} R < \omega_0 = \beta(-1)$.

Assume that the assertion is true if $\operatorname{Ind} K < \alpha$. Let F and G be disjoint closed sets of a space R for which $\operatorname{Ind} K = \alpha$. Then there exists an open set U of R such that

$$F \subset U \subset R - G \quad \text{and} \quad \operatorname{Ind} (K \cap B(U)) = \alpha' < \alpha .$$

Then

$$B(U) = (K \cap B(U)) \cup \left(\bigcup_{n=1}^{\infty} (P_n \cap B(U)) \right)$$

is a decomposition of $B(U)$ satisfying the condition in Theorem VI.7. Hence by use of the induction hypothesis we obtain $\operatorname{Ind} B(U) \leqq \beta(\alpha')$. Thus from i) in D) it follows that $\operatorname{Ind} B(U) < \beta(\alpha)$, which proves that $\operatorname{Ind} R \leqq \beta(\alpha) = \beta(\operatorname{Ind} K)$.

F) If a compact space K has a strong transfinite inductive dimension, then $\operatorname{Ind} K < \omega_1$.

Proof. Assume the contrary and let α be the smallest ordinal, $\alpha \geqq \omega_1$, for which there exists a compact space K of dimension α. Let $\{U_n \mid n = 1, 2, \dots\}$ be an open basis of K. Suppose both V and W are finite sums of sets of $\{U_n\}$ such that $\bar{V} \subset W$. Then to each of such pair $\{V, W\}$ we assign an open set P which satisfies

$$\bar{V} \subset P \subset W \quad \text{and} \quad \operatorname{Ind} B(P) < \alpha .$$

Note that this implies

(1) $\qquad \operatorname{Ind} B(P) < \omega_1 .$

We denote by $\mathcal{P}$ the countable collection of these open sets P. Since K is compact, we can easily verify that for any disjoint closed sets F and G of R there exists $P \in \mathcal{P}$ satisfying

(2) $\qquad F \subset P \subset R - G .$

On the other hand, (1) implies that

$$\sup \{ \operatorname{Ind} B(P) \mid P \in P \} + 1 = \beta < \omega_1 \leqq \alpha .$$

Therefore from (2) we conclude that $\operatorname{Ind} K \leqq \beta$. Thus we achieve a contradiction, proving $\operatorname{Ind} K < \omega_1$.

Theorem VI. 8. *If a space* R *has a strong transfinite inductive dimension, then* $\operatorname{Ind} R < \omega_1$.

Proof. By A), R is weakly infinite-dimensional, and hence by Theorem VI.7 there exists a decomposition

$$R = K \cup (\bigcup_{n=1}^{\infty} P_n) .$$

Since R has $\operatorname{Ind} R$, K also has $\operatorname{Ind} K$. Since K is compact, by F) $\operatorname{Ind} K < \omega_1$. Therefore from E) and ii) in D) we obtain $\operatorname{Ind} R \leqq \beta(\operatorname{Ind} K) < \omega_1$.

Example IV. 1. (Yu. Smirnov [6]) Define a compact metric space Q^α for each $\alpha < \omega_1$ as follows: If α is finite, then Q^α = the α-dimensional (closed) cube. If $\alpha = \beta + 1$, then $Q^\alpha = Q^\beta \times [0,1]$. If α is a limit ordinal number, then Q^α = the Alexandroff one point compactification of the discrete sum $\bigcup_{\beta < \alpha} Q^\beta$. Then it can be proved that $\operatorname{Ind} Q^\alpha = \alpha$.

VI. 5. Sum theorem for transfinite inductive dimension

Generally speaking, properties of transfinite inductive dimension are considerably different from those of finite inductive dimension, and there are not so many beautiful theorems on the former as on the latter. The main purpose of this section is to discuss (finite) sum theorems for transfinite inductive dimension by use of G. H. Toulmin's [1] shuffling sum of ordinal numbers following the work by M. Landau [1] and A. R. Pears [1].

Definition VI. 5.[14] *Let* α *and* β *be ordinal numbers. Then we define the shuffling sums* $\alpha \dotplus \beta$ *and* $\alpha \dotplus \beta$ *as the infimum and supremum of the ordinal numbers of well-ordered sets obtained by shuffling the well-ordered sets* $T_\alpha = \{ x \mid 0 \leqq x < \alpha \}$ *and* $T_\beta = \{ x \mid 0 \leqq x < \beta \}$ *(without changing the order within* T_α *and* T_β *by the shuffling) of ordinal numbers. We make the convention:*

[14] G. H. Toulmin defined the shuffling sum to prove sum theorems for weak transfinite inductive dimension which were extended by Landau and Pears to strong transfinite inductive dimension.

$$-1 \dotplus \alpha = \begin{cases} \alpha - 1 & \textit{if} \quad 0 \leqq \alpha < \omega_0 , \\ \alpha & \textit{if} \quad \alpha \geqq \omega_0 . \end{cases}$$

A) Let α and β be ordinal numbers such that $\alpha = \alpha' + p$, $\beta = \beta' + q$, where α', β' are limit ordinal numbers, and p,q are non-negative integers. Then

$$\alpha \dotplus \beta = \begin{cases} \alpha & \text{if } \alpha' > \beta' , \\ \alpha + q = \beta + p & \text{if } \alpha' = \beta' , \\ \beta & \text{if } \alpha' < \beta' . \end{cases}$$

Proof. Assume $\alpha' > \beta'$; then $\beta < \alpha'$ and T_β is isomorphic with the subset $T'_\beta = \{ x \in T_\alpha \mid x < \beta \}$ of T_α . Shuffle T_β with T_α in such a way that the members of T'_β and those of T_β are placed alternatively. Then the shuffled set thus obtained has the ordinal number α , which obviously yields the minimum in the above definition. Hence $\alpha \dotplus \beta = \alpha$.

Next, assume $\alpha' = \beta'$; then reasoning analogously as above we get a shuffled set whose ordinal number is $\alpha + q$. It is also easy to see that every shuffled set has an ordinal number not less than $\alpha + q$. Thus $\alpha \dotplus \beta = \alpha + q = \beta + p$.

B) Put, for brevity, $\theta(\alpha,\beta) = \alpha \dotplus (\beta + 1)$ for ordinal numbers α,β with $\alpha \geqq \beta$. Then

i) $\theta(\alpha,\beta) < \theta(\alpha,\alpha)$ if $\alpha > \beta$,
ii) $\theta(\gamma,\beta) < \theta(\alpha,\beta)$ if $\alpha > \gamma \geqq \beta$.

Proof. To prove i), assume $\alpha = \alpha' + p$, where α' is a limit ordinal number and p is a non-negative integer. If $\beta < \alpha'$, then by A)

$$\theta(\alpha,\beta) = \alpha < \alpha + p + 1 = \theta(\alpha,\alpha) .$$

If $\beta \geqq \alpha'$, then assume $\beta = \alpha' + q$, where q is a non-negative integer such that $q < p$. Then again by A)

$$\theta(\alpha,\beta) = \alpha + q + 1 < \alpha + p + 1 = \theta(\alpha,\alpha) .$$

To prove ii), we still assume $\alpha = \alpha' + p$. If $\gamma < \alpha'$, then by A) $\theta(\gamma,\beta) < \alpha' \leqq \alpha = \theta(\alpha,\beta)$.

If $\beta < \alpha' \leqq \gamma$, then by A) $\theta(\gamma,\beta) = \gamma < \alpha = \theta(\alpha,\beta)$.

If $\beta = \alpha'$, then put $\beta = \alpha' + q$, $\gamma = \alpha' + r$ for non-negative integers q,r , where $q \leqq r < p$. Now it follows from A) that

$$\theta(\gamma,\beta) = \alpha' + r + q + 1 < \alpha' + p + q + 1 = \theta(\alpha,\beta) .$$

C) If R has $\operatorname{Ind} R$, and F is a closed subset of R, then $\operatorname{Ind} F \leq \operatorname{Ind} R$.

Proof. The easy proof (by use of induction on $\alpha = \operatorname{Ind} R$) is left to the reader.

D) Let $R = F \cup G$, where F and G are disjoint closed sets in R. If $\operatorname{Ind} F$ and $\operatorname{Ind} G$ exist, then $\operatorname{Ind} R = \max(\operatorname{Ind} F, \operatorname{Ind} G)$.

Proof. Defining $\alpha = \max(\operatorname{Ind} F, \operatorname{Ind} G)$ we prove the proposition by induction on α.

D) is obviously true if $\alpha = -1$. So assume that it is true for all $\beta < \alpha$. Then suppose that $\operatorname{Ind} G \leq \operatorname{Ind} F = \alpha$. Let H and K be disjoint closed sets in R. Then there are open sets U and V such that

$$H \cap F \subset U \subset F - K , \quad \operatorname{Ind} B_A(U) < \alpha ,$$
$$H \cap G \subset V \subset G - K , \quad \operatorname{Ind} B_B(V) < \alpha .$$

Now $U \cap V = W$ is an open set satisfying $H \subset W \subset R - K$ and $B(W) = B_A(U) \cup B_B(V)$. By use of the induction hypothesis we get $\operatorname{Ind} B(W) < \alpha$. Thus $\operatorname{Ind} R \leq \alpha$, and hence $\operatorname{Ind} R = \alpha$ follows from C).

E) Let A be a subset of R such that $\operatorname{Ind} A \leq \alpha$. Suppose that F and G are disjoint closed sets in R. Then there is an open set U such that $F \subset U \subset R - G$ and $\operatorname{Ind} B(U) \cap A < \alpha$.

Proof. Select open sets P and Q in R such that $F \subset P$, $G \subset Q$, $\bar{P} \cap \bar{Q} = \emptyset$. Since $\operatorname{Ind} A \leq \alpha$, there is an open set V of the subspace A such that

$$\bar{P} \cap A \subset V \subset A - \bar{Q} \quad \text{and} \quad \operatorname{Ind} B_A(V) < \alpha .$$

Define $C = V \cup F$ and $D = (A - V) \cup G$. Then $\bar{C} \cap D = \emptyset$, $C \cap \bar{D} = \emptyset$, and hence there is an open set U of R such that $C \subset U$ and $\bar{U} \cap D = \emptyset$. Accordingly U satisfies $F \subset U$ and $U \cap G = \emptyset$. Now, since $B(U) \cap A \subset B_A(V)$, we obtain from C) $\operatorname{Ind} B(U) \cap A < \alpha$.

Theorem VI. 9.[15] *Assume* $R = F \cup G$ *for closed subsets* F *and* G *of* R . *If* Ind F *and* Ind G *exist, then*

$$\text{Ind } R \leqq \max(\text{Ind } F\,,\ \text{Ind } G) \dotplus (\text{Ind } F \cap G + 1)\ .$$

Proof. Define $\alpha = \max(\text{Ind } F\,,\ \text{Ind } G)$, $\beta = \text{Ind } F \cap G$. Then by use of the symbol of B) it suffices to prove

$(1)_{\alpha,\beta}$ $\quad \text{Ind } R \leqq \theta(\alpha,\beta)$.

The proof will be carried out by induction on (α,β) as follows.

$(1)_{\alpha,-1}$ is true for every α by virtue of D).

Assume that $(1)_{\alpha,\beta}$ is true whenever $\beta < \beta_0$, or $\beta = \beta_0$ and $\beta \leqq \alpha < \alpha_0$. Then we shall prove that $(1)_{\alpha_0,\beta_0}$ is true.

Remember that

$$\alpha_0 = \max(\text{Ind } F\,,\ \text{Ind } G)\ ,\ \beta_0 = \text{Ind } F \cap G\ ,$$

where F and G are closed subsets of R such that $R = F \cup G$. Let H and K be given disjoint closed sets in R .

Case 1: $\alpha_0 = \beta_0$.

By use of E) we can select an open set U such that

$$H \subset U \subset R - K \quad \text{and} \quad \text{Ind } B(U) \cap F \cap G < \beta_0 = \alpha_0\ .$$

Put $\gamma = \text{Ind } B(U) \cap F \cap G$; then $\gamma < \alpha_0$. Note that

$$\max(\text{Ind } B(U) \cap F\,,\ \text{Ind } B(U) \cap G) \leqq \alpha_0\ ,$$
$$B(U) = (B(U) \cap F) \cup (B(U) \cap G)\ .$$

Using the induction hypothesis we obtain $\text{Ind } B(U) \leqq \theta(\alpha_0,\gamma)$. It follows from B) i) that $\text{Ind } B(U) < \theta(\alpha_0,\alpha_0)$. Thus

$$\text{Ind } R \leqq \theta(\alpha_0,\alpha_0) = \theta(\alpha_0,\beta_0)\ .$$

[15] Actually this theorem and the following corollary hold for hereditarily normal spaces and totally normal spaces (to be defined later) respectively.

Case 2: $\alpha_0 > \beta_0$.
By E) there is an open set U such that $H \subset U \subset R - K$ and

(2) $\quad \operatorname{Ind} B(U) \cap F < \alpha_0$.

Again by use of E) we can find an open set V such that

(3) $\quad H \subset V \subset \bar{V} \subset U$,

(4) $\quad \operatorname{Ind} B(V) \cap G < \alpha_0$.

Put $W = (U - G) \cup V$. Then W is an open set satisfying $H \subset W \subset R - K$. We claim that

(5) $\quad B(W) \subset (B(U) \cap F) \cup (B(V) \cap G) \cup (F \cap G)$.

To prove it assume

$$x \notin (B(U) \cap F) \cup (B(V) \cap G) \cup (F \cap G) .$$

Then either $x \in F - G$ or $x \in G - F$.
Assume first $x \in F - G$. Since $x \notin B(U)$ follows from $x \notin B(U) \cap F$, either $x \in U$ or $x \in R - \bar{U}$ holds. If the former is the case, then $x \in U - G \subset W$. If the latter is the case, then $R - \bar{U}$ is a neighbourhood of x which is disjoint from W . Thus in either cases we obtain $x \notin B(W)$.
Next, assume that $x \in G - F$. Since $x \notin B(V)$ follows from $x \notin B(V) \cap G$, either $x \in V$ or $x \in R - \bar{V}$ holds. If the former is the case, then $x \in W$. Otherwise $(R - \bar{V}) - F$ is a neighbourhood of x which is disjoint from W . Thus we obtain $x \notin B(W)$ in either cases. Therefore our claim (5) is proved.
Put, for brevity, $B_1 = B(U) \cap F$ and $B_2 = B(V) \cap G$. Then observe that $B_1 \cap B_2 = \emptyset$ follows from (3). Hence by use of (2), (4) and D) we obtain $\operatorname{Ind} B_1 \cup B_2 < \alpha_0$. Thus

$$\max(\operatorname{Ind} B_1 \cup B_2 \ , \ \operatorname{Ind} F \cap G) = \delta < \alpha_0 \ ,$$
$$\operatorname{Ind}(B_1 \cup B_2) \cap (F \cap G) = \varepsilon \leqq \beta_0 \ .$$

Thus by the induction hypothesis we obtain

$$\operatorname{Ind}(B_1 \cup B_2) \cup (F \cap G) \leqq \theta(\delta, \varepsilon) \ .$$

It follows from C) and (5) that $\operatorname{Ind} B(W) \leqq \theta(\delta, \varepsilon)$. On the other hand B) ii) implies

$$\theta(\delta, \varepsilon) < \theta(\alpha_0 \ , \varepsilon) \leqq \theta(\alpha_0 \ , \beta_0) \ .$$

Therefore Ind $B(W) < \theta(\alpha_0, \beta_0)$, which proves

$(1)_{\alpha_0,\beta_0}$ Ind $R \leqq \theta(\alpha_0, \beta_0)$,

and the induction is concluded.

Corollary. *Assume* $R = F \cup G$ *for closed sets* F *and* G *. If* Ind $F \cap G$ *is finite, and* Ind F *and* Ind G *exist, then* Ind $R = \max$ (Ind F , Ind G) .

Proof. If both Ind F and Ind G are finite, then the corollary is simply a special case of the sum theorem for finite dimension. If max (Ind F , Ind G) is infinite, then the corollary follows directly from the preceding theorem.[16]

[16] B. T. Levšenko [2] showed that the equality Ind $R = \max$ (Ind F , Ind G) does not generally hold even if R is compact: Assume $R = A \cup B$, Ind $A = \alpha = \alpha' + p$, Ind $B = \beta = \beta' + q$, where A and B are not necessarily closed in R , and α' , β' , p , q are as in A). Then B. T. Levšenko [3] proved that

Ind $R \leqq \max(\alpha, \beta)$ if $\alpha' \neq \beta'$,

Ind $R \leqq \alpha' + p + q + 1$ if $\alpha' = \beta'$.

M. Landau [1] proved

$1 +$ Ind $R \leqq (1 + \alpha) \dotplus (1 + \beta)$.

A. Pears [1] proved

Ind $R \leqq \max(\alpha, \beta) \dotplus (\min(\alpha, \beta) + 1)$,

which is a reformulation of Levšenko's result. (The above three theorems were actually proved for hereditarily normal spaces.) L. A. Luxemburg [1] gave an example of a compact metric space X such that Ind $X = \omega_0 + 2$, ind $X = \omega_0 + 1$, which sharply contrasts the coincidence of Ind and ind in the finite-dimensional case. The discrete sum of the n-dimensional cubes I^n , $n = 1, 2, \ldots$ has ind but not Ind. (However, it is known that if a hereditarily normal compact space has ind, then it has Ind.)

On the other hand D. Henderson [2] defined a new transfinite dimension to prove a subspace theorem, a finite sum theorem, a product theorem etc., which have the same appearance as their counter-parts on finite dimension.

Also note that the locally countable sum of countable-dimensional closed sets is obviously countable-dimensional, and it is easy to prove that the countable sum of weakly infinite-dimensional (in the sense of Alexandroff) closed sets is weakly infinite-dimensional in the same sense. The locally countable sum theorem obviously holds for strong countable-dimensionality, too. See B. T. Levšenko [2] for further results on sum theorems of weakly infinite-dimensional spaces.

VI. 6. General imbedding theorem

The purpose of this section is to extend the imbedding theorems to general metric spaces [17]. Up to the present, one has not succeeded in finding an imbedding theorem for finite-dimensional general metric spaces which is very analogous to the one for the separable case. We can, however, imbed them into a kind of generalized Hilbert cube, i.e. a countable product of star spaces [18]. Although the contents of this section are only partially related with infinite-dimensional spaces, the method used here is analogous to that of Section 2 of this chapter, and this is the reason why this section is located here.

Definition VI. 6. *Let* $\{E_\alpha \mid \alpha \in A\}$ *be a collection of unit segments* $[0,1]$ *. By identifying all zeros in* $\cup\{E_\alpha \mid \alpha \in A\}$ *we get a star-shaped set* $S(A)$ *. We introduce a metric in* $S(A)$ *as follows,*

$$\rho(x,y) = \begin{cases} |x-y| & \text{if } x,y \text{ belong to the same segment } E_\alpha , \\ |x+y| & \text{if } x,y \text{ belong to distinct segments.} \end{cases}$$

Then we obtain a one-dimensional metric space $S(A)$ *called the star space with the index set* A *.*

Now we can assert the following.

Theorem VI. 10. [19] *A metric space* R *with weight* $|A|$ *has* $\dim \leqq n$ *if and only if it can be imbedded in the subset* $K_n(A)$ *of the countable product* $P(A) = \Pi_{m=1}^{\infty} S_m(A)$ *of star spaces* $S_m(A)$, $m = 1,2,\ldots,$ *where we denote by* $K_n(A)$ *the set of points in* $P(A)$ *at most* n *of whose non-vanishing coordinates are rational.*

Proof. To show $\dim K_n(A) \leqq n$, we decompose $K_n(A)$ in the following way

$$K_n(A) = \bigcup_{m=0}^{n} K'_m ,$$

where K'_m is the set of points in $P(A)$ exactly m of whose non-vanishing coordinates are rational. We consider a given class $\{a_j \mid j = 1,\ldots,m\}$ of m rational

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[17] The content of this section is due to J. Nagata [6].

[18] We owe this terminology to H. J. Kowalsky [1].

[19] $|A|$ denotes the cardinal number of the set A . We mean by the *weight* of a topological space the cardinal number of an open basis which has the least cardinal number among the open bases of R .

numbers a_j such that $0 < a_j \leq 1$. Then the set of points in K'_m whose j-th coordinates are equal to a_j for every j is a 0-dimensional closed set of K'_m. This assertion is easily proved by the product theorem, because the set of irrational points and zero in a star space is a 0-dimensional space, and so is the set of a_j from the distinct branches of the star space $S_j(A)$. Hence it follows from the sum theorem that K'_m as the countable sum of those 0-dimensional closed sets is also 0-dimensional. By the decomposition theorem, this implies that $\dim K_n(A) \leq n$.

Conversely, we suppose R is a general metric space with weight $|A|$ and $\dim R \leq n$. By Bing's metrization theorem, there exists a σ-discrete open basis

$$\mathcal{W}_m = \{ W_{m\alpha} \mid \alpha \in A_m \}, \quad m = 1, 2, \ldots .$$

We can assume without loss of generality that there exist open sets $V_{m\alpha}$, $\alpha \in A_m$, $m = 1, 2, \ldots$ such that $\bar{V}_{m\alpha} = F_{m\alpha} \subset W_{m\alpha}$ and such that for every neighbourhood $U(p)$ of every point p of R, there exist m and $\alpha \in A_m$ for which $p \in F_{m\alpha} \subset W_{m\alpha} \subset U(p)$. Moreover, since $A_m \subset A$ can be assumed, we may further assume $A_m = A$ for every A by adding as many empty sets as desired to the original $\mathcal{W}_m$. Putting

$$W_m = \cup \{ W_{m\alpha} \mid \alpha \in A_m \} \quad \text{and} \quad F_m = \cup \{ F_{m\alpha} \mid \alpha \in A_m \},$$

we obtain open sets W_m and closed sets F_m satisfying $F_m \subset W_m$, $m = 1, 2, \ldots$. Now by use of the decomposition theorem we decompose R in the following way:

$$R = \bigcup_{k=1}^{n+1} R_k$$

for 0-dimensional sets R_k, $k = 1, \ldots, n+1$. We define open sets U_{mr}, $m = 1, 2, \ldots$; r = rational numbers with $0 < r < \sqrt{2}/2m$ such that

(1) $$F_m \subset U_{mr} \subset \bar{U}_{mr} \subset U_{mr'} \subset \bar{U}_{mr'} \subset W_m \quad \text{if} \quad r > r',$$

(2) $$\bar{U}_{mr} = \cap \{ U_{mr'} \mid r' < r \}, \quad U_{mr} = \cup \{ \bar{U}_{mr'} \mid r' > r \},$$

(3) $\operatorname{ord}_p \{ B(U_{mr}) \mid m = 1, 2, \ldots;\ r = \text{a rational number with } 0 < r < \sqrt{2}/2m \} \leq k - 1$ for each point p of R_k.

The process to construct U_{mr} is parallel to that used in the argument of 2 A), so it will not be given here. Define

$$U^m_{r\alpha} = U_{mr} \cap W_{m\alpha};$$

then, since each $\{W_{m\alpha} \mid \alpha \in A_m\}$ is discrete, $U^m_{r\alpha}$ is an open set for all $\alpha \in A_m$, $m = 1,2,\ldots$, r = rational numbers with $0 < r < \sqrt{2}/2m$ such that

$$F_{m\alpha} \subset U^m_{r\alpha} \subset \bar{U}^m_{r\alpha} \subset U^m_{r'\alpha} \subset \bar{U}^m_{r'\alpha} \subset W_{m\alpha} \quad \text{if} \quad r > r' ,$$

$$B(U^m_{r\alpha}) = B(U_{mr}) \cap W_{m\alpha} .$$

Hence (3) implies that

$$(4) \qquad \operatorname{ord}_p \left\{ B(U^m_{r\alpha}) \;\middle|\; \begin{array}{l} \alpha \in A_m , \ m = 1,2,\ldots , \text{ and } r \text{ is a} \\ \text{rational number with } 0 < r < \sqrt{2}/2m \end{array} \right\} \leq k-1$$

for each point p of R_k. Now we construct star spaces $S_m(A_m)$ with the index set $A_m = A$ and denote by E_α, $\alpha \in A_m$ the unit segments of which $S_m(A_m)$ consists. Let f_m be the mapping of R into $S_m(A_m)$ defined as follows:

$$f_m(p) = \begin{cases} 0 & \text{if } p \notin W_m , \\ \sup\{r \mid p \in U^m_{r\alpha}\} \in E_\alpha & \text{if } p \in W_{m\alpha} , \text{ and } p \in U^m_{r\alpha} \text{ for some } r , \\ 0 & \text{if } p \in W_{m\alpha} \text{ and } p \notin U^m_{r\alpha} \text{ for every } r . \end{cases}$$

Then it is easy to see that f_m is a continuous mapping with the properties $f_m(R - W_m) = 0$, and, if $x \in F_m$, then $f_m(x) = \sqrt{2}/2m \in E_\alpha$ for some $\alpha \in A_m$. By use of (2) we can also easily see that $f_m(p) = r \in E_\alpha$ if and only if $p \in B(U^m_{r\alpha})$. Hence $f_m(p)$ is non-vanishing and rational if and only if $p \in B(U^m_{r\alpha})$ for some $\alpha \in A_m$ and r.

Now we consider the product

$$P(A) = \prod_{m=1}^{\infty} S_m(A_m)$$

and its subset $K_n(A)$ mentioned in the theorem. Let us define a mapping f of R into $P(A)$ by

$$f(p) = \{ f_m(p) \mid m = 1,2,\ldots \} .$$

Then it follows from the properties of f_m and (4) that f is a homeomorphic mapping of R onto a subset of $K_n(A)$. Thus the proof of the theorem is complete.

As for the imbedding of countable-dimensional general metric spaces we can assert the following theorem.

Theorem VI. 11. *A metric space R with weight $|A|$ is countable-dimensional if and only if it can be imbedded in the subset $K_\infty(A)$ of a countable product $P(A) = \Pi_{m=1}^{\infty} S_m(A)$ of star spaces $S_m(A)$, $m = 1,2,\ldots$, where we denote by $K_\infty(A)$ the set of points in $P(A)$ at most finitely many of whose non-vanishing coordinates are rational.*

Proof. The proof is analogous to the proof for the finite-dimensional case and is therefore left to the reader.

In contrast to the separable case, it is impossible to find a universal space for n-dimensional general metric spaces among finite products of star-spaces, because such a product and accordingly every of its subsets can be decomposed into countably many closed subsets each of which has a σ-star-finite base while, as pointed out by Yu. Smirnov, not every finite-dimensional metric space has that property.
S. Lipscomb [1], however, has recently constructed such a universal space in a finite product of one-dimensional spaces as follows:

Let $N(A)$ denote the generalized Baire 0-dimensional metric space defined on the set A. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots) \in N(A)$. Then define a binary relation R in $N(A)$ by $\alpha R \beta$ if and only if

1) $\alpha = \beta$ or

2) $\alpha \neq \beta$ and there is a natural number j such that
for every $k < j$ $\alpha_k = \beta_k$, and
i) $\alpha_j = \beta_{j+s}$ for all $s \geq 1$,
ii) $\beta_j = \alpha_{j+s}$ for all $s \geq 1$.

Then the quotient space $J(A) = N(A)/R$ is a one-dimensional metric space, whose point is either a singleton or a doubleton class; a former type of point is called an irrational point and a latter type a rational point. Then he proved:

A metric space of weight $|A|$ has $\dim \leq n$ if and only if it is topologically imbedded in the set of points of $J(A)^{n+1}$ which have at most n rational coordinates.

An imbedding theorem for more general (non-metrizable) spaces is due to B. Pasynkov [4] and A. Zarelua [2], who constructed an n-dimensional compact space P with weight τ such that every completely regular space of $\dim \leq n$ and weight $\leq \tau$ is homeomorphic to a subset of P.[20]

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[20] For other types of imbedding theorems see, e.g., B. R. Wenner [1], W. Kulpa [1], A. G. Nemec [1] and L. A. Luxemburg [2].