CHAPTER V

DIMENSION AND METRIZATION

It is interesting to find in the following list a remarkable analogy between dimension theory and metrization theory.

Dimension theory

Theorem II. 2. *A metric space* R *has* $\dim \leq n$ *if and only if there exists a* σ*-locally finite open basis* $\mathcal{V}$ *such that* $\operatorname{Ind} B(V) \leq n-1$ *for every* $V \in \mathcal{V}$.

Theorem III. 9. *A metric space* R *has* $\dim \leq n$ *if and only if there exists a sequence* $\{ F_i \mid i = 1, \dots \}$ *of locally finite closed coverings of* R *which satisfies the following conditions:*

i) *for every neighbourhood* $U(p)$ *of every point* p *of* R, *there exists some* i *with* $S(p, F_i) \subset U(p)$,

ii) $F_i = \{ F(\alpha_1, \dots, \alpha_i) \mid \alpha_k \in \Omega$, $k = 1, \dots, i \}$, *where* $F(\alpha_1, \dots, \alpha_i)$ *may be empty,*

Metrization theory

Theorem I. 3 (Nagata-Smirnov's Metrization Theorem). *A regular space* R *is metrizable if and only if there exists a* σ*-locally finite open basis* $\mathcal{V}$.

A T_1*-space* R *is metrizable if and only if there exists a sequence* $\{ F_i \mid i = 1, 2, \dots \}$ *of locally finite closed coverings of* R *which satisfies the condition i) on the left* [1].

[1] First proved by K. Morita [5].

iii) $F(\alpha_1,\ldots,\alpha_{i-1}) = \cup \{ F(\alpha_1,\ldots,\alpha_{i-1},\beta) \mid \beta \in \Omega \}$,

iv) $\mathrm{ord}\ F_i \leq n+1$.

Theorem II. 1 (The Sum Theorem) . *Let* $\{ F_\gamma \mid \gamma \in \Gamma \}$ *be a locally countable closed covering of a metric space* R *such that* $\dim F_\gamma \leq n$ *for* $\gamma \in \Gamma$. *Then* $\dim R \leq n$.

Let $\{ F_\gamma \mid \gamma \in \Gamma \}$ *be a locally finite closed covering of a topological space* R *such that each* F_γ *is metrizable. Then* R *is metrizable* [2] .

Let f *be a closed continuous mapping of a metric space* R *of* $\dim \leq n$ *onto a metric space* S *such that for each point* $q \in S$ *the set* $B(f^{-1}(q))$ *consists of at most one point. Then* $\dim S \leq n$ [3].

Let f *be a closed continuous mapping of a metric space* R *onto a topological space* S *such that for each point* $q \in S$ *the set* $B(f^{-1}(q))$ *is compact. Then* S *is metrizable* [4].

Theorem I. 2 (Alexandroff-Urysohn's Metrization Theorem). *A* T_1*-space* R *is metrizable if and only if there exists a sequence* $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \ldots$ *of open coverings* $\mathcal{U}_i$ *such that* $\{ S(p,\mathcal{U}_i) \mid i = 1,2,\ldots \}$ *is a neighbourhood basis for each point* p *of* R .

To complete the list we can also find theorems in dimension theory (Corollary to Theorem V.1, Theorem V.2) which correspond to Alexandroff-Urysohn's metrization theorem. In this chapter we shall first establish Theorem V.1 and related theorems as the foundation of further investigations. Then by applying them we shall investigate relations between dimensions and metric functions of metric spaces.

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[2] First proved by J. Nagata [1]. Yu. Smirnov [2] and A. H. Stone [3] proved, under some additional conditions, that the union of countably many closed metrizable spaces is metrizable.

[3] This statement is the special case $m = 0$ of Theorem III.7.

[4] This theorem, a part of Theorem I.9, is due to A. H. Stone [2] and to K. Morita - S. Hanai [1]. To this list we may also add Theorem II.9 corresponding to Theorem I.3. Incidentally, it will be interesting to see the corollaries to Theorem II.6 and Theorem III.2 in comparison with the conditions for paracompactness and normality, respectively.

V. 1. Characterization of dimension by a sequence of coverings

Theorem V. 1 [5]. *A (metric) space* R *has dimension* $\leq n$ *if and only if there exists a sequence* $\{ \mathcal{U}_k \mid k = 1, 2, \dots \}$ *of open coverings such that*

i) $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3, \dots,$

ii) ord $\mathcal{U}_k \leq n+1$, $k = 1,2,\dots,$

iii) mesh $\mathcal{U}_k \to 0$ as $k \to \infty$.

Proof. Since the necessity of these conditions is clear, we shall prove only the sufficiency. We can assume without loss of generality that mesh $\mathcal{U}_k < 2^{-k}$. Let

$$\mathcal{U}_k = \{ U_{k\alpha} \mid \alpha \in A_k \} , \quad N = \{ (k,\alpha) \mid k = 1,2,\dots, \alpha \in A_k \} .$$

With each $\alpha \in A_k$ $(k \geq 2)$ we associate by use of i), an index $\beta = f(\alpha) \in A_{k-1}$ such that $U_{k\alpha} \subset U_{k-1\beta}$. Suppose $\mathcal{Q} = \{ Q_\gamma \mid \gamma \in \Gamma \}$ is a given open covering of R . Then we let

$$V_k = \{ x \mid \rho(x, R - Q_\gamma) > 2^{-k} \text{ for some } \gamma \in \Gamma \} , \; k = 1,2,\dots, \quad V_{-1} = V_0 = \emptyset .$$

It is clear that V_k is open and satisfies $V_k \subset V_{k+1}$. For any $(k,\alpha) \in N$ we obtain

$$(1) \qquad U_{k\alpha} - V_k = \emptyset \text{ or } U_{k\alpha} \cap V_{k-1} = \emptyset ,$$

because the contrary would imply

$$\delta(U_{k\alpha}) = \text{diameter } U_{k\alpha} > 2^{-k}$$

[5] This theorem was first proved by P. Vopenka [2] after C. H. Dowker - W. Hurewicz [1] proved that the existence of a sequence $\{ \mathcal{U}_k \}$ of locally finite open coverings satisfying i)' $\mathcal{U}_1 > \bar{\mathcal{U}}_2 > \mathcal{U}_2 > \bar{\mathcal{U}}_3, \dots,$ ii), iii) implies the n-dimensionality of R and J. Nagata [2] proved the corollary to this theorem.
"Is the existence of a sequence $\{ \mathcal{U}_k \}$ of open coverings satisfying only ii) and iii) sufficient for a space R to be of dimension $\leq n$?" had been a major problem posed by P. Alexandroff until K. Sitnikov [1] gave an example of a 2-dimensional separable metric space which has a sequence $\{ \mathcal{U}_k \}$ satisfying ii) for $n = 1$ and iii).
We define the *metric dimension* $\mu \dim R$ of a space as the smallest number n such that there exists a sequence $\{ \mathcal{U}_k \mid k = 1,2,\dots \}$ of open coverings satisfying ii) and iii). Yu. Smirnov [5], V. Egorov [1] and M. Katětov [4] investigated metric dimension. In particular the last paper established for every space R the following relation between $\mu \dim R$ and $\dim R$: $\mu \dim R \leq \dim R \leq 2 \mu \dim R$.
Various other kinds of metric-dependent dimension functions were studied by K. Nagami, J. H. Roberts and his school; see e.g. K. Nagami - J. H. Roberts [1], J. H. Roberts [2], and J. C. Nicholas - J. C. Smith [1]; the last showed that the finite sum theorem does not hold for μ-dim and other metric-dependent dimensions. J. R. Isbell [1] and Yu. Smirnov [8] studies uniform dimension and proximity dimension, respectively. For further developments in this aspect, see H. Herrlich [1], H. Pust [1].

which contradicts our assumption. Now, we define open sets $W_{k\alpha}$ by $W_{k\alpha} = U_{k\alpha} \cap (V_k - \bar{V}_{k-2})$ for $(k,\alpha) \in N$. Then we can easily see that

(2) $\{ W_{k\alpha} \mid (k,\alpha) \in N \}$ covers R,

because for every $x \in R$ there exists $k \geq 1$ satisfying $x \in V_k - \bar{V}_{k-2}$ and $(k,\alpha) \in N$ satisfying $x \in U_{k\alpha}$. We divide $N_0 = \{ (k,\alpha) \mid (k,\alpha) \in N ,\ W_{k\alpha} \neq \emptyset \}$ into three disjoint parts A, B and C as follows:

$$A = \{(k,\alpha) \mid (k,\alpha) \in N_0 ,\quad W_{k\alpha} \cap V_{k-1} = \emptyset \},$$
$$B = \{(k,\alpha) \mid (k,\alpha) \in N_0 ,\quad W_{k\alpha} \subset V_{k-1}\},$$
$$C = \{(k,\alpha) \mid (k,\alpha) \in N_0 ,\quad W_{k\alpha} - V_{k-1} \neq \emptyset ,\ W_{k\alpha} \cap V_{k-1} \neq \emptyset \}.$$

Furthermore we let

$$\mathcal{A} = \{ W_{k\alpha} \mid (k,\alpha) \in A \} ,$$
$$\mathcal{B} = \{ W_{k\alpha} \mid (k,\alpha) \in B \} ,$$
$$\mathcal{C} = \{ W_{k\alpha} \mid (k,\alpha) \in C \} .$$

Then we can prove the following

(3) If $\alpha \in A_1$ and $W_{1\alpha} \neq \emptyset$, then $(1,\alpha) \in A$.

(4) If $(k,\alpha) \in B$, then $(k-1 ,\ f(\alpha)) \in A \cup C$ and $W_{k\alpha} \subset W_{k-1\ f(\alpha)}$.

(5) If $(k,\alpha) \in C$, then $(k-1 ,\ f(\alpha)) \in A$.

Since $V_0 = \emptyset$, (3) is obvious. If $(k,\alpha) \in B$, then $k > 1$, $W_{k\alpha} \subset V_{k-1}$, and hence

$$U_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) = W_{k\alpha} \neq \emptyset .$$

This implies

$$U_{k-1\ f(\alpha)} \cap (V_{k-1} - \bar{V}_{k-2}) \neq \emptyset ,$$

because $U_{k-1\ f(\alpha)} \supset U_{k\alpha}$. So $W_{k-1\ f(\alpha)} - \bar{V}_{k-2} \neq \emptyset$; thus $(k-1 , f(\alpha)) \in A \cup C$, and

$$W_{k\alpha} = U_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) \subset U_{k-1\ f(\alpha)} \cap (V_{k-1} - \bar{V}_{k-2}) \subset W_{k-1\ f(\alpha)} .$$

Now, let us turn to the proof of (5). First note that $(k,\alpha) \in C$ implies $W_{k\alpha} \cap V_{k-1} \neq \emptyset$. Since by the definition of $W_{k\alpha}$, $W_{k\alpha} \cap \bar{V}_{k-2} = \emptyset$, we obtain

$$W_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) \neq \emptyset .$$

Therefore

$$U_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) \neq \emptyset ,$$

and hence

$$U_{k-1\,f(\alpha)} \cap (V_{k-1} - \bar{V}_{k-3}) \neq \emptyset ,$$

which means $W_{k-1\,f(\alpha)} \neq \emptyset$. Next, we note that $(k,\alpha) \in C$ implies $W_{k\alpha} \cap (R - V_{k-1}) \neq \emptyset$. Since by the definition of $W_{k\alpha}$, $W_{k\alpha} \subset V_k$, we obtain

$$W_{k\alpha} \cap (V_k - V_{k-1}) \neq \emptyset .$$

On the other hand

$$W_{k\alpha} \cap (V_k - V_{k-1}) \subset U_{k\alpha} - V_{k-1}$$

follows from $W_{k\alpha} \subset U_{k\alpha}$. Thus we obtain $U_{k\alpha} - V_{k-1} \neq \emptyset$, which implies $U_{k-1\,f(\alpha)} - V_{k-1} \neq \emptyset$.

Hence it follows from (1) that $U_{k-1\,f(\alpha)} \cap V_{k-2} = \emptyset$. Thus we get $(k-1, f(\alpha)) \in A$ proving (5).

We define open sets $H_{k\beta}$ by

$$H_{k\beta} = W_{k\beta} \cup [\, \cup \{ W_{k+1\,\alpha} \mid (k+1, \alpha) \in C ,\ f(\alpha) = \beta \}]$$

for $(k, \beta) \in A$. Then it follows from the definitions of $H_{k\beta}$, $W_{k\beta}$ and A that

(6) $\quad H_{k\beta} \subset U_{k\beta}$, i.e. $\delta(H_{k\beta}) < 2^{-k}$,

(7) $\quad H_{k\beta} \subset V_{k+1} - \bar{V}_{k-1}$.

To see (7) we should note that $(k, \beta) \in A$ implies $W_{k\beta} \cap V_{k-1} = \emptyset$ and accordingly $W_{k\beta} \cap \bar{V}_{k-1} = \emptyset$. Furthermore we claim that

(8) $\quad H_{k\beta} \cap V_k \subset W_{k\beta}$.

For, if $f(\alpha) = \beta$, then

$$W_{k+1\,\alpha} \cap V_k = U_{k+1\,\alpha} \cap (V_k - \bar{V}_{k-1}) \subset U_{k\beta} \cap (V_k - \bar{V}_{k-2}) = W_{k\beta} .$$

Hence (8) follows from the definition of $H_{k\beta}$.

We know, in view of (3) and (4), that $\mathcal{B} < \mathcal{A} \cup \mathcal{C}$. On the other hand, the definition of $H_{k\beta}$, combined with (3) and (5), implies that $\mathcal{A} \cup \mathcal{C} < \{ H_{k\beta} \mid (k, \beta) \in A \}$. Hence, by (2) $\{H_{k\beta}\}$ covers R. Let us denote this open covering by H.

Now, let $H_{k\beta}$ be a given element of H. Then we choose a point $x \in W_{k\beta}$. Evidently $x \in V_k$, and hence

$$\rho(x, R - Q_\gamma) > 2^{-k} \quad \text{for some } \gamma \in \Gamma .$$

Since by the definition of $H_{k\beta}$ we have $x \in H_{k\beta}$, we obtain from (6) $H_{k\beta} \subset Q_\gamma$. Therefore we conclude that $H < Q = \{ Q_\gamma \mid \gamma \in \Gamma \}$.

Finally, let us prove that $\operatorname{ord} H \leqq n+1$.

Let x be a given point of R; then we choose r such that $x \in V_r - V_{r-1}$. If $x \in H_{k\beta}$, then from (7) we obtain either $k = r$ or $k = r-1$. We suppose

(9) $\quad x \in H_{r-1\,\beta}$ for p distinct indices β,

(10) $\quad x \in H_{r\alpha'}$ for q distinct indices α',

It follows from $W_{r-1\,\beta} \subset V_{r-1}$ that $x \notin W_{r-1\,\beta}$. Hence by virtue of (9) and the definition of $H_{k\beta}$ we have $x \in W_{r\alpha}$ for at least p distinct indices α with $(r,\alpha) \in C$, because the mapping $f(\alpha) = \beta$ is unique. On the other hand, by (10) and (8) we have $x \in W_{r\alpha'}$ for at least q distinct indices α' with $(r,\alpha') \in A$, because $x \in V_r$. Therefore $x \in W_{r\alpha} \subset U_{r\alpha}$ for at least $p+q$ distinct indices α.

Since $\operatorname{ord} U_r \leqq n+1$, this implies $p+q \leqq n+1$.

Therefore $\operatorname{ord}_x H \leqq n+1$, i.e. $\operatorname{ord} H \leqq n+1$. Thus we conclude $\dim R \leqq n$.

Corollary. *A T_1-space R is a metrizable space of dimension $\leqq n$ if and only if there exists a sequence*

$$U_1 > U_2^* > U_2 > U_3^* > \dots$$

of open coverings U_i such that $\{ S(p, U_i) \mid i = 1,2,\dots \}$ is a neighbourhood basis of each point p of R, $\operatorname{ord} U_i \leqq n+1$, $i = 1,2,\dots$.

Proof. Combine Theorem I.2 with Theorem V.1.

We shall need later the following proposition which is a slight modification of the "only if" part of the above corollary.

A) A space R of dimension $\leqq n$ has a sequence

$$U_1 > U_2^* > U_2 > U_3^* > \dots$$

of open coverings such that $\{ S(p, U_i) \mid i = 1,2,\dots \}$ is a neighbourhood basis of each point p of R, and such that each element of $\mathcal{U}_{i+1}$ meets at most $n+1$ elements of $\mathcal{U}_i$.

Proof. Since R is a metric space of dimension $\leqq n$, by use of the paracompactness of R we can construct a locally finite open covering $\mathcal{U}_1'$ with mesh $\mathcal{U}_1' < 1$ and ord $\mathcal{U}_1' \leqq n+1$.

Suppose $\mathcal{U}_1' = \{ U_\gamma' \mid \gamma \in \Gamma \}$; then by I.1 A) we can find an open covering $\mathcal{U}_1 = \{ U_\gamma \mid \gamma \in \Gamma \}$ with $\bar{U}_\gamma \subset U_\gamma'$. Since $\bar{\mathcal{U}}_1$ is a locally finite closed covering with ord $\bar{\mathcal{U}}_1 \leqq n+1$, we obtain from the corollary to Theorem II.6 a locally finite open covering $\mathcal{U}_2'$ such that each element of $\mathcal{U}_2'$ meets at most $n+1$ elements of $\mathcal{U}_1$ and such that

$$\mathcal{U}_2'^* < \mathcal{U}_1 \ , \ \text{mesh } \mathcal{U}_2' < \tfrac{1}{2} \ , \ \text{and} \ \ \text{ord } \mathcal{U}_2' \leqq n+1 \ .$$

Then we construct a locally finite open covering $\mathcal{U}_2$ satisfying $\bar{\mathcal{U}}_2 < \mathcal{U}_2'$. $\bar{\mathcal{U}}_2$ is a locally finite closed covering with

$$\text{mesh } \mathcal{U}_2 < \tfrac{1}{2} \ , \ \text{ord } \bar{\mathcal{U}}_2 \leqq n+1 \ , \ \text{and} \ \ \mathcal{U}_2^* < \mathcal{U}_1 \ ,$$

such that each element of $\mathcal{U}_2$ meets at most $n+1$ elements of $\mathcal{U}_1$. By repeating this process we get the desired sequence $\mathcal{U}_1 > \mathcal{U}_2^* > \dots$ of open coverings.

Theorem V. 2. *Let R be a metrizable space. Then there is a metric ρ of R such that the completion $\langle R^*, \rho^* \rangle$ of $\langle R, \rho \rangle$ satisfies* $\dim R^* = \dim R$. *If moreover R is separable, then we can choose as ρ a totally bounded metric.*

Proof. Assume that $\dim R \leqq n$. Then it is easy to see that we can introduce a metric ρ (a totally bounded metric ρ if R is separable) such that there is a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \dots$ of uniform coverings of $\langle R, \rho \rangle$ with

$$\text{mesh } \mathcal{U}_i \to 0 \ \text{ as } \ i \to \infty \ , \ \text{and} \ \ \text{ord } \mathcal{U}_i \leqq n+1 \ .$$

Construct the completion $\langle R^*, \rho^* \rangle$ of $\langle R, \rho \rangle$ (see I.2.). Now, suppose $\mathcal{U}_i = \{ U_\gamma \mid \gamma \in \Gamma_i \}$, and put

$$\mathcal{V}_i = \{ R^* - \overline{R - U_\gamma}^* \mid \gamma \in \Gamma_i \} \ ,$$

where the symbol $\overline{}^*$ denotes closure in R^* .

Then by I.2 C) each $\mathcal{V}_i$ is a uniform covering of $\langle R^*, \rho^* \rangle$. By use of I.2 B) and the definition of ρ^* we can easily prove that $\{ \mathcal{V}_i \}$ satisfies

$$\mathcal{V}_1 > \mathcal{V}_2 > \dots \ , \ \text{ord } \mathcal{V}_i \leqq n+1 \ , \ \text{and} \ \ \text{mesh } \mathcal{V}_i = \text{mesh } \mathcal{U}_i \ .$$

The last equality implies that mesh $V_i \to 0$ as $i \to \infty$. Hence from Theorem V.1 it follows that $\dim R^* \leqq n$.

On the other hand $\dim R \leqq \dim R^*$ is obviously true, and thus we obtain $\dim R^* = \dim R$.

V. 2. Length of coverings

We shall apply the result of Section 1 to the notion of the "length of a multiplicative covering" (due to P. Alexandroff and A. Kolmogoroff) to establish another theorem for n-dimensionality.

Definition V. 1. *We call a covering* $\mathfrak{U}$ *a multiplicative covering if* $\emptyset \notin \mathfrak{U}$ *and if every non-empty intersection* $\bigcap_{i=1}^{k} U_i$ *of elements* U_i, $i = 1,\ldots,k$, *of* $\mathfrak{U}$ *is also an element of* $\mathfrak{U}$.

Definition V. 2. *Let* $\mathfrak{U}$ *be a multiplicative covering. We mean by the length of an element* U *of* $\mathfrak{U}$ *the greatest number* r *such that there exists a sequence* $U = U_1 \supsetneqq U_2 \supsetneqq \cdots \supsetneqq U_r$ *of elements of* $\mathfrak{U}$. *We mean by the length of* $\mathfrak{U}$ *the greatest length of elements of* $\mathfrak{U}$, *i.e.* length $\mathfrak{U} = \max\{\text{length } U \mid U \in \mathfrak{U}\}$. *It is easy to see that for every multiplicative covering* $\mathfrak{U}$ length $\mathfrak{U} \leqq \operatorname{ord} \mathfrak{U}$. *We can easily construct examples of multiplicative coverings* $\mathfrak{U}$ *for which length* $\mathfrak{U} < \operatorname{rank} \mathfrak{U}$, $\operatorname{rank} \mathfrak{U} <$ length $\mathfrak{U}$ *and* length $\mathfrak{U} = \operatorname{rank} \mathfrak{U}$ *respectively hold.*

Theorem V. 3 [6]. *A* T_1*-space* R *is a metrizable space of dimension* $\leqq n$ *if and only if there exists a sequence*

$$\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \cdots$$

of multiplicative open coverings of length $\leqq n+1$ *such that* $\{S(p,\mathfrak{U}_i) \mid i = 1,2,\ldots\}$ *is a neighbourhood basis of each point* p *of* R $(n \geqq 0)$.

Proof. If $\dim R \leqq n$, then by the corollary to Theorem V.1 we can construct a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \cdots$ of open coverings of order $\leqq n+1$ such that

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[6] It remains open whether we can replace the condition ii) in Theorem V.1 by length $\mathfrak{U}_k \leqq n+1$ to improve Theorem V.3. K. Nagami [2] generalized this theorem as follows: A T_1-space R is a metrizable space of dimension $\leqq n$ if and only if there exists a sequence $\{\mathfrak{F}_i \mid i = 1,2,\ldots\}$ of locally finite multiplicative coverings such that $\mathfrak{F}_{i+1} < \mathfrak{F}_i$, length $\mathfrak{F}_i \leqq n+1$, for every neighbourhood $U(p)$ of every point p of R there exists i with $S(p,\mathfrak{F}_i) \subset U(p)$.

$\{S(p,\mathfrak{U}_i) \mid i=1,2,\dots\}$ is a neighbourhood basis of p. Since $\mathfrak{U}_i$ plus all the intersections of a finite number of elements of $\mathfrak{U}_i$ is obviously a multiplicative covering of length $\leq n+1$, the necessity is proved.

Let us assume the existence of a sequence satisfying the condition of the proposition. First we shall construct an open covering $\mathfrak{M}_1$ such that

$$\mathfrak{U}_{n+2} < \mathfrak{M}_1 < \mathfrak{U}_1 \ , \ \text{ord}\ \mathfrak{M}_1 \leq n+1 \ .$$

As a matter of fact, this is the principal part of the proof. We denote by $\{U_{r\alpha} \mid \alpha \in A_r\}$ all the elements of $\mathfrak{U}_1$ of length r. Then

$$\mathfrak{U}_1 = \{U_{r\alpha} \mid \alpha \in A_r,\ r=1,\dots,n+1\}$$

because length $\mathfrak{U}_1 \leq n+1$. We define open sets $V_{r\alpha}^{(i)}$, $i=1,\dots,n+1$ by

$$V_{r\alpha}^{(1)} = U_{r\alpha}\ ,\quad V_{r\alpha}^{(i)} = \text{Int}\ \{x \mid S(x,\mathfrak{U}_i) \subset V_{r\alpha}^{(i-1)}\},\quad i=2,\dots,n+1\ ,$$

where we denote by Int S the interior of the set S. It follows directly from the above definition and $\mathfrak{U}_i^* < \mathfrak{U}_{i-1}$ that

(1) $$V_{r\alpha}^{(n+1)} \subset \dots \subset V_{r\alpha}^{(2)} \subset V_{r\alpha}^{(1)} = U_{r\alpha}\ ,$$

(2) $$\mathfrak{U}_i < \{V_{r\alpha}^{(i)} \mid \alpha \in A_r\ ,\ r=1,\dots,n+1\},\quad i=1,\dots,n+1\ ,$$

(3) $$S(V_{r\alpha}^{(i)},\mathfrak{U}_i) \subset V_{r\alpha}^{(i-1)},\quad i=2,\dots,n+1\ .$$

Next we define open sets $M_{r\alpha}$, $\alpha \in A_r$, $r=1,\dots,n+1$ by

$$M_{1\alpha} = V_{1\alpha}^{(1)} = U_{1\alpha}\ ,$$

(4) $$\begin{aligned} M_{r\alpha} = V_{r\alpha}^{(r)} - \overline{[\cup\{S(V_{1\alpha}^{(r)},\ \mathfrak{U}_{n+2}) \mid \alpha \in A_1\}]} \cup \\ \overline{[\cup\{S(V_{2\alpha}^{(r)},\ \mathfrak{U}_{n+2}) \mid \alpha \in A_2\}]} \cup \dots \cup \\ \overline{[\cup\{S(V_{r-1\,\alpha}^{(r)},\ \mathfrak{U}_{n+2}) \mid \alpha \in A_{r-1}\}]},\quad r=2,\dots,n+1\ . \end{aligned}$$

Let us show that

(5) $$\mathfrak{U}_{n+2} < \mathfrak{M}_1 = \{M_{r\alpha} \mid \alpha \in A_r,\ r=1,\dots,n+1\}\ .$$

Let U be a given element of $\mathcal{U}_{n+2}$; then by using (2) for $i = n+1$, we can find $V^{(n+1)}_{r\alpha}$ with $U \subset V^{(n+1)}_{r\alpha}$. Hence from (1) we can deduce that $U \subset V^{(r)}_{r\alpha}$, which implies

$$\mathcal{U}_{n+2} < \{ V^{(r)}_{r\alpha} \mid \alpha \in A_r \, , \; r = 1, \dots, n+1 \} .$$

Therefore for every $U \in \mathcal{U}_{n+2}$ we can find the least number r such that

(6) $\qquad U \subset V^{(r)}_{r\alpha}$.

To prove (5) we shall show that

(7) $\qquad U \cap S(V^{(r)}_{k\alpha} , \mathcal{U}_{n+2}) = \emptyset$

for this r and every k with $1 \leqq k \leqq r-1$ and for every $\alpha \in A_k$. If we assume the contrary:

$$U \cap S(V^{(r)}_{k\alpha} , \mathcal{U}_{n+2}) \neq \emptyset$$

for some k , α with $1 \leqq k \leqq r-1$, $\alpha \in A_k$, then from $\mathcal{U}^*_{n+2} < \mathcal{U}_r$, (1) and (3), it follows that

$$U \subset S(V^{(r)}_{k\alpha} , \mathcal{U}_r) \subset V^{(r-1)}_{k\alpha} \subset V^{(k)}_{k\alpha} .$$

This contradicts the definition of r because $k < r$. Hence we obtain (7). This combined with (6) and (4) implies $U \subset M_{r\alpha}$, which proves (5).

Now, to show that

(8) $\qquad \operatorname{ord} \mathcal{M}_1 \leqq n+1$,

we prove

(9) $\qquad M_{r\alpha} \cap M_{r\beta} = \emptyset$ for $\alpha \neq \beta$ and for every r .

In case $\alpha, \beta \in A_1$, $\alpha \neq \beta$ clearly implies

$$M_{1\alpha} \cap M_{1\beta} = U_{1\alpha} \cap U_{1\beta} = \emptyset$$

because length $U_{1\alpha}$ = length $U_{1\beta}$ = 1. To prove the same assertion in case $r > 1$, we show that

(10) $\qquad U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma}$ for $\alpha, \beta \in A_r$ and $\gamma \in A_{r'}$,

implies

$$V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} = V^{(r)}_{r'\gamma} .$$

First, by the definition of $V^{(i)}_{r\alpha}$, it is obvious that

$$V^{(2)}_{r'\gamma} \subset V^{(2)}_{r\alpha} \cap V^{(2)}_{r\beta} .$$

Conversely, suppose $x \in V^{(2)}_{r\alpha} \cap V^{(2)}_{r\beta}$; then there exist neighbourhoods $P(x)$ and $Q(x)$ of x such that

$$S(P(x),\mathfrak{U}_2) \subset U_{r\alpha} , \; S(Q(x),\mathfrak{U}_2) \subset U_{r\beta} .$$

Hence

$$S(P(x) \cap Q(x),\mathfrak{U}_2) \subset U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma} .$$

This means that $x \in V^{(2)}_{r'\gamma}$, proving

$$V^{(2)}_{r'\gamma} = V^{(2)}_{r\alpha} \cap V^{(2)}_{r\beta} .$$

Repeating this process, we conclude that

$$V^{(r)}_{r'\gamma} = V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} \text{ for every } r .$$

We now turn to the proof of (9). By use of (10) and (4) we obtain

$$M_{r\alpha} \cap M_{r\beta} \subset V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} - \overline{[\cup\{S(V^{(r)}_{1\alpha},\ \mathfrak{U}_{n+2}) \mid \alpha \in A_1\}] \cup \ldots \cup}$$

$$\overline{[\cup\{S(V^{(r)}_{r-1\,\alpha},\mathfrak{U}_{n+2}) \mid \alpha \in A_{r-1}\}]} \subset V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} - S(V^{(r)}_{r'\gamma},\mathfrak{U}_{n+2}) = \emptyset$$

for r' determined by $U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma}$, because $r' < r$ and in consequence

$$S(V^{(r)}_{r'\gamma},\mathfrak{U}_{n+2}) \subset [\cup\{S(V^{(r)}_{1\alpha},\ \mathfrak{U}_{n+2}) \mid \alpha \in A_1\}] \cup \ldots \cup$$

$$[\cup\{S(V^{(r)}_{r-1\,\alpha},\ \mathfrak{U}_{n+2}) \mid \alpha \in A_{r-1}\}]$$

holds. Thus (9) is proved for $r = 1,\ldots,n+1$.

Since

$$\mathfrak{M}_1 = \{M_{r\alpha} \mid \alpha \in A_r,\ r = 1,\ldots,n+1\} ,$$

the assertion (8) follows directly from (9). Since $\mathfrak{M}_1 < \mathfrak{U}_1$ is clear, from (5) combined with (8) we obtain the desired open covering $\mathfrak{M}_1$ satisfying

$$\mathcal{U}_{n+2} < \mathcal{M}_1 < \mathcal{U}_1 \ , \ \text{ord } \mathcal{M}_1 \leqq n+1 \ .$$

Repeating this process, we get a sequence $\mathcal{M}_i$, $i = 1,2,\ldots$ of open coverings such that

$$\text{ord } \mathcal{M}_i \leqq n+1 \ , \ \text{and} \ \ \mathcal{U}_{1+i(n+1)} < \mathcal{M}_i < \mathcal{U}_{1+(i-1)(n+1)} \ .$$

Therefore from the corollary to Theorem V.1 we can deduce $\dim R \leqq n$ and the metrizability of R .

Corollary [7]. *A space R has dimension $\leqq n$ if and only if for every open covering $\mathcal{U}$ of R there exists a multiplicative open covering $\mathcal{V}$ of R such that* $\mathcal{V} < \mathcal{U}$, length $\mathcal{V} \leqq n+1$.

V. 3. Dimension and metric function

It is an interesting problem to characterize the dimension of a metric space by the property of its metric function. We know that if $\rho(x,y_i) < \varepsilon$, $i = 1,2,3$ on the 1-dimensional Euclidean space E^1 , then $\rho(y_i,y_j) < \varepsilon$ for some two points y_i , y_j of the three points y_1 , y_2 , y_3 and the same is also true for seven points y_i , $i = 1,\ldots,7$ and a point x of E^2 . The number of points y_i having such a property will increase with the dimension number n of E^n . This example leads us to a new idea to characterize the dimension of a metric space as follows.

The main theorem (Theorem V.4) of this section was first proved by J. Nagata [7] and P. A. Ostrand [2] independently. The proof to be presented here is a modification of Ostrand's proof.

A) If $\dim R \leqq n$, then for each open covering $\mathcal{U}$ of R , there is an open covering $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$ and $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$, where each $\mathcal{V}_i$ is a discrete open collection [8].

Proof. By use of the method of the proof of II.7 B) we can construct an open covering $\mathcal{W}$ such that $\mathcal{W} < \mathcal{U}$ and $\mathcal{W} = \bigcup_{i=1}^{n+1} \mathcal{W}_i$ with open collections $\mathcal{W}_i$ of order $\leqq 1$. Let $\mathcal{W}_i = \{ W_{i\alpha} \mid \alpha \in A_i \}$. Select an open covering $\mathcal{P}$ such that $\mathcal{P}^* < \mathcal{W}$ and put

$$V_{i\alpha} = \cup \{ P \mid P \in \mathcal{P},\ S(P,\mathcal{P}) \subset W_{i\alpha} \} ,$$

[7] P. Alexandroff - A. Kolmogoroff [1] proved this theorem for a normal space R and for finite open coverings $\mathcal{U}$, $\mathcal{V}$.

$$V_i = \{ V_{i\alpha} \mid \alpha \in A_i \} \ , \ V = \bigcup_{i=1}^{n+1} V_i \ .$$

Then V is obviously an open refinement of U because $V_{i\alpha} \subset W_{i\alpha}$ and $W < U$. Each V_i is discrete, because each member of P meets at most one member of V_i .

Definition V. 3. *Let* U *be a covering of a space* R. *Then for each non-negative integer* n *we define*

$$[U]^n = \{ S^n(U,U) \mid U \in U \} \ ,$$

where $S^0(U,U) = U$.

B) Let $\dim R \leqq n$. Then there is a sequence U_1 , $U_2, \ldots$ of open coverings of R such that

i) $U_j = \bigcup_{i=1}^{n+1} U_j^i$, where each U_j^i is a discrete collection,
ii) mesh $U_j < 1/j$,
iii) $[U_{j+1}]^{20} < U_j$,
iv) if $j < k$ and $1 \leqq i \leqq n+1$, then each member of $[U_k]^{20}$ meets at most one member of U_j^i .

---

[8] Obviously the converse of this proposition is also true. P. A. Ostrand [1] obtained a similar characterization of dimension as follows:
A metric space R has $\dim \leqq n$ if and only if for each open covering U of R and each integer $k \geqq n+1$ there are k discrete open collections $V_1, \ldots, V_k$ such that $\bigcup_{i=1}^{k} V_i < U$, and the union of any $n+1$ of the V_i is a covering of R .
He applied this theorem to prove the following theorem concerning the representation of a function of many variables as a composite of functions of one variable, which gives a good example of an application of dimension theory. (Another such example was presented in IV.3.):
Let X^p , $p = 1, \ldots, m$ be compact metric spaces such that $\dim X^p = d_p < \infty$ and let $n = \sum_{p=1}^{m} d_p$. Then there are continuous functions $\psi^{pq} : X^p \to [0,1]$, $p = 1, \ldots, m$, $q = 1, \ldots, 2n+1$ such that every real-valued continuous function f defined on $\prod_{p=1}^{m} X^p$ is representable in the form

$$f(x_1, \ldots, x_m) = \sum_{q=1}^{2n+1} \varphi_q \left(\sum_{p=1}^{m} \psi^{pq}(x_p) \right) ,$$

where φ_q , $q = 1, \ldots, 2n+1$ are real-valued continuous functions with a real variable.
E. Ščepin [2] and R. Mañé [1] also obtained interesting results in dimension theory related with other fields: The former proved that every finite-dimensional compact ANR (see J. Nagata [8] for the definition) is metrizable, and the latter proved that if a compact metric space R has a homeomorphism $f : R \to R$ and there exists $\varepsilon > 0$ such that $\rho(f^i(x) , f^i(y)) < \varepsilon$, $i = 1, 2, \ldots$ implies $x = y$, then $\dim R < \infty$.

Proof. By use of A) we find an open covering $\mathcal{U}$ satisfying i) and ii) for $j = 1$. Then each point x of R has a neighbourhood $P(x)$ which meets at most one member of $\mathcal{U}_1^i$ for each i and is contained in some member of $\mathcal{U}_1$. Select an open covering $\mathcal{Q}$ such that $[\mathcal{Q}]^{20} < \{ P(x) \mid x \in R \}$. By A) there is an open covering $\mathcal{U}_2$ such that $\mathcal{U}_2 < \mathcal{Q}$, mesh $\mathcal{U}_2 < 1/2$ and $\mathcal{U}_2 = \bigcup_{i=1}^{n+1} \mathcal{U}_2^i$ for discrete collections $\mathcal{U}_2^i$, $i = 1,\dots, n+1$.

Then $\{ \mathcal{U}_1, \mathcal{U}_2 \}$ satisfies i) - iv). Continue the same process to get a sequence $\{ \mathcal{U}_1, \mathcal{U}_2, \dots \}$ satisfying the desired conditions.

C) Let Q^* denote the set of all rationals of the form $2^{-m_1} + 2^{-m_2} + \dots + 2^{-m_t}$, where m_i are natural numbers satisfying $1 \leq m_1 < m_2 < \dots < m_t$. If $\dim R \leq n$, then we can define open coverings $S(m)$ for all $m \in Q^*$ such that

i) $S(m) = \bigcup_{i=1}^{n+1} S^i(m)$, where each $S^i(m)$ is a discrete collection,

ii) $\{ S(x, S(m)) \mid m \in Q^* \}$ is a neighbourhood base at each $x \in R$,

iii) if $m, p \in Q^*$ satisfy $m < p$, then $S(m) < S(p)$,

iv) if $m, p \in Q^*$, $m < p$, $1 \leq i \leq n+1$, and $S_1 \in S^i(m)$, $S_2 \in S^i(p)$, then either $S_1 \subset S_2$ or $S_1 \cap S_2 = \emptyset$,

v) if $m, p \in Q^*$ satisfy $m + p < 1$, and if $S_1 \in S(m)$, $S_2 \in S(p)$ satisfy $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2 \subset S_3$ for some $S_3 \in S(m+p)$.

Proof. The proof given here is somewhat sketchy to leave the detail to the reader. To begin with, construct open coverings $\mathcal{U}_1$, $\mathcal{U}_2, \dots$ which satisfy the four conditions of B). (We assume that each $\mathcal{U}_i$ contains no empty set.) Then define open collections by

(1) $$\mathcal{V}_j^i = \{ S^{20}(U, \mathcal{U}_{j+1}) \mid U \in \mathcal{U}_j^i \}, \quad 1 \leq i \leq n+1, \quad j = 1, 2, \dots$$

$$\mathcal{V}_j = \bigcup_{i=1}^{n+1} \mathcal{V}_j^i .$$

For $A \subset R$, $1 \leq i \leq n+1$, $j \geq 1$ and $k \geq 0$, we define sets $T^k(A, i, j)$ and $T(A, i, j)$ as follows

(2) $$T^0(A, i, j) = S(A, \mathcal{V}_j^i),$$

$$T^k(A, i, j) = S(T^{k-1}(A, i, j), \mathcal{V}_{j+k}^i) \quad \text{for } k \geq 1,$$

$$T(A, i, j) = \bigcup_{k=0}^{\infty} T^k(A, i, j).$$

Let $m = 2^{-m_1} + \dots + 2^{-m_t} \in Q^*$, where $1 \leq m_1 < \dots < m_t$, and $1 \leq i \leq n+1$; then for each $A \subset R$ we define an open set $S(A, i, m)$ as follows.

(3) $S(A,i,m) = T(A,i,m_1+1)$ if $t=1$,

$S(A,i,m) = T(S(S(A,i,m'),V_{m_t}),i,m_t)$ if $t>1$,

where $m' = 2^{-m_1}+\dots+2^{-m_{t-1}} = m-2^{-m_t}$.

Now we define open collections $S(m)$ for $m=2^{-m_1}+\dots+2^{-m_t}\in Q^*$ by

$$S^i(m) = \{ S(U,i,m) \mid U\in \mathcal{U}^i_{m_1} \} , \quad 1\leqq i\leqq n+1 , \quad S(m) = \bigcup_{i=1}^{n+1} S^i(m) .$$

In the rest of the proof we denote by m and p members of Q^* such that $m = 2^{-m_1}+\dots+2^{-m_t}$, $p = 2^{-p_1}+\dots+2^{-p_u}$, $1\leqq m_1<\dots<m_t$, and $1\leqq p_1<\dots<p_u$.

Since $\mathcal{U}^i_{m_1} < S^i(m)$ and since $\mathcal{U}_{m_1}$ covers R , $S(m)$ is a covering of R . From iii) of B) and (1) it follows that $V_j<\mathcal{U}^*_j$, and accordingly by (2) we obtain $T(A,i,j)\subset S^4(A,\mathcal{U}_j)$. Thus from (3) it follows that

(4) $S(U,i,m)\subset S^4(U,\mathcal{U}_{m_1+1})$ if $t=1$.

Hence by iv) of B), each member of $\mathcal{U}_{m_1+1}$ meets at most one member of $S^i(m)$.

In the general case $t>1$, we can prove without much difficulty that

(5) $S(U,i,m)\subset S^{10}(U,\mathcal{U}_{m_1+1})$,

and thus each member of $\mathcal{U}_{m_1+1}$ meets at most one member of $S^i(m)$ because of iv) of B). Therefore each $S^i(m)$ is discrete.

From (4) and iii) of B) it follows that $S(x,S(m))\subset S(x,\mathcal{U}_{m_1-1})$ if $m = 2^{-m_1}$. Thus ii) of C) follows from ii) of B).

Now, to prove iii) of C), suppose $m<p$.

Case a): $m_1=p_1$, $m_2=p_2$, ..., $m_t=p_t$, and $t<u$.

Then $S(m)<S(p)$ is obvious because

(6) $S(U,i,m)\subset S(U,i,p)$ for each i and $U\in\mathcal{U}^i_{m_1}=\mathcal{U}^i_{p_1}$.

Case b): $m_1=p_1,\dots,m_{s-1}=p_{s-1}$, $m_s>p_s$ for some $s>1$.

Then put

$$p' = 2^{-p_1}+\dots+2^{-p_s} ,$$

$$m' = 2^{-m_1}+\dots+2^{-m_{s-1}} = 2^{-p_1}+\dots+2^{-p_{s-1}} .$$

Now, by use of iii) of B), we obtain

$$S(U,i,m) \subset S^6(S(U,i,m'),\mathcal{U}_{m_s}) \subset S(S(U,i,m'),\mathcal{U}_{p_s}) .$$

On the other hand by (3) we get

$$S(U,i,p') \supset S(S(U,i,m'),\mathcal{V}_{p_s}) \supset S(S(U,i,m'),\mathcal{U}_{p_s}) .$$

Thus

(7) $S(U,i,m) \subset S(U,i,p') \subset S(U,i,p)$

holds for each $U \in \mathcal{U}^i_{m_1} = \mathcal{U}^i_{p_1}$, and we have proved $S(m) < S(p)$.

Case c): $m_1 > p_1$.

Let $U \in \mathcal{U}^i_{m_1}$; then by (4) and (5) $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1})$. Hence by iii) of B) $S(U,i,m) \subset U'$ for some $U' \in \mathcal{U}_{p_1}$.

Suppose $U' \in \mathcal{U}^j_{p_1}$; then

$$S(U,i,m) \subset U' \subset S(U',j,p) \in S^j(p) \subset S(p) .$$

This implies $S(m) < S(p)$.

To prove iv), we first note the following fact.

(8) If a) or b) (defined in the above) is the case, then $S^i(m) < S^i(p)$ for each i .

Because $m_1 = p_1$ holds in this case, and, as shown in the above ((6) and (7)), for each $U \in \mathcal{U}^i_{m_1} = \mathcal{U}^i_{p_1}$, we have $S(U,i,m) \subset S(U,i,p)$. Now iv) follows from (8) and the discreteness of $S^i(p)$ if a) or b) is the case.

Assume the case c) and also that $p_s \leqq m_1 < p_{s+1}$, $m_1 = p_s + k$ (where $s \geqq 1$, $k \geqq 0$) .

Suppose that $S(U,i,m) \in S^i(m)$ and $S(U',i,p) \in S^i(p)$ meet each other, where $U \in \mathcal{U}^i_{m_1}$, $U' \in \mathcal{U}^i_{p_1}$.

Assume $s > 1$. Then by (3) there are sequences $\sigma_1,\dots,\sigma_n$ of open sets such that

$$\sigma_1 = \{ U', S_{11}, S_{12}, \dots\dots \} ,$$
$$\sigma_2 = \{ V_2, S_{20}, S_{21}, \dots\dots \} ,$$
$$\dots\dots$$
$$\sigma_s = \{ V_s, S_{s0}, S_{s1}, \dots, S_{sk}, \dots \} ,$$
$$\dots\dots$$
$$\sigma_u = \{ V_u, S_{u0}, S_{u1}, \dots\dots \} ,$$

where $V_j \in \mathcal{V}_{p_j}$, $S_{jl} \in \mathcal{V}^i_{p_j+l}$ or $S_{jl} = \emptyset$, each non-empty member of a sequence meets its non-empty successor in the same sequence, the sum X_j of the members of σ_j meets V_{j+1}, and $\cup^u_{j=1} X_j$ meets $S(U,i,m)$.

Case α): $S(U,i,m) \cap X_j \neq \emptyset$ for some $j < s$ or $S(U,i,m) \cap V_s \neq \emptyset$.

Then we may assume $S(U,i,m) \cap V_s \neq \emptyset$, because $\mathcal{V}_{p_s}$ is a covering. By (4) and (5)

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) \subset S^{20}(U,\mathcal{U}_{p_s+k+1}) \in \mathcal{V}^i_{p_s+k},$$

because $U \in \mathcal{U}^i_{p_s+k}$. Since $S^{20}(U,\mathcal{U}_{p_s+k+1}) \cap V_s \neq \emptyset$,

$$S(U,i,m) \subset S^{20}(U,\mathcal{U}_{p_s+k+1}) \subset S(U',i,p).$$

Case β): $S(U,i,m) \cap S_{sl} \neq \emptyset$ for some l with $0 \leq l \leq k-1$.

By an argument similar to the above $S(U,i,m) \subset S(U',i,p)$ can be proved.

Case γ): $S(U,i,m) \cap S_{sk} \neq \emptyset$.

Let

$$S_{sk} = S^{20}(U_0,\mathcal{U}_{p_s+k+1}),\ U_0 \in \mathcal{U}^i_{p_s+k}.$$

Since $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{p_s+k+1})$, there is a member of $[\mathcal{U}_{p_s+k+1}]^{20}$ which meets both U and U_0. Thus by iv) of B) $U = U_0$. Hence

$$S(U,i,m) \subset S_{sk} \subset S(U',i,p).$$

Case δ): $S_{sk} \neq \emptyset$, and $S(U,i,m) \cap S_{sl} \neq \emptyset$ for some $l > k$
or $S(U,i,m) \cap X_j \neq \emptyset$ for some $j > s$.

Let

$$S_{sk} = S^{20}(U_0,\mathcal{U}_{p_s+k+1}),\ U_0 \in \mathcal{U}^i_{p_s+k}.$$

Then there is a chain of ten members of $\mathcal{U}_{p_s+k+1}$ which connects S_{sk} and $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{p_s+k+1})$, where we mean by a chain a finite sequence of sets each member of which meets its successor. Hence there is a member of $[\mathcal{U}_{p_s+k+1}]^{20}$ which meets both U and U_0. Thus $U = U_0$, and $S(U,i,m) \subset S(U',i,p)$.

Case ε): $S_{sk} = \emptyset$, and $S(U,i,m) \cap S_{sl} \neq \emptyset$ for some $l > k$,
or $S(U,i,m) \cap X_j \neq \emptyset$ for some $j > s$.

Assume that S_{sk} is the last non-empty member of the sequence

$$S_{s0}, S_{s1}, \ldots, S_{sk-1} .$$

Then there is a chain of ten members of $\mathcal{U}_{p_s+k+1}$ which connects S_{sk} with $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{p_s+k+1})$. Thus

$$S^{20}(U,\mathcal{U}_{p_s+k+1}) \in \mathcal{V}^i_{p_s+k} , \text{ and } S^{20}(U,\mathcal{U}_{p_s+k+1}) \cap S_{sk} \neq \emptyset .$$

Hence

$$S(U,i,m) \subset S^{20}(U,\mathcal{U}_{p_s+k+1}) \subset S(U',i,p) .$$

If $S_{s0},\ldots,S_{sk-1}$ are all empty, we can prove the same by use of V_s in place of S_{sk} . In the above argument we have assumed $s > 1$. However a similar argument is valid in the special case $s = 1$. Thus iv) of C) is established in all cases.

Finally, to prove v), let $m+p<1$. Suppose $S(U,i,m) \cap S(U',j,p) \neq \emptyset$, where $U \in \mathcal{U}^i_{m_1}$, $U' \in \mathcal{U}^j_{p_1}$.

Case 1: $m_1 = p_1$.

It follows from (4) and (5) that

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) ,$$

$$S(U',j,p) \subset S^{10}(U',\mathcal{U}_{m_1+1}) ,$$

$$S^{10}(U,\mathcal{U}_{m_1+1}) \cap S^{10}(U',\mathcal{U}_{m_1+1}) \neq \emptyset .$$

Thus by iii) of B)

$$S(U,i,m) \cup S(U',j,p) \subset W \subset S_0$$

for some $W \in \mathcal{U}_{m_1-1}$ and $S_0 \in S(m+p)$, because $m+p = 2^{-(m_1-1)} + \ldots .$

Case 2: $m_1 > p_1$ (and thus $m < p$) .

Assume that $p_s < m_1 \leq p_{s+1}$ $(s \geq 1)$.

Case 2.1: $m_1 < p_{s+1}$.

Assume e.g. $s = 1$. It follows from (4) and (5) that

$$S^{10}(U,\mathcal{U}_{m_1+1}) \cap S(U',j,p) \neq \emptyset .$$

Since $m_1+1 \leq p_2$, by (3) there is a chain of six members of $\mathcal{U}_{m_1+1}$ which connects $T(U',j,p_1+1)$ with $S^{10}(U,\mathcal{U}_{m_1+1})$. Thus

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) \subset S^{20}(U,\mathcal{U}_{m_1+1}) \in \mathcal{V}_{m_1} ,$$

and

$$S^{20}(U,\mathcal{U}_{m_1+1}) \cap T(U',j,p_1+1) \neq \emptyset .$$

Hence

$$S(U,i,m) \subset S(T(U',j,p_1+1) , \mathcal{V}_{m_1}) \subset S(U',j,q) ,$$

where $q = 2^{-p_1}+2^{-m_1}+2^{-p_2}+\ldots+2^{-p_n}$. On the other hand we obtain $S(U',j,p) \subset S(U',j,q)$ from (7). Thus

$$S(U,i,m) \cup S(U',j,p) \subset S_0 \quad \text{for some} \quad S_0 \in \mathcal{S}(p+m) ,$$

because $q \leq p+m$.

In case of $s>1$, too, we can prove the same by a similar argument. Also note that a similar argument can be applied to the special case $p_u < m_1$ if (6) instead of (7) is used.

Case 2.2: $m_1 = p_{s+1}$.

Denote by k the number satisfying $0 \leq k < s$, $p_{s+1}-1 = p_s$, $p_s - 1 = p_{s-1}, \ldots, p_{s-k+2}-1 = p_{s-k+1}$, and $p_{s-k+1}-1 > p_{s-k}$. (Note that $k=0$ means $p_{s+1}-1 > p_s$, and $k=s$ means that $p_{j+1}-1 = p_j$ for all $j=1,2,\ldots,s$.)

Case 2.2.1: $k=s$.

Then note that

$$2^{-p_1}+\ldots+2^{-p_{s+1}}+2^{-m_1} = 2^{-(p_1-1)} \leq m+p < 1 .$$

Since $S(U',j,p) \subset S^{10}(U',\mathcal{U}_{p_1+1})$, and

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) \subset S^{10}(U,\mathcal{U}_{p_1+1}) ,$$

it is true that

$$S^{10}(U',\mathcal{U}_{p_1+1}) \cap S^{10}(U,\mathcal{U}_{p_1+1}) \neq \emptyset .$$

Hence

$$S(U',j,p) \cup S(U,i,m) \subset U_0 \subset S_0$$

for some $U_0 \in \mathcal{U}_{p_1 - 1}$ and $S_0 \in S(m+p)$.

Case 2.2.2: $k < s$.

Put

$$p' = 2^{-p_1} + \ldots + 2^{-p_{s-k}} \quad \text{and} \quad p'' = p' + 2^{-(p_{s-k+1} - 1)} .$$

(Note that $p'' \leqq m + p$) . Then it follows from (4) and (5) that $S(U,i,m) \subset S^{10}(U, \mathcal{U}_{m_1 + 1})$, and there is a chain of six members of $\mathcal{U}_{p_{s-k+1}}$ which connects $S(U',j,p')$ with $S^{10}(U,\mathcal{U}_{m_1+1})$. Denote by C the sum of the members of this chain. Then there is $U_0 \in \mathcal{U}_{p_{s+k+1} - 1}$ such that

$$S(U,i,m) \cup C \subset S^{10}(U,\mathcal{U}_{m_1 + 1}) \cup C \subset U_0$$

because $p_{s-k+1} \leqq m_1$. Since

$$U_0 \cap S(U',j,p') \neq \emptyset \quad \text{and} \quad S^{20}(U_0, \mathcal{V}_{p_{s-k+1}}) \in \mathcal{V}_{p_{s-k+1} - 1} ,$$

we obtain

$$S(U,i,m) \subset U_0 \subset S^{20}(U_0, \mathcal{V}_{p_{s-k+1}}) \subset S(U',j,p'') .$$

On the other hand $S(U',j,p) \subset S(U',j,p'')$ follows from (7). Hence

$$S(U,i,m) \cup S(U',j,p) \subset S_0 \quad \text{for some} \quad S_0 \in S(m+p) ,$$

because $p'' \leqq m + p$. Thus all conditions for $S(m)$ are verified.

D) Let $\dim R \leqq n$ $(n \geqq 0)$. Then there is a topology-preserving metric ρ of R such that for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of R there is a pair of indices i,j such that $i \neq j$, and $\rho(y_i, y_j) \leqq \rho(x, y_j)$.

Proof. Define $S(m)$, $m \in Q^*$ satisfying the five conditions of C). Then define $\rho : R \times R \to [0,1]$ by

$$\rho(x,y) = \inf \{ m \mid m \in Q^*, \ y \in S(x, S(m)) \} \quad \text{if} \quad y \in S(X, S(m)) \quad \text{for some} \quad m ,$$

$$\rho(x,y) = 1 \quad \text{otherwise.}$$

It is obvious that $\rho(x,y) = \rho(y,x)$ and $\rho(x,x) = 0$. From ii) and iii) of C) it follows that $\rho(x,y) > 0$ if $x \neq y$.

Assume $\rho(x,y) < 1$ and $\rho(x,z) < 1$ for $x,y,z \in R$. Then for every $\varepsilon > 0$ there are $m,p \in Q^*$ such that $y \in S(x,S(m))$, $z \in S(y,S(p))$, $\rho(x,y) \leqq m < \rho(x,y) + \varepsilon$, and $\rho(y,z) \leqq p < \rho(y,z) + \varepsilon$.

By v) of C) there is $W \in S(m+p)$ such that $x,z \in W$. Hence

$$\rho(x,z) \leqq m + p < \rho(x,y) + \rho(y,z) + 2\varepsilon \quad .$$

Thus it is proved that $\rho(x,z) \leqq \rho(x,y) + \rho(y,z)$. Therefore ρ is a metric function.

Note that

$$S(x,S(m/2)) \subset S_m(x) \subset S(x,S(m)) \quad \text{for each } m \in Q^* \text{ .}$$

Thus by virtue of ii) the metric ρ is compatible with the topology of R .

Finally, let $x,y_1,\ldots,y_{n+2}$ be any $n+3$ points of R . If $\rho(x,y_k) = 1$ for some k , then $\rho(y_i,y_k) \leqq \rho(x,y_k)$ for any i . So assume that $\rho(x,y_k) < 1$ for $k = 1,\ldots,n+2$. Let $\varepsilon > 0$ be given. Then choose $m(j) \in Q^*$, $j = 1,\ldots,n+2$ such that $\rho(x,y_j) \leqq m(j) < \rho(x,y_j) + \varepsilon$, and $U_j \in S(m(j))$, $j = 1,\ldots,n+2$ such that $x,y_j \in U_j$.

Suppose $U_j \in S^{i(j)}(m(j))$, $j = 1,\ldots,n+2$. Then for some pair (j,k) of distinct indices we have $i(j) = i(k)$ because $1 \leqq i(j) \leqq n+1$ for $j = 1,\ldots,n+2$. Suppose $m(j) \leqq m(k)$; then from i) and iv) of C) it follows that $U_j \subset U_k$, and thus y_j , $y_k \in U_k$. Hence

$$\rho(y_j,y_k) \leqq m(k) < \rho(x,y_k) + \varepsilon \text{ .}$$

The pair (y_j,y_k) may differ as ε differs. However, since there are only finitely many such pairs, we can fix a pair (y_j,y_k) satisfying

$$\rho(y_j,y_k) < \rho(x,y_k) + 1/n \quad \text{for} \quad n = 1,2,\ldots \text{ .}$$

Thus $\rho(y_j,y_k) \leqq \rho(x,y_k)$.

E) If one can introduce a metric ρ into R which satisfies the condition of D), then $\dim R \leqq n$.

Proof. Actually we shall prove the following slightly stronger statement by use of induction on n : If one can introduce a metric ρ on R such that for a definite number $\delta > 0$ and for every $n+3$ points $x,y_1,\ldots,y_{n+2}$ of R with $\rho(x,y_j) < \delta$, $j = 1,\ldots,n+2$, there is a pair of distinct indices i,j for which $\rho(y_i,y_j) \leqq \rho(x,y_j)$, then $\dim R \leqq n$.

Let ρ be a metric satisfying the condition for $n = 0$. Then for any $x,y \in R$ and

$\varepsilon > 0$ with $\varepsilon < \delta$, $S_\varepsilon(x)$ and $S_\varepsilon(y)$ are disjoint or equal. More precisely $S_\varepsilon(x) = S_\varepsilon(y)$ if $y \in S_\varepsilon(x)$, and $S_\varepsilon(x) \cap S_\varepsilon(y) = \emptyset$ if $y \notin S_\varepsilon(x)$.

Thus $\mathcal{U}_k = \{ S_{\frac{1}{k}}(x) \mid x \in R \}$ for sufficiently large k's are disjoint open coverings of R which satisfy the conditions of Theorem V.1 for $n = 0$. Therefore $\dim R \leq 0$.

Assume validity of the statement for $n = m - 1$. Suppose that ρ is a metric satisfying the condition for $n = m$ and $\delta > 0$. Then we claim that for each positive number $\varepsilon < \delta$ and for each closed set F of R $\dim B(S_\varepsilon(F)) \leq m - 1$, where $S_\varepsilon(F) = \bigcup \{ S_\varepsilon(x) \mid x \in F \}$.

Assume the contrary; then by the induction hypothesis we can select $m + 2$ points $x, y_1, \ldots, y_{m+1} \in B(S_\varepsilon(F))$ such that

$$\rho(x, y_j) < \varepsilon \text{ , } j = 1, \ldots, m + 1 \text{ ,}$$
$$\rho(y_i, y_j) > \rho(x, y_j) \quad \text{for every } i, j \text{ with } i \neq j \text{ .}$$

Now, choose a small neighbourhood $U(x)$ of x such that for every $x' \in U(x)$

$$\rho(x', y_j) < \varepsilon \quad \text{and} \quad \rho(y_i, y_j) > \rho(x', y_j) \quad \text{if} \quad i \neq j \text{ .}$$

Then there is $y_{m+2} \in F$ satisfying $S_\varepsilon(y_{m+2}) \cap U(x) \neq \emptyset$ because $x \in B(S_\varepsilon(F))$. Select a point $x' \in S_\varepsilon(y_{m+2}) \cap U(x)$. Then

$$\rho(x', y_j) < \varepsilon < \delta \text{ , } j = 1, \ldots, m + 2 \text{ ,}$$
$$\rho(y_i, y_j) > \rho(x', y_j) \quad \text{if} \quad i \neq j \quad \text{and} \quad 1 \leq i, j \leq m + 1 \text{ ,}$$
$$\rho(y_i, y_{m+2}) \geq \varepsilon > \rho(x', y_{m+2}) \text{ , } i = 1, \ldots, m + 1 \text{ ,}$$
$$\rho(y_{m+2}, y_j) \geq \varepsilon > \rho(x', y_j), \; j = 1, \ldots, m + 1 \text{ .}$$

This contradicts the property of ρ . Thus we have proved

$$\dim B(S_\varepsilon(F)) \leq m - 1 \text{ .}$$

Now, let F and G be disjoint closed sets of R and m_0 a natural number satisfying $1/m_0 < \delta$. Then put

$$S = \bigcup_{m = m_0}^{\infty} (S_{\frac{1}{m}}(F) - \overline{S_{\frac{1}{m}}(G)}) \text{ .}$$

It is easy to see that S is an open set such that

$$F \subset S \subset R - G \quad \text{and} \quad B(S) \subset \left(\bigcup_{m=m_0}^{\infty} B(S_{\frac{1}{m}}(F)) \right) \cup \left(\bigcup_{m=m_0}^{\infty} B(S_{\frac{1}{m}}(G)) \right) .$$

It follows from the sum theorem and the claim proved above that dim $B(S) \leq m - 1$. Therefore dim $R \leq m$ which completes the induction.

Combining D) and E) we obtain the following main theorem.

Theorem V. 4. *A metric space* R *has* dim $\leq n$ $(n \geq 0)$ *if and only if one can introduce a topology-preserving metric* ρ *on* R *such that for every* $n+3$ *points* $x, y_1, \ldots, y_{n+2}$ *of* R , *there is a pair of distinct indices* i, j *for which* $\rho(y_i, y_j) \leq \rho(x, y_j)$.

Definition V. 4. *A metric* ρ *is called non-archimedean if it satisfies* $\rho(x,z) \leq \max(\rho(x,y), \rho(y,z))$ *for every points* x, y *and* z *of the space.*

Corollary. [9] *A metric space* R *is 0-dimensional if and only if one can introduce a topology-preserving non-archimedean metric on* R .

Proof. This is a direct consequence of Theorem V.4 because the condition there for $n=0$ is precisely the one for a non-archimedean metric.

We can slightly weaken the condition of Theorem V.4 if R is separable.

Theorem V. 5. [10] *A separable metric space* R *has* dim $\leq n$ *if and only if one can introduce a topology-preserving totally bounded metric* ρ *into* R *such that for every* $n+3$ *points* $x, y_1, \ldots, y_{n+2}$ *of* R , *there is a triplet of indices* i, j, k *satisfying* $\rho(y_i, y_j) \leq \rho(x, y_k)$, *and* $i \neq j$.

Proof. The 'only if' part follows directly from Theorem V.4.

We can prove the 'if' part as follows if R is compact. Let $\varepsilon > 0$ and M a maximal subset of R such that $\rho(x,y) \geq \varepsilon$ for every $x, y \in M$ with $x \neq y$. Then $\mathcal{U}_\varepsilon = \{ S_\varepsilon(x) \mid x \in M \}$ is obviously an open covering of R . Suppose $x \in S_\varepsilon(x_k)$, $k = 1, \ldots, n+2$; then $\rho(x, x_k) < \varepsilon$, $k = 1, \ldots, n+2$.

Hence for some distinct i and j we obtain $\rho(x_i, x_j) < \varepsilon$ which implies $x_i = x_j$. Thus ord $\mathcal{U}_\varepsilon \leq n+1$.

Since R is compact, we can construct a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open coverings which satisfies the condition of Theorem V.1. Hence dim $R \leq n$.

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[9] This theorem is due to J. de Groot [1].
[10] This theorem is due to J. de Groot [2].

Now, let R be a separable metric space with a totally bounded metric ρ satisfying the condition. Then the completion $\langle R^*, \rho^* \rangle$ of $\langle R, \rho \rangle$ is a compact metric space. We shall prove that ρ^* satisfies the same condition as ρ. Assume that $x^*, y^*, \ldots, y^*_{n+2}$ are $n+3$ points of R^* such that

$$\rho^*(y^*_i, y^*_j) > \rho^*(x^*, y^*_k) \quad \text{whenever} \quad i \neq j .$$

Since R is dense in R^*, we can choose points x and y_j of R which are so close to x^* and y^*_j, respectively that

$$\rho(y_i, y_j) > \rho(x, y_k) \quad \text{whenever} \quad i \neq j ,$$

which is a contradiction. Thus ρ^* satisfies the required condition, and hence, as proved in the above, $\dim R^* \leq n$. Therefore it follows from the subspace theorem that $\dim R \leq n$.

Corollary.[11] *A compact metric space R has $\dim \leq n$ if and only if one can introduce a metric ρ into R such that for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of R, there is a triplet i, j, k of indices satisfying $\rho(y_i, y_j) \leq \rho(x, y_k)$, and $i \neq j$.*

V. 4. Another metric that characterizes dimension

Every spherical neighbourhood of any point of E^n has a boundary of dimension $n-1$. J. H. Roberts' improvement of the corollary to Theorem IV.12 (p. 85) shows that we can characterize dimension of separable metric spaces in this direction. But we must face difficult circumstances to characterize analogously dimension of general metric spaces. Because the existence of a metric which satisfies the condition of the corollary to Theorem IV.12 may not be sufficient for a general metric space R to have dimension $\leq n$ (even if it holds for every $r > 0$) since

[11] It is unknown yet if this theorem is true for every metric space. In the following is another characterization of dimension by a metric satisfying a stronger condition:

A metric space R has $\dim \leq n$ if and only if one can introduce a topology-preserving metric ρ on R such that if $\rho(S_{\frac{\varepsilon}{2}}(x), y_i) < \varepsilon$, $i = 1, \ldots, n+2$ for $\varepsilon > 0$ and points $x, y_1, \ldots, y_{n+2}$ of R, then $\rho(y_i, y_j) < \varepsilon$ for some distinct indices i, j.

This theorem was proved by J. Nagata [2] and a simpler proof was given by S. Buzâsi [1].

ind $R \leq n$ is not sufficient for that. However, we can develop this idea in general metric cases as seen in the following.

Theorem V. 6 [12]. *A space R has dimension $\leq n$ if and only if we can introduce into R a topology-preserving metric ρ such that the spherical neighbourhoods $S_\varepsilon(p)$, $\varepsilon > 0$ of every point p of R have boundaries of dimension $\leq n-1$ and such that $\{S_\varepsilon(p) \mid p \in R\}$ is closure-preserving for every $\varepsilon > 0$.*

Proof. To show the "if" part let us note that

$$B(S_\varepsilon(A)) = B(\cup\{S_\varepsilon(p) \mid p \in A\}) \subset \cup\{B(S_\varepsilon(p)) \mid p \in A\}$$

for every subset A of R because $\{S_\varepsilon(p) \mid p \in R\}$ is closure-preserving. Therefore

$$\{B(S_\varepsilon(A)) \cap B(S_\varepsilon(p)) \mid p \in A\}$$

is a closure-preserving closed covering of $B(S_\varepsilon(A))$. Since $\dim B(S_\varepsilon(p)) \leq n-1$ for every $p \in A$, from the corollary to Theorem II.1 we can conclude that $\dim B(S_\varepsilon(A)) \leq n-1$.

Now, let F and G be given disjoint closed sets of R; then by the above remark we can construct two sequences

$$U_1 \supset \bar{U}_2 \supset U_2 \supset \dots \supset F,$$
$$W_1 \supset \bar{W}_2 \supset W_2 \supset \dots \supset G$$

of open sets U_i and W_i such that

$$F = \bigcap_{i=1}^{\infty} U_i, \quad G = \bigcap_{i=1}^{\infty} W_i,$$

$$\dim(\bar{U}_i - U_i) \leq n-1, \quad \dim(\bar{W}_i - W_i) \leq n-1.$$

It is clear that $U = \cup_{i=1}^{\infty}(U_i - \bar{W}_i)$ is an open set and satisfies

$$F \subset U \subset R - G, \quad \bar{U} - U \subset \bigcup_{i=1}^{\infty} [(\bar{U}_i - U_i) \cup (\bar{W}_i - W_i)].$$

[12] This theorem is due to J. Nagata [5]. S. Buzási's [1] idea is used here to simplify the proof. Also note that as indicated by the proof, one can introduce into every metrizable space R a topology-preserving metric ρ such that $\{S_\varepsilon(p) \mid p \in R\}$ is closure preserving for every $\varepsilon > 0$.

The latter relation implies $\dim(\bar{U} - U) \leqq n-1$ by virtue of the sum theorem. Thus we can conclude that $\dim R \leqq n$.

To show the 'only if' part let $\dim R \leqq n$; then we can choose a sequence $\{ \mathcal{U}_i \mid i = 1,2,\ldots \}$ of open coverings such that

a) $\mathcal{U}_1 > \mathcal{U}_2^{***} > \mathcal{U}_2 > \mathcal{U}_3^{***} > \ldots$,

b) mesh $\mathcal{U}_m \to 0$ as $m \to \infty$,

c) $S^2(x, \mathcal{U}_{m+1}^*)$ intersects at most $n+1$ members of $\mathcal{U}_m$ for every $x \in R$.

Note that for every $A \subset R$ and integers $1 \leqq m_2 < \ldots < m_p$, the following holds:

$$(1) \qquad S^2(\ldots S^2(A, \mathcal{U}_{m_2}) \ldots \mathcal{U}_{m_p}) \subset S^3(A, \mathcal{U}_{m_2}) .$$

The easy proof is left to the reader.

Now we define open coverings $S(m_1, \ldots, m_p)$ for integers $m_1, \ldots, m_p$ satisfying $1 \leqq m_1 < m_2 < \ldots < m_p$ as follows.

$$S(U; m_1) = U \quad \text{for each} \quad U \in \mathcal{U}_{m_1} ,$$

$$S(U; m_1, \ldots, m_p) = S^2(\ldots S^2(U; \mathcal{U}_{m_2}) \ldots, \mathcal{U}_{m_p}) ,$$

$$S(m_1, \ldots, m_p) = \{ S(U; m_1, \ldots, m_p) \mid U \in \mathcal{U}_{m_1} \} .$$

It follows from (1) and a) that

$$S(U; m_1, \ldots, m_p) \subset S^3(U, \mathcal{U}_{m_2}) \subset S(U, \mathcal{U}_{m_1}) .$$

Hence we obtain

$$(2) \qquad S(m_1) = \mathcal{U}_{m_1} < S(m_1, \ldots, m_p) < \mathcal{U}_{m_1}^* .$$

Now we can prove that

$$S(l_1, \ldots, l_q) < S(m_1, \ldots, m_p)$$

holds whenever

$$1/2^{l_1} + \ldots + 1/2^{l_q} < 1/2^{m_1} + \ldots + 1/2^{m_p} .$$

To do so assume that

(3) $m_1 = l_1, \ldots, m_{i-1} = l_{i-1}$, $m_i < l_i$ for some i with $1 \leq i \leq p, q$,

and put $C = S(U; l_1, \ldots, l_{i-1})$ for any $U \in \mathfrak{U}_{m_1}$ ($C = U$ if $i = 1$). Then it follows from (1) and a) that

$$S(U; l_1, \ldots, l_q) \subset S^3(C, \mathfrak{U}_{l_i}) \subset S^2(C, \mathfrak{U}_{m_i}).$$

Since

$$S^2(C, \mathfrak{U}_{m_i}) \in S(m_1, \ldots, m_i)$$

follows from (3) and $S(m_1, \ldots, m_i) < S(m_1, \ldots, m_p)$ is obvious, we have $S(l_1, \ldots, l_q) < S(m_1, \ldots, m_p)$.

Now, we define a mapping $\rho : R \times R \to [0,1]$ by

(4) $\rho(x,y) = \inf \{ 1/2^{m_1} + \ldots + 1/2^{m_p} \mid y \in S(x, S(m_1, \ldots, m_p)) \}$,
$\rho(x,y) = 1$ if $y \notin S(x, S(m_1, \ldots, m_p))$ for every $S(m_1, \ldots, m_p)$.

First of all $\rho(x,y) = \rho(y,x)$ is obvious. It is almost obvious that $\rho(x,y) = 0$ if and only if $x = y$. To prove the triangle axiom for ρ, assume that

$$1 > \rho(x,y) = a \geq b = \rho(z,y),$$

because the case of $\rho(x,y) = 1$ is trivial. Let $\varepsilon > 0$ be given; then by (4) we can choose $1 \leq m_1 < \ldots < m_p$ and $1 \leq l_1 < \ldots < l_q$ such that $2 \leq p$, $2 \leq q$, $l_1 < m_p$, and

(5) $x \in S(y, S(m_1, \ldots, m_p))$, $z \in S(y, S(l_1, \ldots, l_q))$,
$a < 1/2^{m_1} + \ldots + 1/2^{m_p} < a + \varepsilon$,
$b < 1/2^{l_1} + \ldots + 1/2^{l_q} < b + \varepsilon$,
$1/2^{l_1} + \ldots + 1/2^{l_q} < 1/2^{m_1} + \ldots + 1/2^{m_p}$.

Then $m_i \leq l_1 \leq m_{i+1}$ holds for some i with $1 \leq i < p$.

Case 1: $m_i < l_1 < m_{i+1}$.
Let $U \in \mathfrak{U}_{m_1}$, $V \in \mathfrak{U}_{l_1}$, and

(6) $x, y \in S(U; m_1, \ldots, m_p) = A$, $y, z \in S(V; l_1, \ldots, l_q) = B$

be given. Further, put $D = S(U;m_1,\ldots,m_i)$. Then we obtain

(7) $$A \subset S^3(D,U_{m_i+1}) \subset S(D,U^*_{l_1+1}) .$$

Since

$$B \subset S^3(V,U_{l_2}) \subset S(V,U^*_{l_1+1}) ,$$

there is $W \in U^*_{l_1+1}$ such that $y \in W$ and $W \cap V \neq \emptyset$. From (7) it follows that

$$y \in W \cap S(D,U^*_{l_1+1}) \neq \emptyset .$$

Thus

$$x,z \in A \cup V \subset S(S^2(D,U^*_{l_1+1}),U_{l_1}) \subset S^2(D,U_{l_1}) ,$$

which implies that

$$x,z \in S(U;m_1,\ldots,m_i,l_1,\ldots,l_q) .$$

Therefore we conclude that

$$\rho(x,z) \leqq 1/2^{m_1} + \ldots + 1/2^{m_i} + 1/2^{l_1} + \ldots + 1/2^{l_q} < a+b+2\varepsilon .$$

Case 2: $m_i = l_1$, and $m_1,\ldots,m_i$ are integers in succession, i.e. $m_i = m_{i-1}+1 = m_{i-2}+2 = \ldots = m_1+(i-1)$.

Case 2.1: $m_1 = 1$.

Then

$$\rho(x,z) \leqq 1 = 1/2^{m_1} + \ldots + 1/2^{m_i} + 1/2^{l_1} < a+b+2\varepsilon .$$

Case 2.2: $m_1 > 1$.

Combining (2) and (5) we obtain

$$x \in S(y,U^*_{m_1}) \quad \text{and} \quad z \in S(y,U^*_{l_1}) .$$

Hence

$$x \in S(z,U^{**}_{m_1}) \subset S(z,U_{m_1-1}) = S(z,S(m_1-1)) ,$$

which implies

$$\rho(x,z) \leq 1/2^{m_1 - 1} = 1/2^{m_1} + \dots + 1/2^{m_i} + 1/2^{l_1} < a + b + 2\varepsilon .$$

Case 3: $m_i = l_1$; $m_s, m_{s+1}, \dots, m_i$ are integers in succession, and $m_{s-1} + 1 < m_s$, where $2 \leq s \leq i$.

We may suppose (6) holds. Further put $E = S(U; m_1, \dots, m_{s-1})$. Then it follows from (1) and a) that

$$A \subset S^3(E, \mathfrak{U}_{m_s}) \subset S(E, \mathfrak{U}_{m_s - 1}) \text{ and}$$
$$B \subset S(V, \mathfrak{U}_{l_1}) \in \mathfrak{U}^*_{l_1} = \mathfrak{U}^*_{m_i} < \mathfrak{U}_{m_s - 1} .$$

Since $y \in A \cap B \neq \emptyset$, these relations imply that

$$x, z \in A \cup B \subset S^2(E, \mathfrak{U}_{m_s - 1}) = S(U; m_1, \dots, m_{s-1}, m_s - 1) .$$

Hence

$$\rho(x,z) \leq 1/2^{m_1} + \dots + 1/2^{m_{s-1}} + 1/2^{m_s - 1} = 1/2^{m_1} + \dots + 1/2^{m_i} + 1/2^{l_1} < a + b + 2\varepsilon .$$

Thus in every case we obtain $\rho(x,z) < a + b + 2\varepsilon$, which proves the triangle axiom for ρ . Since

$$S(x, \mathfrak{U}_{m+1}) \subset S_{\frac{1}{2^m}}(x) \subset S(x, \mathfrak{U}_m)$$

is obvious, the metric ρ agrees with the topology of R .

Now for any sequence $m_1, m_2, \dots$ of countably many integers with $1 \leq m_1 < m_2 < \dots$ and for $U \in \mathfrak{U}_{m_1}$ we define an open set $S(U; m_1, m_2, \dots)$ by

$$S(U; m_1, m_2, \dots) = \bigcup_{k=1}^{\infty} S(U; m_1, \dots, m_k)$$

and also an open covering $\mathfrak{S}(m_1, m_2, \dots)$ by

$$\mathfrak{S}(m_1, m_2, \dots) = \{ S(U; m_1, m_2, \dots) \mid U \in \mathfrak{U}_{m_1} \} .$$

Let us assume in the remainder of the proof that

$$0 < \varepsilon = 1/2^{m_1} + 1/2^{m_2} + \dots, \text{ and } 1 \leq m_1 < m_2 < \dots .$$

Then we can assert that

(8) $S_\varepsilon(x) = S(x, S(m_1, m_2, \dots))$ for every $x \in R$.

For, if $y \notin S(x, S(m_1, m_2, \dots))$, then $y \notin S(x, S(m_1, \dots, m_k))$ for $k = 1, 2, \dots$. Thus $\rho(x,y) \geqq 1/2^{m_1} + 1/2^{m_2} + \dots$, which means $y \notin S_\varepsilon(x)$. Namely we obtain

$$S_\varepsilon(x) \subset S(x, S(m_1, m_2, \dots)) .$$

Conversely, if $y \in S(x, S(m_1, m_2, \dots))$, then there is $U \in \mathcal{U}_{m_1}$ such that $x, y \in S(U; m_1, m_2, \dots)$. In view of the definition of $S(U; m_1, m_2, \dots)$ we get $x, y \in S(U; m_1, \dots, m_k)$ for some $k \geqq 1$. Hence

$$\rho(x,y) \leqq 1/2^{m_1} + \dots + 1/2^{m_k} < \varepsilon$$

which means $y \in S_\varepsilon(x)$. Hence

$$S(x, S(m_1, m_2, \dots)) \subset S_\varepsilon(x) .$$

Thus we conclude that (8) is true.

To show $\dim B(S_\varepsilon(x)) \leqq n-1$, we note that (1) implies

$$S(U; m_1, \dots, m_k) \subset S^3(U, \mathcal{U}_{m_2}) \subset S^3(U, \mathcal{U}_{m_1+1}) .$$

Thus we obtain

(9) $S(U; m_1, m_2, \dots) \subset S^3(U, \mathcal{U}_{m_1+1})$.

Further note that by c) each $S(x, \mathcal{U}^*_{m_1+1})$ intersects at most $n+1$ members of $\mathcal{U}_{m_1}$. Hence each $x \in R$ is contained in $S^3(U, \mathcal{U}_{m_1+1})$ for at most $n+1$ distinct members U of $\mathcal{U}_{m_1}$. This note combined with (9) implies

(10) $\operatorname{ord} S(m_1, m_2, \dots) \leqq n+1$.

Now, let us turn to the proof of $\dim B(S_\varepsilon(x)) \leqq n-1$.

Let $y \in B(S_\varepsilon(x))$ be given. Then we can prove

(11) $\operatorname{ord}_y \mathcal{U}_{m_k} \leqq n$, $k = 1, 2, \dots$.

For, suppose

$$y \in U_i \in \mathcal{U}_{m_k}, \quad i = 1, \ldots, n+1 .$$

Then, since $\cap_{i=1}^{n+1} U_i$ is a neighbourhood of y, by (8)

$$(\bigcap_{i=1}^{n+1} U_i) \cap S(x, S(m_1, m_2, \ldots)) \neq \emptyset .$$

Hence there exists $U' \in \mathcal{U}_{m_1}$ satisfying

$$x \in S(U'; m_1, m_2, \ldots) \subset S(x, S(m_1, m_2, \ldots)) = S_\varepsilon(x) \quad \text{and}$$

$$S(U'; m_1, m_2, \ldots) \cap (\bigcap_{i=1}^{n+1} U_i) \neq \emptyset .$$

Since

$$S(U'; m_1, m_2, \ldots) = \cup \{ S(U; m_k, m_{k+1}, \ldots) \mid U \in \mathcal{U}_{m_k}, \ U \subset S(U'; m_1, \ldots, m_k) \},$$

there is $U \in \mathcal{U}_{m_k}$ such that

(12) $$U \subset S(U'; m_1, m_2, \ldots) \subset S_\varepsilon(x) ,$$

(13) $$S(U, m_k, m_{k+1}, \ldots) \cap (\bigcap_{i=1}^{n+1} U_i) \neq \emptyset .$$

Recall that

$$y \in U_i \cap B(S_\varepsilon(x)) \neq \emptyset , \quad i = 1, \ldots, n+1 .$$

Thus it follows from (12) that U is distinct from each of U_i, $i = 1, \ldots, n+1$. Hence (13) implies that

$$\text{ord } S(m_k, m_{k+1}, \ldots) \geqq n+2 ,$$

which contradicts (10). Therefore (11) is proved.

This implies that the restrictions $\mathcal{U}'_{m_k}$ of $\mathcal{U}_{m_k}$ $(k = 1, 2, \ldots)$ to $B(S_\varepsilon(x))$ are open coverings of the boundary satisfying

$$\mathcal{U}'_{m_1} > \mathcal{U}'_{m_2} > \ldots , \ \text{mesh } \mathcal{U}'_{m_k} \to 0 \ \text{as} \ k \to \infty , \ \text{and} \ \text{ord } \mathcal{U}'_{m_k} \leqq n .$$

Hence by Theorem V.1 we conclude that $\dim B(S_\varepsilon(x)) \leqq n-1$.

Finally we shall prove that $\{ S_\varepsilon(x) \mid x \in R \}$ is closure-preserving for every $\varepsilon > 0$. (Note that we still assume $\varepsilon = 1/2^{m_1} + 1/2^{m_2} + \ldots$.) From (8) and (10) it follows that each $S_\varepsilon(x)$ is a finite sum of members of $S(m_1, m_2, \ldots)$. Hence it suffices to prove that $S(m_1, m_2, \ldots)$ is closure-preserving. To do so we note that because of the condition c) of $\{ \mathcal{U}_m \}$ each element U_0 of $\mathcal{U}_{m_1+1}$ intersects $S^3(U, \mathcal{U}_{m_1+1})$ for at most $n+1$ distinct members U of $\mathcal{U}_{m_1}$. Hence by (9) U_0 intersects at most $n+1$ distinct elements of $S(m_1, m_2, \ldots)$. The covering $S(m_1, m_2, \ldots)$ must therefore be locally finite and accordingly closure-preserving. This proves $\{ S_\varepsilon(x) \mid x \in R \}$ to be closure-preserving and the proof of the theorem is complete.

The following corollary is a direct consequence of the theorem.

Corollary. *A space R has $\dim \leqq n$ if and only if we can introduce on R a topology-preserving metric such that $\dim B(S_\varepsilon(F)) \leqq n-1$ for every $\varepsilon > 0$ and for every closed set F of R.*