CHAPTER IV

DIMENSION OF SEPARABLE METRIC SPACES

Although we are now in a position to establish most of the main theorems of dimension theory for general metric spaces, there are yet some theorems which make it preferable to devote one chapter to separable metric spaces. We can extend some of them to more general spaces than separable metric spaces, but it is often true that their best form is achieved only in separable metric spaces. We can extend, for example, Theorem IV.9 about relations between dimension and measure to non-metrizable spaces in a modified form. As for the imbedding theorem we can extend it to general metric spaces [1], but imbedding into Euclidean spaces is possible only for separable metric spaces.

The purpose of this chapter is to discuss theorems in which separability has its own meaning and to give an account of classical theorems in Euclidean spaces.

IV. 1. Cantor manifolds

Theorem IV. 1. Ind R = ind R = dim R *for every separable metric space* R .

Proof. Since Ind $R \geq$ ind R is clear, let us show Ind $R \leq$ ind R .

If ind $R = -1$, then this assertion is obvious. Suppose that Ind $R \leq$ ind R has been established for every R with ind $R \leq n-1$. Now, assume ind $R = n$ for a separable metric space R with a countable open basis $\{U_i \mid i = 1,2,\dots\}$. We define open sets W_{ij}, $i,j = 1,2,\dots$ as follows. If there exist open sets W satisfying $\bar{U}_i \subset W \subset U_j$ and ind $B(W) \leq n-1$, i.e. Ind $B(W) \leq n-1$ by virtue of the induction hypothesis, then we put $W_{ij} = W$ for one of those W . If there exists no open set satisfying this condition, then we put $W_{ij} = \emptyset$ . .

[1] See Chapter VI.6.

It follows from $\operatorname{ind} R = n$ that $\{ W_{ij} \mid i,j = 1,2,\dots \}$ is an open basis with $\operatorname{Ind} B(W_{ij}) \leq n-1$. Hence R satisfies the condition of Theorem II.2, and hence $\operatorname{Ind} R \leq n$. This means $\operatorname{Ind} R \leq \operatorname{ind} R$.

Since $\operatorname{Ind} R = \dim R$ is established for every metric space in Theorem II.7, we can conclude $\operatorname{Ind} R = \operatorname{ind} R = \dim R$ for every separable metric space R.

By virtue of this theorem $\operatorname{Ind} R$, $\operatorname{ind} R$ and $\dim R$ have the same meaning throughout this chapter.

It is known that $\operatorname{ind} R = \dim R$ for any space R which can be expressed as a countable sum of closed subspaces with the star-finite property (or, more generally, which has a σ-star-finite open basis)[2]. On the other hand P. Roy [1] gave an example of a metric space R such that $\operatorname{ind} R = 0$ and $\operatorname{Ind} R = 1$ and thus negatively solved the famous problem on the equivalence between ind and Ind for general metric spaces [3].

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[2] See K. Morita [4] and A. Zarelua [1].

[3] We shall give here only a description of his example, a complete metric space R such that $\operatorname{ind} R = 0$ but $\operatorname{Ind} R = 1$. For that purpose we introduce the following notations. We mean by a sequence a function defined on either the set of non-negative integers or the set of positive integers or any initial segment of either of them.

X = the set of all finite sequences $\{x_i\}$ of real numbers defined on an initial segment of the non-negative integers such that $x_i = 0$ only in case $i = 0$. If $x = \{ x_i \mid i = 0,\dots,n \}$, then we denote by $|x|$ the integer n. Y = the set of all sequences $\{y_i\}$ of positive numbers defined on the set of all positive integers such that $y_i \neq y_j$ if $i \neq j$. If r is a positive number, then Y_r denotes the set of all members of Y which take the value r and we denote by F_r a one-to-one mapping from the positive numbers onto Y_r. Z = the set of all sequences $\{z_i\}$ of positive numbers defined on the set of all positive integers.

Now we define the desired space R as follows: $R = R_1 \cup R_2$, where R_1 = the set of all sequences of non-zero real numbers defined on the set of all positive integers, $R_2 = X \times Y \times Z$ with $p(X)$, $p(Y)$ and $p(Z)$ denoting the coordinates of a point p of R_2. The topology of R is defined by the open basis $\mathcal{U}$ consisting of two types of subsets U_x and $U_{p,n}$ of R. Let $x = \{x_i\} \in X$; then we define $U_x = U^1 \cup U^2$ as follows:

$$U^1 = \{ p \in R_1 \mid p_i = x_i \ \text{ for } \ i = 1,\dots,|x| \ , \text{ if } \ |x| > 0 \} ,$$

$$U^2 = \{ p \in R_2 \mid |p(X)| \geq |x| \ ; \ p(X)_i = x_i \ \text{ for } \ i = 1,\dots,|x| \ , \text{ if } \ |x| > 0 \} ,$$

where we denote by $p(X)_i$ the i-th term of the sequence $p(X)$.

Let n be a positive integer and p a point of R_2; then we define $U_{n,p} = U^0 \cup U^+ \cup U^-$ as follows:

$$U^0 = \{ q \in R_2 \mid q(X) = p(X) \ ; \ q(Y) = p(Y) \ ; \ q(Z)_i = p(Z)_i \ \text{ for } \ i = 1,\dots,n-1, \ \text{ if } \ n > 1 \} ,$$

$$U^+ = \bigcup_{j=1}^{\infty} U_{\gamma(p,n,+,j)} , \quad U^- = \bigcup_{j=1}^{\infty} U_{\gamma(p,n,-,j)} ,$$

where $\{ \gamma(p,n,+,j) \mid j = 1,2,\dots \}$ and $\{ \gamma(p,n,-,j) \mid j = 1,2,\dots \}$ are two infinite sequences of members of X such that

$$|\gamma(p,n,\pm,j)| = |p(X)| + n + 1 ,$$

$$\gamma(p,n,\pm j)_i = \begin{cases} p(X)_i & \text{for } \ i = 0,\dots,|p(X)| , \\ \pm p(Y)_{n+j-1} & \text{for } \ i = |p(X)| + 1 , \\ \mp F_r^{-1}(p(Y)) \ \text{ with } \ r = p(Y)_{n+j-1} & \text{for } \ i = |p(X)| + 2, \end{cases}$$

and if $n > 1$, then $\gamma(p,n,\pm,j)_i = \mp p(Z)_{i'}$ with $i' = i - |p(X)| - 2$ for $i = |p(X)| + 3,\dots,|p(X)| + n + 1$.

Definition IV. 1. *For $n>0$ a compact n-dimensional metric space R is called an n-dimensional Cantor manifold if $R-A$ is connected for every at most $(n-2)$-dimensional subset A of R.*

Note that the above condition preceded by 'if' implies $\dim R \geqq n$ if R contains at least two points.

A) Let R be a compact metric space and f a continuous mapping of a closed subset F into S^n. If f cannot be extended over R, then there exists a closed subset G of R such that

i) f cannot be extended over $F \cup G$, but

ii) f can be extended over $F \cup H$ for every proper closed subset H of G.

Proof. By use of Zorn's lemma [4] we obtain a maximal family $\{G_\alpha \mid \alpha \in A\}$ of closed sets such that $\{G_\alpha\}$ is totally ordered with respect to the order $\subset$, i.e. either $G_\alpha \subset G_\beta$ or $G_\alpha \supset G_\beta$ for every $\alpha, \beta \in A$, and

(1) f cannot be extended over $F \cup G_\alpha$ for every $\alpha \in A$.

If we let $G = \bigcap\{G_\alpha \mid \alpha \in A\}$, then G is a closed subset of R. We can see that f cannot be extended over $F \cup G$. For suppose f were extended over $F \cup G$, then there would be, by the corollary to Theorem I.7, an open set U containing $F \cup G$ over which f could be extended. By virtue of the compactness of R there would be G_α satisfying $G_\alpha \subset U$, and hence $F \cup G_\alpha \subset U$. But this contradicts (1).

If follows from the maximality condition of $\{G_\alpha\}$ that f can be extended over $F \cup H$ for every proper closed subset H of G.

Theorem IV. 2. *Any compact n-dimensional space R contains an n-dimensional Cantor manifold as a subset $(n>0)$.*

Proof. Since $\dim R = n$, there exists, by Theorem III.2, a closed subset F of R and a mapping f of R into S^{n-1} which cannot be extended over R. We get, by use of A), a closed set G satisfying i) and ii) there.

Let us show that G is an n-dimensional Cantor manifold. For if we assume the contrary, then there exist proper closed sets G_1 and G_2 of G such that $G = G_1 \cup G_2$ and $\dim G_1 \cap G_2 \leqq n-2$. It follows from ii) that f can be extended to mappings f_1 and f_2 defined over $F \cup G_1$ and $F \cup G_2$ respectively. By virtue of

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[4] By Zorn's lemma we mean the following: A partially ordered system P each of whose totally ordered subsystems has an upper bound, contains a maximal element. In the present case we consider the system of all the families of closed sets satisfying the conditions concerned.

III.3.D) f_1 can be extended to a mapping of $F \cup G$ into S^{n-1} which is also an extension of f over $F \cup G$. But this contradicts i) of A). (Note that $\dim G \geq n$ automatically follows as remarked right after Definition IV.1.).

IV. 2. Dimension of E^n

A) Let $\{ S_i \mid i = 0, \ldots, n \}$ be the faces of an n-simplex T^n and $\{ F_i \mid i = 0, \ldots, n \}$ a covering of T^n for which $F_i \subset T^n - S_i$. If K is a given geometrical complex triangulating T^n, then K contains an n-simplex which intersects every member of $\{ F_i \mid i = 0, \ldots, n \}$ [5].

Proof. We assign to each vertex p of K a number $i(p)$ for which $p \in F_{i(p)}$. To each r-simplex $T = [p_0, \ldots, p_r]$ of K we assign the set $\{ i(p_0), \ldots, i(p_r) \}$ of numbers, which is denoted by $i(T) = \{ i(p_0), \ldots, i(p_r) \}$. Suppose $\{ T_j \mid j = 1, \ldots, k \}$ are all the n-simplexes belonging to K. Denote by s the number of T_j's such that $i(T_j) = \{ 0, 1, \ldots, n \}$. Then we shall prove by induction on n that s is odd.

In case of $n = 1$ this assertion is clearly true, because 0 is assigned to a vertex of T^1 and 1 to another vertex.

Now we assume this assertion for $(n-1)$-simplexes to prove it for the n-simplex T^n. Let s_j be the number of the $(n-1)$-dimensional faces T' of T_j such that $i(T') = \{ 1, \ldots, n \}$. Then it is easily seen that $s_j = 0$, 1 or 2. For, if $s_j \neq 0$, then there exists a face $T' = [p_1, \ldots, p_n]$ of $T_j = [p_0, p_1, \ldots, p_n]$ for which $i(T') = \{ 1, \ldots, n \}$. If $i(p_0) = 0$, then T' is the only face satisfying this condition, and hence $s_j = 1$. In this case we note that $i(T_j) = \{ 0, 1, \ldots, n \}$.

If $1 \leq i(p_0) \leq n$, then clearly $s_j = 2$. We denote by t the number of such T_j's. Thus we obtain

$$(1) \qquad \sum_{j=1}^{k} s_j = s + 2t .$$

On the other hand, we denote by u the number of $(n-1)$-simplexes T' of K for which

$$(2) \qquad \begin{cases} i(T') = \{ 1, \ldots, n \} , \\ T' \text{ is contained in } \bigcup_{i=0}^{n} S_i , \end{cases}$$

[5] This proposition is due to E. Sperner [1] and called Sperner's lemma. As for elementary knowledge of combinatorial topology see P. Alexandroff - H. Hopf [1] or J. Nagata - Y. Kodama [1]. See also the introductory portion of VIII.1 especially the footnote to Example VIII.1 for the definition of a complex.

and by v the number of the $n-1$-simplexes T'' of K such that $i(T'') = \{1,\dots,n\}$, and T'' is not contained in $\cup_{i=0}^{n} S_i$. The former type of simplex faces just one of $\{T_j \mid j=1,\dots,k\}$, and the latter type faces just two of them. Therefore we obtain $\sum_{j=1}^{k} s_j = u+2v$, which combined with (1) implies $s+2t=u+2v$.

Thus, to prove that s is odd it suffices to show that u is odd. We note that if T' is an $(n-1)$-simplex of K which satisfies (2), then T' must be contained in S_0. For, if T' is contained in S_i for some i different from 0, then from $F_i \subset T^n - S_i$ it follows that $i(T')$ cannot contain i contradicting (2). Hence u is the number of the $(n-1)$-simplexes T' of K contained in S_0 with $i(T') = \{1,\dots,n\}$. Applying the induction hypothesis to the covering $\{F_i \cap S_0 \mid i = 1,\dots,n\}$ of the $(n-1)$-simplex S_0, we conclude that u is odd. This completes the proof that s is odd, and hence K contains at least one n-simplex T_j with $i(T_j) = \{0,1,\dots,n\}$, i.e. a simplex which intersects each of $\{F_i \mid i=0,\dots,n\}$.

Now we are in a position to prove

Theorem IV. 3. $\dim T^n = n$.

Proof. Since $\dim E^n \leqq n$ is deduced by the product theorem from $\dim E^1 \leqq 1$, in view of $T^n \subset E^n$ we obtain $\dim T^n \leqq n$. Hence it suffices to show that $\dim T^n \geqq n$.

Let $\{S_i \mid i=0,\dots,n\}$ be the faces of T^n. Then we consider the open covering $\{T^n - S_i \mid i=0,\dots,n\}$ of T^n. Suppose $V = \{V_i \mid i=0,\dots,n\}$ is a given open covering of T^n for which $V_i \subset T^n - S_i$. Then there exists a closed covering $\{F_i \mid i=0,\dots,n\}$ of T^n such that $F_i \subset V_i$.

Now we shall show that $\cap_{i=0}^{n} F_i \neq \emptyset$.

If the contrary is true, then since T^n is compact, $\{T^n - F_i \mid i=0,\dots,n\}$ is a uniform covering of T^n. Hence there exists $\varepsilon > 0$ such that for every $p \in T^n$ $S_\varepsilon(p) \subset T^n - F_i$ for some i.

We construct a triangulation K of T^n each of whose simplexes has diameter $< \varepsilon$. Then every simplex of K is contained in some $T^n - F_i$, but this contradicts A). Thus we obtain $\cap_{i=0}^{n} F_i \neq \emptyset$, which implies $\cap_{i=0}^{n} V_i \neq \emptyset$. This means $\operatorname{ord} V \geqq n+1$.

Therefore we conclude that $\dim T^n \geqq n$, which completes the proof.

Theorem IV.3 implies $\dim I^n = n$, $\dim S^n = n$ and especially

Theorem IV. 4. $\dim E^n = n$.

B) Let A and B be countable subsets which are dense in E^n. Then there exists a topological mapping of E^n onto itself which maps A onto B.

Proof. In case $n=1$ we write $A = \{a_i \mid i=1,2,\dots\}$ and $B = \{b_i \mid i=1,2,\dots\}$ to establish a one-to-one mapping f between A and B. First we define $f(a_1) = b_1$.

Suppose we have defined $f(a_i) = b_{m_i}$, $i=1,\dots,k-1$, so that f preserves the natural order, i.e. $a_i < a_j$ implies $f(a_i) < f(a_j)$. Then we define $f(a_k)$ as follows. We denote by m_k the first number of $N_k = \{ m \mid m \neq m_1, \dots, m_{k-1}$; if $i \leq k-1$, then $a_i < a_k$ or $a_i > a_k$ implies $b_{m_i} < b_m$ or $b_{m_i} > b_m$ respectively $\}$. Note that N_k is not empty because B is dense in E^1. Then we put $f(a_k) = b_{m_k}$.

It is easy to see that f is a one-to-one mapping between A and B preserving the natural order. To prove by induction that the mapping is onto, assume that $\{ b_1, \dots, b_{i-1} \} \subset f(A)$. Then $\{ 1, \dots, i-1 \} \subset \{ m_1, \dots, m_k \}$ for some k. Assume $i \notin \{ m_1, \dots, m_k \}$ and let e.g.

$$b_{m_1} < b_{m_2} < \dots < b_{m_{j-1}} < b_i < b_{m_j} < \dots < b_{m_k} .$$

Then, since A is dense, we obtain $a_{j-1} < a_\omega < a_j$ for some $\omega > k$. If ω is the first number satisfying this condition, then $m_\omega = i$, i.e. $f(a_\omega) = b_i \in f(A)$.

We can extend f to a topological mapping from E^1 onto itself as follows. For a given point x of $E^1 - A$, we put $A_1 = \{ a \mid a < x, a \in A \}$ and $A_2 = \{ a \mid a > x , a \in A \}$. Then A_1 and A_2 are subsets of A such that $B = f(A_1) \cup f(A_2)$ and such that $b < b'$ whenever $b \in f(A_1)$ and $b' \in f(A_2)$. Note that $f(A_1)$ has no maximum, and $f(A_2)$ has no minimum. Since $\bar{B} = E^1$, there is one and only one number y such that $b \in f(A_1)$ or $b \in f(A_2)$ implies $b < y$ or $b > y$, respectively. Setting $f(x) = y$, we obtain a topological mapping (= homeomorphism) f from E^1 onto itself which maps A onto B.

In case $n = 2$ we suppose $A = \{ a_i = (a_i^1, a_i^2) \mid i = 1,2,\dots \}$ and $B = \{ b_i = (b_i^1, b_i^2) \mid i = 1,2,\dots \}$.

We may assume that A satisfies $a_i^1 \neq a_j^1$ and $a_i^2 \neq a_j^2$ whenever $i \neq j$, because, using the previous argument on E^1, we can map A to a dense subset of $Q \times Q$ by a homeomorphism φ from E^2 onto itself, where Q denotes the set of all rational numbers. Then consider the rotation θ of E^2 around the origin by $\pi/6$. Now $\theta \circ \varphi$ is a homeomorphism of E^2 onto itself which maps A to a dense subset satisfying the above condition. We assume the same condition for B, too.

To define a one-to-one, order-preserving mapping f from A onto B, first let $f(a_1) = b_1$. (The following definition of N_k will indicate what we mean by 'order-preserving'.) Assume that we have defined $f(a_i) = b_{m_i}$ for $i \leq k-1$. Define $N_k = \{ m \mid m \neq m_1, \dots, m_{k-1}$; if $i \leq k-1$, then $a_i^1 < a_k^1$ $(a_i^1 > a_k^1)$ implies $b_{m_i}^1 < b_m^1$ $(b_{m_i}^1 > b_m^1)$, and $a_i^2 < a_k^2$ $(a_i^2 > a_k^2)$ implies $b_{m_i}^2 < b_m^2$ $(b_{m_i}^2 > b_m^2) \}$. Denote by m_k the first number of N_k and define $f(a_k) = b_{m_k}$.

$f(A) = B$ can be proved analogously to the one-dimensional case. Then it is also easy to extend f to a homeomorphism from E^2 onto itself. We can prove the assertion for E^n in a similar way, but the details are left to the reader.

Theorem IV. 5. *Let S be a subset of E^n. Then* $\dim S = n$ *if and only if S contains an open set of E^n.*

Proof. Since the "if" part is clear, we suppose S contains no open set to show $\dim S \leq n-1$. Then $E^n - S$ is dense in E^n , and hence we can choose a countable subset A of $E^n - S$ such that A is dense in E^n . By E^n_k we denote the set of all points of E^n exactly k of whose coordinates are irrational. Note that $\dim E^n_k = 0$ for each k follows from the sum theorem. Then E^n_0 is a countable set which is dense in E^n . Hence, by use of B) we can construct a topological mapping f of E^n onto itself which maps A onto E^n_0 . Hence S is mapped into $E^n - E^n_0$ by f . Since $E^n - E^n_0 = \bigcup_{i=1}^n E^n_i$, and $\dim E^n_i = 0$, $i = 1,\dots,n$, it follows from the decomposition theorem that $\dim (E^n - E^n_0) \leq n-1$. Therefore $\dim S \leq n-1$, which proves the "only if" part of this theorem.

IV. 3. Some theorems in Euclidean space

In this section we shall give an account of some classical theorems due to Brouwer as examples of interesting applications of dimension theory.

A) The identity mapping f from I^n onto itself is essential.

Proof. If we assume f to be inessential, then by use of a projection to the boundary S^{n-1} of I^n from an interior point of I^n we can construct a continuous mapping g from I^n into S^{n-1} such that $g(q) = q$ for every $q \in S^{n-1}$. Then, since $\dim I^n = n$, by Theorem III.5 there is an essential mapping h from I^n onto itself. Now $g \cdot h$ is a continuous mapping from I^n into $S^{n-1} \subsetneqq I^n$. If $p \in h^{-1}(S^{n-1})$, then $g \cdot h(p) = h(p)$, which contradicts the fact that h is essential. Thus f must be essential.

In a similar way we can also prove that the identity mapping from S^n onto itself is essential.

Theorem IV. 6 (Brouwer's fixed-point theorem). *Every continuous mapping f of I^n into itself has a fixed point, i.e. a point $p \in I^n$ such that $f(p) = p$.*

Proof. Let us assume the contrary; then there exists a continuous mapping f of I^n into itself such that $f(x) \neq x$ for every point $x \in I^n$. We regard S^{n-1} as the boundary of I^n . Then for $x \in I^n$ we denote by $g(x)$ the intersection of S^{n-1} with the ray beginning at $f(x)$ and passing through x . One can easily see that g is a continuous mapping of I^n into S^{n-1} and leaves each point of S^{n-1} fixed. This contradicts A).

B) Let S be a closed subspace of E^n . Then a point p of S is contained in the boundary $B(S)$ of S with respect to E^n if and only if for every neighbourhood $V(p)$ of p in S there exists a neighbourhood $U(p)$ of p in S such that

$U(p) \subset V(p)$ and such that every continuous mapping of $S - U(p)$ into S^{n-1} can be continuously extended over S.

Proof. To show the "only if" part, for the given neighbourhood $V(p)$ of a point $p \in S$ we choose a spherical neighbourhood $S(p)$ in E^n such that $U(p) = S(p) \cap S \subset V(p)$.

Let f be a given continuous mapping of $S - U(p)$ into S^{n-1}. Since $B(S(p)) = S^{n-1}$, we obtain $\dim B(S(p)) \leqq n-1$. Hence by use of the corollary to Theorem III.2 we can extend f to a mapping g of $(S - U(p)) \cup B(S(p))$ into S^{n-1}. Now, choose a point $q \in S(p) - S$ and define φ as the projection from q on $B(S(x))$. Then we get an extension h of f over S by putting

$$h(x) = g(\varphi(x)) \quad \text{for every } x \in U \cap S,$$
$$h(x) = f(x) \quad \text{for every } x \in S - U.$$

To show the "if" part we suppose p is an inner point of S. Then there exists a spherical neighbourhood $S(p)$ of p satisfying $\overline{S(p)} \subset S$. Let $U(p)$ be an arbitrary neighbourhood of p contained in $S(p)$. We denote by f the projection of $S - U(p)$ from p onto S^{n-1} which we regard as the boundary of $S(p)$. Then f cannot be extended over S. To show this we suppose g is a continuous extension of f over S. By restricting g to $\overline{S(p)} = I^n$ we get a continuous mapping g' of I^n onto S^{n-1} which fixes every point of S^{n-1}. But this contradicts A).

Let f be a topological mapping of a space R onto a space S; then f maps each open set A of R to an open set B of S. On the other hand if f is a topological mapping of an open set A of R onto a subset B of S, then B may not be open in S (e.g. let $R = A = E^1$, $S = E^2$, $B = \{(x,y) \in E^2 \mid y = 0\}$). However, if $R = S = E^n$, then B must be open as the following theorem asserts.

Theorem IV. 7. *Let f be a topological mapping of a subset A of E^n onto a subset B of E^n. If $p \in A$ is an inner point of A, then $f(p)$ is also an inner point of B. In particular, if A is open, then B is also open.*

Proof. This theorem is a direct consequence of B) because B) characterizes the inner points of a closed subset S of E^n only by the topological properties of S itself. To be more precise, choose $p \in \operatorname{Int} A$; then there is $\varepsilon > 0$ for which $\overline{S_\varepsilon(p)} \subset A$. Note that $\overline{S_\varepsilon(p)}$ and $f(\overline{S_\varepsilon(p)})$ are both closed sets of E^n. Hence by B) we obtain that $f(p) \in \operatorname{Int} f(\overline{S_\varepsilon(p)}) \subset \operatorname{Int} B$.

IV. 4. Imbedding

The purpose of this section is to topologically imbed a given separable metric space R of dimension $\leqq n$ in a $2n+1$-dimensional Euclidean space. More precisely, we can construct a universal n-dimensional set in E^{2n+1}, i.e. a set in which every separable metric space of dimension $\leq n$ can be imbedded, but in which no space of dimension $> n$ can be imbedded. Generally speaking, we cannot imbed R in a Euclidean space of lower dimension as will be shown in the following.

Example IV. 1. We consider a tetrahedron $[a_0, a_1, a_2, a_3]$ in E^3. We take two points b, c on the edges $[a_0, a_3]$ and $[a_1, a_2]$ respectively such that b and c do not coincide with any vertex. Then the six edges of the tetrahedron plus the segment $[b,c]$ are intuitively seen to form a one-dimensional space which cannot be imbedded in E^2.

Definition IV. 2. *Let f be a continuous mapping of a space R into a space S and V a covering of R. If for every point y of S there exists a neighbourhood $U(y)$ of y such that $f^{-1}(u(y)) \subset V$ for some $V \in V$, then f is called a V-mapping.*

A) Let us denote the metric space of all continuous mappings of a separable metric space R into I^{2n+1} by $C(R, I^{2n+1})$. Let L be an n-dimensional plane of I^{2n+1} and V a finite open covering of R. Then

$$D(L,V) = \{ f \mid f \in C(R, I^{2n+1}),\ \overline{f(R)} \cap L = \emptyset,\ f \text{ is a } V\text{-mapping} \}$$

is an open subset of $C(R, I^{2n+1})$.

Proof. To prove A) it suffices to show that a given point f of $D(L,V)$ is an interior point of it.

For every point x of I^{2n+1} we fix an open neighbourhood $U(x)$ of x such that $f^{-1}(U(x)) \subset V$ for some $V \in V$. Since I^{2n+1} is compact, there exists a finite subcovering $\{ U(x_j) \mid j = 1, \ldots, s \}$ of $\{ U(x) \mid x \in I^{2n+1} \}$, i.e.

$$I^{2n+1} = \bigcup_{j=1}^{s} U(x_j) .$$

By virtue of the compactness of I^{2n+1} there exists a positive number δ such that for every $x \in I^{2n+1}$ and for some j

$$(1) \qquad S_\delta(x) \subset U(x_j) .$$

Since $\overline{f(R)}$ and L are compact because of the compactness of I^{2n+1}, for the metric ρ of I^{2n+1} we have $\rho(\overline{f(R)}, L) > 0$. Hence we can choose δ such that

$$(2) \qquad \delta < \rho(\overline{f(R)}, L) .$$

Now, let g be any mapping satisfying

(3) $\rho'(f,g) < \frac{1}{6}\delta$,

where we denote by ρ' the metric of $C(R,I^{2n+1})$. Let x be a given point of I^{2n+1} and put $N = g^{-1}(S_{\frac{1}{6}\delta}(x))$. We denote by y a fixed point of N . Then we assert that

(4) $f(N) \subset S_\delta(f(y)) \subset U(x_j)$

for some j . For, if $z \in N$, then $g(y) \in g(N) \subset S_{\frac{1}{6}\delta}(x)$, and $g(z) \in g(N) \subset S_{\frac{1}{6}\delta}(x)$. This implies $\rho(g(y), g(z)) < \frac{1}{3}\delta$.

On the other hand, from (3) it follows that $\rho(f(y), g(y)) < \frac{1}{6}\delta$, and $\rho(g(z), f(z)) < \frac{1}{6}\delta$.

Therefore we get $\rho(f(y),f(z)) < \frac{2}{3}\delta < \delta$, which means $f(z) \in S_\delta(f(y))$. Since z is a given point of N , this means $f(N) \subset S_\delta(f(y))$. The latter half of the relation (4) follows from (1). Thus from (4) it follows that

$$N = g^{-1}(S_{\frac{1}{6}\delta}(x)) \subset f^{-1}(U(x_j)) \subset V$$

for some $V \in \mathcal{V}$. The last part of this relation is deduced from the definition of $U(x)$. Hence g is a $\mathcal{V}$-mapping.

Furthermore, we can deduce from (2) and (3) that $\overline{g(R)} \cap L = \emptyset$. Consequently $g \in D(L,\mathcal{V})$, which proves $S_{\frac{1}{6}\delta}(f) \subset D(L,\mathcal{V})$. Therefore f is an interior point of $D(L,\mathcal{V})$.

B) Under the same assumption as in A), $D(L,\mathcal{V})$ is dense in $C(R,I^{2n+1})$ if $\dim R \leq n$.

Proof. Let f be a given point of $C(R,I^{2n+1})$ and δ a given positive number. We shall construct g such that $\rho'(f,g) < \delta$, $g \in D(L,\mathcal{V})$. First we let

(1) $I^{2n+1} = \bigcup_{j=1}^{m} S_{\frac{1}{4}\delta}(x_j)$, and $\mathcal{W} = \{ f^{-1}(S_{\frac{1}{4}\delta}(x_j)) \mid j = 1,\dots,m \}$.

Then, since $\dim R \leq n$, we can choose a finite open covering $\mathcal{N}$ of R such that

(2) $\mathcal{N}^\Delta < \mathcal{V} \wedge \mathcal{W}$,

(3) $\operatorname{ord} \mathcal{N} \leq n+1$.

Suppose $\mathcal{N} = \{ N_i \mid i = 1,\dots,s \}$; then by use of the property of I^{2n+1} we can choose vertices $x(N_i)$ in I^{2n+1} and $p_1,\dots,p_{n+1}$ in L such that

(4) $\rho(f(N_i), x(N_i)) < \frac{1}{4}\delta$, and

(5) the vertices $x(N_1),\dots,x(N_s)$, $p_1,\dots,p_{n+1}$, are in a *general position*, i.e. if $0 \leq m \leq 2n$, then any $m+2$ vertices do not lie in an m-dimensional plane of E^{2n+1}.

(Note that L is spanned by $p_1,\dots,p_{n+1}$.)

Now we define a *Kuratowski mapping* $g \in C(R, I^{2n+1})$ by

$$(6) \qquad g(x) = \frac{\sum_{i=1}^{s} \rho(x, R - N_i) x(N_i)}{\sum_{i=1}^{s} \rho(x, R - N_i)},$$

where each $x(N_i)$ should be regarded as a position vector. To show $\rho'(f,g) < \delta$, we suppose x is a given point of R. If $x \notin N_i$, then $\rho(x, R - N_i) = 0$. If $x \in N_i$ then we obtain from (1) and (2) that $\rho(f(x), f(y)) < \frac{1}{2}\delta$ for every $y \in N_i$. Hence from (4) it follows that $\rho(f(x), x(N_i)) < \frac{3}{4}\delta$. Hence by (6) $\rho(f(x), g(x)) < \frac{3}{4}\delta$ holds for every $x \in R$. This proves $\rho'(f,g) \leq \frac{3}{4}\delta < \delta$.

It only remains to prove $g \in D(L, V)$.

Suppose $N_{i_1},\dots,N_{i_t}$ are all the members of N which contain a given point p of R. Then we denote by $L(p)$ the $(t-1)$-dimensional plane spanned by $x(N_{i_1}),\dots,x(N_{i_t})$. Since there is only a finite number of distinct planes $L(p)$ for p running through R,

$$\eta = \min\{\rho(L(p), L(p')) \mid L(p) \cap L(p') = \emptyset\} > 0 .$$

Then any two of those planes $L(p)$ and $L(p')$ either meet or else are at a distance $\geq \eta$ from each other.

We consider a given point x of I^{2n+1}. Let $p, p' \in g^{-1}(S_{\frac{1}{2}\eta}(x))$; then $\rho(g(p), g(p')) < \eta$. Since $g(p) \in L(p)$ and $g(p') \in L(p')$ follows from (6), we have $\rho(L(p), L(p')) < \eta$. Therefore from the property of η we obtain

$$(7) \qquad L(p) \cap L(p') \neq \emptyset .$$

Suppose $L(p)$ is spanned by $x(N_{i_1}),\dots,x(N_{i_t})$ and $L(p')$ is spanned by $x(N_{j_1}),\dots,x(N_{j_u})$. Then by (7) $x(N_{i_1}),\dots,x(N_{i_t})$, $x(N_{j_1}),\dots,x(N_{j_u})$ lie on a $(t+u-2)$-dimensional plane. On the other hand, it follows from (3) that $t \leq n+1$, $u \leq n+1$; hence $t+u \leq 2n+2$. Thus, in view of (5) we conclude that at least one of the $x(N_{i_1}),\dots,x(N_{i_t})$ coincides with one of the $x(N_{j_1}),\dots,x(N_{j_u})$. Suppose, for example $x(N_{i_1}) = x(N_{j_1})$, i.e. $N_{i_1} = N_{j_1}$. Then this implies $p,\ p' \in N_{i_1}$, and hence $p' \in S(p, N)$ for every $p,\ p'$ with $p, p' \in g^{-1}(S_{\frac{1}{2}\eta}(x))$.

Therefore, by (2) $g^{-1}(S_{\frac{1}{2}\eta}(x)) \subset S(p,N) \subset V$ for some $V \in V$. Hence g is a V-mapping.

Now, let p be a given point of R and $N_{i_1},\dots,N_{i_t}$ all the members of N which contain p. Since $x(N_{i_1}),\dots,x(N_{i_t})$, $p_1,\dots,p_{n+1}$ are in a general position by (5), we obtain $\rho(L(p), L) > 0$. Since there is only a finite number of distinct $L(p)$,

$$\eta' = \min \{ \rho(L(p),L) \mid p \in R \} > 0 .$$

Then $\rho(L(p),L) \geqq \eta'$ for every $p \in R$. Since $g(p) \in L(p)$, we have $\rho(g(p),L) \geqq \eta'$ for every $p \in R$. Therefore we can conclude that $\overline{g(R)} \cap L = \emptyset$, which proves $g \in D(L,V)$. Thus the proof of this assertion is complete.

We should note that this proof implies the following assertion, which will be used later, because the mapping (6) maps R into an n-dimensional polyhedron in I^{2n+1} whose vertices are $x(N_1),\dots,x(N_s)$.

C) The set of all mappings in $C(R,I^{2n+1})$ which map R into an n-dimensional polyhedron in I^{2n+1} is dense in $C(R,I^{2n+1})$ if $\dim R \leqq n$.

Theorem IV. 8 (Imbedding theorem) [6]. *A separable metric space R has dimension $\leqq n$ if and only if R is homeomorphic to a subset of*

$$S = I^{2n+1} \cap (E^{2n+1}_{n+1} \cup E^{2n+1}_{n+2} \cup \dots \cup E^{2n+1}_{2n+1}) ,$$

where we denote by E^{2n+1}_i the set of points in E^{2n+1} exactly i of whose coordinates are irrational.

Proof. To begin with, from the product theorem and the sum theorem we can deduce $\dim E^{2n+1}_i = 0,\ i = n+1,\dots,2n+1$. Hence by use of the decompositon theorem we obtain $\dim S \leqq n$, which proves the "if" part of this theorem.

Conversely, let $\dim R \leqq n$ to prove the "only if" part. For every subset $\{ i_1,\dots,i_{n+1} \}$ of $\{ 1,\dots,2n+1 \}$ and every set $r_1,\dots,r_{n+1}$ of $n+1$ rational numbers, we define an n-dimensional plane

$$L = \{ (x_1,\dots,x_{2n+1}) \mid x_{i_1} = r_1 ,\dots, x_{i_{n+1}} = r_{n+1} \} .$$

Then we denote by $L_1,L_2,\dots$ the totality of those planes. We note that

$$S = I^{2n+1} - \bigcup_{i=1}^{\infty} L_i .$$

Moreover, we take a countable basis $\{ V_1,V_2,\dots \}$ of R and construct, for any pair V_i , V_j with $\bar V_i \subset V_j$, an open covering $\{ V_j, R - \bar V_i \}$. Then we denote by $\mathcal{V}_1,\mathcal{V}_2,\dots$ the totality of those open coverings. Now, we denote by $D(L_k,\mathcal{V}_k)$ the subspace of $C(R,I^{2n+1})$ defined by

$$D(L_k,\mathcal{V}_k) = \{ f \mid f \in C(R,I^{2n+1}) ,\ \overline{f(R)} \cap L_k = \emptyset ,\ f \text{ is a } \mathcal{V}_k\text{-mapping} \} .$$

[6] This theorem is due to K. Menger [1], G. Nöbeling [1], L. Pontrjagin - G. Tolstowa [1], and W. Hurewicz [4]. The present proof is due to C. Kuratowski [3].

Since by A) and B) each $D(L_k, V_k)$ is open and dense in the complete metric space $C(R, I^{2n+1})$, from Theorem I.5 we obtain $\bigcap_{k=1}^{\infty} D(L_k, V_k) \neq \emptyset$.

Let $f \in \bigcap_{k=1}^{\infty} D(L_k, V_k)$; then f is clearly a continuous mapping of R into S.

Let $W(p)$ be a given neighbourhood of a point p of R. Then there exist open sets V_i and V_j with $p \in V_i \subset \bar{V}_i \subset V_j \subset W(p)$. Since f is a V_k-mapping for every k, there exists a neighbourhood $U(f(p))$ of $f(p)$ such that $f^{-1}(U(f(p))) \subset V_j$ or $f^{-1}(U(f(p))) \subset R - \bar{V}_i$. Since from $p \in V_i$ it follows that

$$p \in f^{-1}(U(f(p))) \cap V_i \neq \emptyset,$$

we conclude $f^{-1}(U(f(p))) \subset V_j$. Hence, if $q \notin W(p)$ $(\supset V_j)$, then $f(q) \notin U(f(p))$. This means that f is a homeomorphism between R and a subset $f(R)$ of S.

We shall see in the following that the same technique of imbedding produces various interesting results.

IV. 5. ε-mappings

It is natural to try to approximate n-dimensional compact sets in Euclidean space by elementary figures like n-dimensional polyhedra. In the present section we shall give an account of Alexandroff's theory [7] in this area which is also of great importance in the history of topology itself.

Definition IV. 3. *A continuous mapping f of a space R into a space S is called an ε-mapping if the inverse image $f^{-1}(q)$ of each point q of S has diameter* $< \varepsilon$.

A) Let R be a compact metric space with $\dim R \geqq n$. Then there exists a positive number ε such that R cannot be mapped by any ε-mapping onto any metric space S of dimension $\leqq n-1$.

Proof. Since $\dim R \geqq n$, there exists such a positive number ε that any open refinement $\mathcal{U}$ of $\mathcal{S}_\varepsilon = \{S_\varepsilon(x) \mid x \in R\}$ has order $\geqq n+1$. Suppose f is an ε-mapping which maps R onto a space S of dimension $\leqq n-1$. Then by virtue of the compactness of R, we can choose $\delta(\varepsilon) > 0$ such that $\rho(x,y) \geqq \varepsilon$ in R implies $\rho'(f(x), f(y)) \geqq \delta(\varepsilon)$ in S.

[7] See P. Alexandroff [1]. In this connection there is another interesting aspect of investigations using the concept of inverse limit. H. Freudenthal [1] proved that a compact metric space R has $\dim \leqq n$ if and only if it is homeomorphic with the inverse limit of an inverse sequence of compact polyhedra P_i of $\dim \leqq n$. S. Mardešić [1] extended Freudenthal's result to compact T_2-spaces. See also K. Nagami [3], [4], B. Pasynkov [1], [3], and V. Ključin [1].

In view of $\dim S \leqq n-1$ we can choose an open covering $\mathcal{V}$ of S satisfying

(1) $\qquad \operatorname{ord} \mathcal{V} \leqq n$ and $\mathcal{V} < \mathcal{S} = \{ S_{\delta(\varepsilon)}(y) \mid y \in S \}$

Let V be a given element of $\mathcal{V}$; then $V \subset S_{\delta(\varepsilon)}(y)$ for some $y \in S$. Choose a fixed point $x \in f^{-1}(y)$; then by the property of $\delta(\varepsilon)$ $f^{-1}(V) \subset S_{\varepsilon}(x)$.
Hence $\{ f^{-1}(V) \mid V \in \mathcal{V} \}$ is an open refinement of $\mathcal{S}_{\varepsilon}$ and has order $\leqq n$ by virtue of (1). But this contradicts the property of $\mathcal{S}_{\varepsilon}$.

B) Let R be an n-dimensional compact set in a Euclidean space E^m . Then for a given positive number ε , R can be mapped onto a polyhedron of dimension $\leqq n$ by such a continuous mapping f that $\rho(x, f(x)) < \varepsilon$ for every $x \in R$.

Proof. The first half of the proof is quite analogous to that of the imbedding theorem. For a given $\varepsilon > 0$ we denote by $\mathcal{V} = \{ V_i \mid i = 1, \dots, s \}$ an open covering of R such that

(1) $\qquad \mathcal{V} < \mathcal{S} = \{ S_{\frac{1}{8}\varepsilon}(p) \mid p \in R \}$,

(2) $\qquad \operatorname{ord} \mathcal{V} \leqq n+1$.

With any member V_i of $\mathcal{V}$ we associate a vertex $x(V_i)$ satisfying

(3) $\qquad \rho(x(V_i), V_i) < \frac{1}{4}\varepsilon$

such that $n+1$ of the $x(V_i)$, $i = 1, \dots, s$, never lie on an $(n-1)$-dimensional plane. Then we define a Kuratowski mapping g by

$$g(x) = \frac{\sum_{i=1}^{s} \rho(x, R - V_i)\, x(V_i)}{\sum_{i=1}^{s} \rho(x, R - V_i)} .$$

Let $V_{i_1}, \dots, V_{i_t}$ be all the members of $\mathcal{V}$ which contain a given point x of R . Then $g(x)$ is contained in the simplex $[x(V_{i_1}), \dots, x(V_{i_t})]$. Since from (1), (3) and $x \in V_{i_k}$ it follows that $\rho(x, x(V_{i_k})) < \frac{1}{2}\varepsilon$, we obtain

(4) $\qquad \rho(x, g(x)) < \frac{1}{2}\varepsilon$ for every $x \in R$.

Thus g is a continuous mapping which maps R into the polyhedron P composed of such simplexes $[x(V_{i_1}), \dots, x(V_{i_t})]$.
Now, we construct a triangulation of P into simplexes $T_1, \dots, T_r$ with diameters less than $\frac{1}{2}\varepsilon$. We assume

$$\dim T_1 \geqq \dim T_2 \geqq \dots \geqq \dim T_r .$$

If $T_1 \not\subset g(R)$, then by the compactness of $g(R)$, $T_1 - g(R)$ contains an interior point p of T_1. Hence we denote by g_1 the mapping which projects $T_1 \cap g(R)$ onto the faces of T_1 from p while it leaves the other points of $g(R)$ fixed. If $T_1 \subset g(R)$, then we regard g_1 as the identity mapping. We define $g_2, \dots, g_r$ for $T_2, \dots, T_r$ in the same way. Then $f = g_r \dots g_1 g$ is obviously a continuous mapping of R onto a polyhedron P' contained in P.

Since (2) implies $t \leqq n+1$ and accordingly $\dim P \leqq n$, we have $\dim P' \leqq n$. On the other hand, since

$$\delta(T_j) = \text{diameter } T_j < \tfrac{1}{2}\varepsilon, \quad j = 1, \dots, r,$$

we easily see that $\rho(g(x), f(x)) < \frac{1}{2}\varepsilon$. This combined with (4) proves that $\rho(x, f(x)) < \varepsilon$ for every $x \in R$.

Theorem IV. 9. *A compact set R in a Euclidean space has dimension $\leqq n$ if and only if for every $\varepsilon > 0$ it can be mapped onto a polyhedron of dimension $\leqq n$ by such a continuous mapping f that $\rho(x, f(x)) < \varepsilon$ for every $x \in R$* [8].

Proof. The "only if" part is a direct consequence of B). On the other hand, a mapping f satisfying this condition is clearly a 2ε-mapping. Therefore the "if" part is a direct consequence of A).

Theorem IV. 10. *A compact metric space R has dimension $\leqq n$ if and only if for every $\varepsilon > 0$ it can be mapped onto a polyhedron of dimension $\leqq n$ by an ε-mapping.*

Proof. Let $\dim R \leqq n$; then by the imbedding theorem R can be mapped by a topological mapping g onto a compact subset S of E^{2n+1}. Since R is compact, there exists $\delta > 0$ such that $g^{-1}(S_\delta(y)) \subset S_{\frac{1}{3}\varepsilon}(x)$ for every $y \in E^{2n+1}$ and some $x \in R$. Then by use of B) we can map S onto a polyhedron P of dimension $\leqq n$ by a continuous mapping f which satisfies $\rho(y, f(y)) < \delta$ for every $y \in S$. Thus f satisfies $f^{-1}(z) \subset S_\delta(z)$ for every $z \in P$. Hence we obtain $g^{-1} f^{-1}(z) \subset S_{\frac{1}{3}\varepsilon}(x)$ for every $z \in P$ and for some $x \in R$. Thus $f \cdot g$ is an ε-mapping of R onto P.

The "if" part is a direct consequence of A).

IV. 6. Pontrjagin-Schnirelmann's theorem

In the following we describe another good example of an application of the imbedding technique.

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[8] It is easy to see that this theorem is valid for every compact set R in a Hilbert space, too.

Definition IV. 4. *Let* R *be a metric space with metric* ρ *and let* $\varepsilon > 0$. *Then* $N(\varepsilon, R, \rho) = \inf \{ |a| \mid a$ *is a finite covering of* R *with* mesh $a \leq \varepsilon \}$, *where* $|a|$ *denotes the cardinality of* a .

The following proposition is easy to prove, so that the proof can be left to the reader.

A) Let P be an n-dimensional polyhedron with Euclidean metric ρ . Then

$$c/\varepsilon^n \leq N(\varepsilon, P, \rho) \leq c'/\varepsilon^n$$

for some positive constants c and c' if $\varepsilon > 0$ is sufficiently small.

The above proposition implies that

$$n = \dim P = \lim_{\varepsilon \to 0} \frac{-\log N(\varepsilon, P, \rho)}{\log \varepsilon} .$$

Generally we can characterize dimension of separable metric spaces by use of the function $N(\varepsilon, R, \rho)$.

B) Let S be a compact subset of E^k with $\dim S \geq n$, where $k \geq n$. Then for some n-dimensional coordinate plane E^n of E^k $\dim \pi(S) = n$ holds, where π denotes the projection from E^k onto E^n .

Proof. Let L be a covering of E^k by congruent k-dimensional closed cubes with faces parallel to the $(k-1)$-dimensional coordinate planes such that the interiors of any two elements of L do not intersect and such that $\operatorname{ord} L = k+1$. Such a covering may be called a *Lebesgue brick covering*, and an example is the covering of E^2 by the squares whose vertices are $(2m, 4n)$, $(2m+2, 4n)$, $(2m+2, 4n+2)$, $(2m, 4n+2)$ or $(2m+1, 4n+2)$, $(2m+3, 4n+2)$, $(2m+3, 4n+4)$, $(2m+1, 4n+4)$, where m and n are arbitrary integers. Obviously we can construct a Lebesgue brick covering of arbitrarily small mesh.

Now, denote by $E_1, \ldots, E_\ell$ all n-dimensional coordinate planes in E^k and by π_i the projection from E^k onto E_i . Assume that $\dim \pi_i(S) \leq n-1$, $i = 1, \ldots, \ell$. Denote by $F_1, F_2, \ldots$ the collection of the intersections of $n+1$ elements of L . Since S is compact, only finitely many of them, say $F_1, \ldots, F_s$ intersect S , and $\rho(S, \bigcup_{j=s+1}^{\infty} F_j) > 0$.

Now, F_1 is contained in a $(k-n)$-dimensional Euclidean space E^{k-n} which is perpendicular to some E_i . Thus $\pi_i(F_1)$ is a single point of E_i . Since $\dim \pi_i(S) \leq n-1$, $E_i - \pi_i(S)$ is dense in E_i . Hence we can map S to $t_1(S) \subset E^k$ by an arbitrarily small translation t_1 parallel to E_i in such a way that

$$t_1(S) \cap F_1 = \emptyset \text{ and } \rho(t_1(S), \bigcup_{j=s+1}^{\infty} F_j) > 0 .$$

By a similar argument we can find a translation t_2 such that

$$t_2t_1(S) \cap F_2 = \emptyset \text{ and } \rho(t_2t_1(S), F_1 \cup \bigcup_{j=s+1}^{\infty} F_j) > 0 .$$

Continuing this process, we obtain a translation t such that $t(S) \cap (\bigcup_{j=1}^{\infty} F_j) = \emptyset$. Then the restriction L' of L to $t(S)$ is a closed covering with order $\leq n$. It is easy to see that L can be slightly swelled to an open covering U of $t(S)$ with order $\leq n$. Since mesh U can be arbitrarily small and $t(S)$ is compact, this implies that $\dim t(S) \leq n-1$, which is a contradiction. Thus the proposition is proved.

C) Let R be a compact metric space with $\dim R \geq n$. Then there is a continuous mapping ψ from R onto an n-dimensional subset R' of E^n such that $\rho(x,y) \geq \rho'(\psi(x), \psi(y))$ for every $x,y \in R$, where ρ and ρ' denote the metric of R and E^n, respectively.

Proof. Let $a_1,\ldots,a_k \in R$ be fixed points. Then define $\varphi : R \to E^k$ by

$$\varphi(x) = \left(\frac{\rho(a_1,x)}{\sqrt{k}}, \ldots, \frac{\rho(a_k,x)}{\sqrt{k}} \right) .$$

It is easy to see that φ satisfies $\rho(x,y) \geq \rho'(\varphi(x), \varphi(y))$. Now we can select $a_1,\ldots,a_k$ as follows to make φ satisfy another condition, namely $\dim \varphi(R) \geq n$.

Let ε be a positive number which satisfies the condition of 5 A). Select $a_1,\ldots,a_k$ in such a way that for each $x \in R$ there is a_i satisfying $\rho(a_i,x) < \varepsilon/2$, which is possible because R is compact. Now, let x and y be two points of R such that $\rho(x,y) > \varepsilon$. Then choose a_i satisfying the above condition, which implies that

$$\rho(a_i,y) \geq \rho(x,y) - \rho(a_i,x) > \varepsilon/2 ,$$

and accordingly $\rho(a_i,x) \neq \rho(a_i,y)$. Thus $\varphi(x) \neq \varphi(y)$. This proves that φ is an ε-mapping. Hence by 5 A) we get $\dim \varphi(R) \geq n$. By B) there is an n-dimensional coordinate plane E^n such that the projection $\pi(\varphi(R)) = R'$ of $\varphi(R)$ in E^n has $\dim \geq n$. Now the mapping $\psi = \pi \circ \varphi$ satisfies the desired condition.

D) Let $\langle R, \rho \rangle$ be a compact metric space with $\dim R \geq n$. Then $N(\varepsilon, R, \rho) \geq c/\varepsilon^n$ holds for a positive constant c and sufficiently small $\varepsilon > 0$.

Proof. Let φ be a continuous mapping of R onto $R' \subset E^n$ satisfying the condition of C). Then since $\dim R' \geqq n$, by Theorem IV.5 R' contains an n-dimensional simplex. Hence it follows from A) that $N(\varepsilon, R, \rho) \geqq N(\varepsilon, R', \rho') \geqq c/\varepsilon^n$ for some $c > 0$.

E) Let R be a compact metric space with $\dim R \leqq n$. Then there is a topological mapping g from R into I^{2n+1} such that

$$N(\varepsilon_i, g(R), \rho) \leqq 1/\varepsilon_i^{n+\delta_i}, \quad i = 1, 2, \ldots$$

for some monotone decreasing sequences $\{\varepsilon_i\}$ and $\{\delta_i\}$ of positive numbers, both converging to 0, where ρ denotes the Euclidean metric of I^{2n+1}.

Proof. The proof will be carried out by repeated use of 4 A) and 4 B) (actually the method used to prove 4 B)).

Let $V_1 > V_2 > \ldots$ be a sequence of finite open coverings of I^{2n+1} with mesh $V_i \to 0$ and ord $V_i \leqq n+1$. Denote by $D(V_i)$ the set of all V_i-mappings in $C(R, I^{2n+1})$. First select a Kuratowski mapping $g_1 \in D(V_1)$ (see the proof of 4 B)). Then $g_1(R)$ is a compact subset of an n-dimensional polyhedron in I^{2n+1}. Hence by A) $N(\varepsilon, g_1(R), \rho) \leqq c/\varepsilon^n$ holds for a constant c and sufficiently small $\varepsilon > 0$. Thus we can select ε_1 and δ_1 such that

$$0 < \varepsilon_1 < 1, \quad 0 < \delta_1 < 1, \quad \text{and} \quad N(\varepsilon_1, g_1(R), \rho) < 1/\varepsilon_1^{n+\delta_1}$$

and a finite collection U_1 of open sets of I^{2n+1} which covers $g_1(R)$ satisfying mesh $U_1 \leqq 2\varepsilon_1$ and $|U_1| = N(\varepsilon_1, g_1(R), \rho)$. Put

$$\eta_1 = \rho(g_1(R), I^{2n+1} - \cup U_1) > 0 .$$

Choose $\xi_1 > 0$ such that

$$\xi_1 < \min\{1/2, \eta_1\}, \quad \text{and} \quad \overline{S_{\xi_1}(g_1)} \subset D(V_1) \quad \text{(see 4 A))}.$$

Next select a Kuratowski mapping $g_2 \in D(V_2) \cap S_{\xi_1}(g_1)$. Then $g_2(R)$ is a compact subset of the intersection of an n-dimensional polyhedron and $S_{\eta_1}(g_1(R)) \subset \cup U_1$. Thus we can find ε_2 and δ_2 such that

$$0 < \varepsilon_2 < \min\{1/2, \varepsilon_1\}, \quad 0 < \delta_2 < \min\{1/2, \delta_1\}, \quad \text{and}$$

$$N(\varepsilon_2, g_2(R), \rho) < 1/\varepsilon_2^{n+\delta_2}$$

and a finite open collection U_2 in I^{2n+1} which covers $g_2(R)$ satisfying $\bar{U}_2 < U_1$, mesh $U_2 \leqq 2\varepsilon_2$, and $|U_2| = N(\varepsilon_2, g_2(R), \rho)$. Put $\eta_2 = \rho(g_2(R), I^{2n+1} - \cup U_2)$ and choose $\xi_2 > 0$ satisfying

$$\xi_2 < \min\{1/2^2, \eta_2\}, \text{ and } \overline{S_{\xi_2}(g_2)} \subset D(V_2) \cap S_{\xi_1}(g_1).$$

Continuing this process we obtain monotone decreasing positive sequences $\{\varepsilon_i\}$ and $\{\delta_i\}$, a positive sequence $\{\xi_i\}$, $g_i \in D(V_i)$, $i = 1,2,\ldots$ and finite open collections $\mathcal{U}_i$, $i = 1,2,\ldots$ in I^{2n+1} such that

(1) $\lim \varepsilon_i = \lim \delta_i = 0$,

(2) $\xi_i < 1/2^i$,

(3) $\overline{S_{\xi_i}(g_i)} \subset D(V_i) \cap S_{\xi_{i-1}}(g_{i-1})$,

(4) $g_i(R) \subset \cup \mathcal{U}_i$,

(5) $\bar{\mathcal{U}}_{i+1} < \cup \mathcal{U}_i$,

(6) mesh $\mathcal{U}_i \leqq 2\varepsilon_i$,

(7) $|\mathcal{U}_i| = N(\varepsilon_i, g_i(R), \rho) < 1/\varepsilon_i^{n+\delta_i}$

Then by (2) and (3) $\{g_i\}$ converges to g in $C(R, I^{2n+1})$. It follows from (3) that $g \in \bigcap_{i=1}^{\infty} D(V_i)$. Hence g is a topological mapping from R into I^{2n+1}. From (4) and (5) it follows that $g(R)$ is covered by every $\mathcal{U}_i$, $i = 1,2,\ldots$. By (6) and (7) we get

$$N(2\varepsilon_i, g(R), \rho) < 1/\varepsilon_i^{\,n+\delta_i}, \quad i = 1,2,\ldots$$

By modifying the values of ε_i and δ_i slightly, we obtain the validity of E).

Definition IV. 5. *Let* $\langle R, \rho \rangle$ *be a metric space. Then* $k(R,\rho) = \sup\{\inf\{-\frac{\log N(\varepsilon, R, \rho)}{\log \varepsilon} \mid 0 < \varepsilon < \varepsilon_0\} \mid \varepsilon_0 > 0\}$.

The following theorem, which we owe to L. Pontrjagin-L. Schnirelmann [1], is interesting in the sense that dimension is characterized in terms of the global cardinality of coverings while it was originally defined in terms of local cardinality, order of covering.

Theorem IV. 11. *If* R *is a compact metrizable space, then*

$$\dim R = \inf\{k(R,\rho) \mid \rho \text{ is a metric for } R\}.$$

Proof. The inequality $\inf\{k(R,\rho)\} \geqq \dim R$ follows from D), and so does $\inf\{k(R,\rho)\} \leqq \dim R$ from E).

Corollary [9]. *If* R *is a separable metrizable space, then*

$$\dim R = \inf \{ k(R,\rho) \mid \rho \text{ is a totally bounded metric for } R \}.$$

Proof. Generally, let $\langle X', \rho' \rangle$ be a totally bounded metric space and X a dense subset of X'. Then it is almost obvious that $N(\varepsilon, X', \rho') = N(\varepsilon, X, \rho)$ holds for every $\varepsilon > 0$, where ρ is the restriction of ρ' to X. Thus $k(X', \rho') = k(X, \rho)$ follows.

By Theorem V.2 (to be proved later) there is a completion R^* of R (with an appropriate totally bounded metric) such that $\dim R^* = \dim R$. Then by I.2 D) R^* is compact. Hence by Theorem IV.11 there is a metric ρ^* for R^* such that $\dim R^* = k(R^*, \rho^*)$. Thus $\dim R = k(R, \rho)$, where ρ is the restriction of ρ^* to R, and hence $\dim R \geqq \inf \{ k(R,\rho) \}$.

On the other hand, let ρ be a given totally bounded metric for R. Then the completion $\langle R', \rho' \rangle$ of $\langle R, \rho \rangle$ is compact. Hence by the preceding Theorem

$$\dim R \leqq \dim R' \leqq k(R', \rho') = k(R, \rho),$$

which implies that $\dim R \leqq \inf \{ k(R,\rho) \}$. Thus we obtain the desired equality.

IV. 7. Dimension and measure

There is a remarkable connection between the concept of dimension and the concept of measure. Measure is not a topological concept, but a separable metric space of dimension $\leq n$ is characterized as a space which is homeomorphic to a space with n-dimensional measure zero (defined in the following). The purpose of this section is to show this connection and apply it to an interesting metrization of n-dimensional separable metric spaces.

Definition IV. 6. *Let* R *be a separable metric space and* p *a non-negative number. Define*

$$m_p^\varepsilon(R) = \inf \{ \sum_{i=1}^{\infty} [\delta(A_i)]^p \mid R = \bigcup_{i=1}^{\infty} A_i,\ \delta(A_i) < \varepsilon,\ i = 1,2,\dots \}, \text{ and}$$

$$m_p(R) = \sup \{ m_p^\varepsilon(R) \mid \varepsilon > 0 \},$$

where we denote by $\delta(A)$ *the diameter of* A *and put* $[\delta(A)]^0 = 0$ *if* $A = \emptyset$, *and* $[\delta(A)]^0 = 1$ *if* $A \neq \emptyset$. *Then we call* $m_p(R)$ *the p-dimensional measure of* R.

[9] This generalization of Pontrjagin-Schnirelmann's theorem is due to J. Bruijning [1].

We can easily verify that this measure is an outer measure and that for a subset R of E^n $m_n(R)=0$ holds if and only if the Lebesgue outer measure $m^*(R)$ of R vanishes [10]. The verifications of the following assertions A) and B) are left to the reader.

A) $m_0(R) = n$ if R is a set of n points.

B) An n-dimensional polyhedron has $n+1$-dimensional measure zero.

C) Let $m_{p+1}(R) = 0$ for a separable metric space R and x a given point of R. Then $m_p(B(S_r(x))) = 0$ holds for almost every positive number r, i.e. $m^*(A) = 0$ for $A = \{ r \mid r>0,\ m_p(B(S_r(x)))>0 \}$.

Proof. Since $m_{p+1}(R)=0$, there exist coverings $\{ A_i^m \mid i=1,2,\dots \}$, $m=1,2,\dots$ of R such that $\delta(A_i^m) < 1/m$, and

$$(1) \qquad \lim_{m\to\infty} \sum_{i=1}^{\infty} [\delta(A_i^m)]^{p+1} = 0$$

Define $a_{mi} = \sup\{ \rho(x,y) \mid y \in A_i^m \}$ and $b_{mi} = \inf\{ \rho(x,y) \mid y\in A_i^m \}$. Then it is clear that

$$(2) \qquad \delta(A_i^m) \geqq a_{mi} - b_{mi}$$

We define functions d_{mi} and d_m on $[0,\infty)$ as follows:

$$(3) \qquad d_{mi}(r) = \begin{cases} 0 & \text{if } 0 \leqq r < b_{mi} \text{ or } r > a_{mi}, \\ [\delta(A_i^m)]^p & \text{if } b_{mi} \leqq r \leqq a_{mi}, \end{cases}$$

$$d_m(r) = \sum_{i=1}^{\infty} d_{mi}(r) .$$

From (2) and (3) it follows that for any $k>0$

$$\int_0^k d_{mi}(r)dr \leqq [\delta(A_i^m)]^{p+1}$$

which combined with (1) implies

$$\lim_{m\to\infty} \sum_{i=1}^{\infty} \int_0^k d_{mi}(r)dr = 0 .$$

[10] As for measure theory, see for example S. Saks [1] or H. Royden [1].

Since $d_{mi}(r) \geqq 0$, we can interchange integration and summation in the preceding formula and thus obtain

$$\lim_{m\to\infty} \int_0^k \sum_{i=1}^{\infty} d_{mi}(r)dr = \lim_{m\to\infty} \int_0^k d_m(r)dr = 0 .$$

Hence we can find a subsequence $\{m_j \mid j=1,2,\dots\}$ such that

(4) $$\lim_{j\to\infty} d_{m_j}(r) = 0 \text{ for almost all } r .$$

The definition (3) of d_{mi} implies

$$[\delta(A_i^{m_j} \cap B(S_r(x)))]^p \leqq d_{m_j i}(r) \text{ for every } r ,$$

and hence from (4) we can deduce

$$\lim_{j\to\infty} \sum_{i=1}^{\infty} [\delta(A_i^{m_j} \cap B(S_r(x)))]^p = 0 \text{ for almost all } r .$$

Since $\{A_i^{m_j} \cap B(S_r(x)) \mid i=1,2,\dots\}$, $j = 1,2,\dots$ are coverings of $B(S_r(x))$ with $\delta(A_i^{m_j} \cap B(S_r(x))) < 1/m_j$ we get $m_p(B(S_r(x))) = 0$ for almost all r .

D) If $m_{n+1}(R) = 0$ for a separable metric space R , then $\dim R \leqq n$.

Proof. If $n = -1$, then this assertion is obviously true. We assume D) for every space R' with $m_n(R') = 0$ in an inductive proof. Let $m_{n+1}(R) = 0$; then from C) it follows that for every neighbourhood $U(x)$ of every point x of R there exists a spherical neighbourhood $S_r(x)$ satisfying $S_r(x) \subset U(x)$ and $m_n(B(S_r(x))) = 0$. By use of the induction hypothesis we obtain $\dim B(S_r(x)) \leqq n-1$. This means $\operatorname{ind} R \leqq n$. Since R is a separable metric space, we conclude $\dim R \leqq n$.

E) A separable metric space R of dimension $\leqq n$ is homeomorphic to a subset S of I^{2n+1} such that $m_{n+1}(S) = 0$.

Proof. We still denote by $C(R, I^{2n+1})$ the complete metric space of all continuous mappings of R into I^{2n+1} as in the proof of the imbedding theorem. Moreover, let

$$D_k = \{ f \mid f \in C(R, I^{2n+1}),\ m_{n+1}^{1/k}(\overline{f(R)}) < 1/k \} .$$

Then every D_k contains the set D consisting of all continuous mappings f such that $m_{n+1}(\overline{f(R)}) = 0$. Since by B) any n-dimensional polyhedron has $n+1$-dimensional measure zero, by 4 C) D_k is dense in $C(R, I^{2n+1})$.

To show that D_k is open, we suppose f is a given point of D_k . Then we can choose ε satisfying

$$m_{n+1}^{1/k}(\overline{f(R)}) < \varepsilon < 1/k \ .$$

Thus we can construct a covering $\{A_i \mid i = 1,2,\dots\}$ of $\overline{f(R)}$ such that $\delta(A_i) < 1/k$ and

(1) $$\sum_{i=1}^{\infty} [\delta(A_i)]^{n+1} < \varepsilon \ .$$

Now it is easy to see that we can construct open sets U_i, $i = 1,2,\dots$ in I^{2n+1} such that

(2) $$A_i \subset U_i \ , \quad \delta(U_i) < 1/k \ , \text{ and } \ [\delta(U_i)]^{n+1} < [\delta(A_i)]^{n+1} + (\tfrac{1}{k} - \varepsilon)/2^i$$

which combined with (1) implies

(3) $$\sum_{i=1}^{\infty} [\delta(U_i)]^{n+1} < 1/k \ .$$

Since $\overline{f(R)}$ is compact

$$\rho(\overline{f(R)}, \ I^{2n+1} - \bigcup_{i=1}^{\infty} U_i) = \eta > 0 \ .$$

Let g be an arbitrary point of $C(R, I^{2n+1})$ satisfying $\rho'(f,g) < \frac{1}{2}\eta$, where we denote by ρ' the metric of $C(R, I^{2n+1})$. Then we can easily verify $\overline{g(R)} \subset \bigcup_{i=1}^{\infty} U_i$. Hence from (2) and (3) it follows that

$$m_{n+1}^{1/k}\overline{(g(R))} < 1/k \ .$$

This proves $g \in D_k$. In consequence $S_{\frac{1}{2}\eta}(f) \subset D_k$. Thus each D_k is an open dense set of $C(R, I^{2n+1})$. Now we denote by $D(V_k)$ the set of all V_k-mappings with V_k as defined in the proof of 6 E). Then we obtain, by use of Theorem I.5,

$$\bigcap_{k=1}^{\infty} [D_k \cap D(V_k)] \neq \emptyset \ .$$

Any mapping contained in this intersection is a homeomorphism of R into I^{2n+1} which satisfies $m_{n+1}(f(R)) = 0$.

Combining E) with D), we obtain

Theorem IV. 12. [11] *A separable metric space R has dimension $\leq n$ if and only if R is homeomorphic to a subset S of I^{2n+1} with $m_{n+1}(S) = 0$.*

[11] This theorem is due to E. Szpilrajn [1]. Szpilrajn's theorem and Pontrjagin-Schnirelmann's theorem are based on somewhat similar ideas. Let $\langle R, \rho \rangle$ be a separable metric space; then $hd(R,\rho) = \sup\{p \mid m_p(R) > 0\}$ is called the *Hausdorff dimension* of $\langle R, \rho \rangle$. J. Hawkes [1] proved $hd(R,\rho) \leq k(R,\rho)$ for every compact metric space. But they do not coincide in general.

Corollary. *A separable metric space* R *has dimension* $\leq n$ *if and only if one can introduce a metric* ρ *in* R *such that for almost all* $r>0$ *the boundary* $B(S_r(x))$ *of the spherical neighbourhood* $S_r(x)$ *of each point* x *has dimension* $\leq n-1$.

Proof. This statement is directly deduced from Theorem IV.12 in combination with C) and Theorem IV.1.

Another interesting application of the imbedding technique can be found in L. Janos' theory to characterize dimension by use of "bisectors". A metric d for X is called *strongly rigid* if $d(x,y) \neq d(u,v)$ whenever (x,y) and (u,v) are different pairs of different points. L. Janos [1] proved that: $\dim X = 0$ *holds for a non-empty separable metrizable space* X *if and only if* X *admits a compatible strongly rigid metric.*

Let Y and Z be subsets of a metric space $\langle X,d \rangle$. Then Z is called a *bisector* of Y (denoted by $Y \triangleright Z$) if $Y \supset Z$ and there are distinct points $y_1, y_2 \in Y$ such that $Z = \{ z \in Y \mid d(z,y_1) = d(z,y_2) \}$. A chain $X = X_0 \triangleright X_1 \triangleright X_2 \triangleright \ldots \triangleright X_{n-1} \triangleright X_n$ of subsets of X is called a *reduced bisector chain* if $\dim X_{n-1} > 0$ and $\dim X_n \leq 0$, and n is called the *length* of the chain. Denote by $r(X,d)$ the maximal length of reduced bisector chains in $\langle X,d \rangle$, and let $r(X) = \min \{ r(X,d) \mid d$ is a metric for $X \}$. Then L. Janos [2] proved:

$r(X) = \dim X$ *if* X *is a non-empty compact metrizable space.*

An important role in his theory is played by J. H. Roberts' [1] theorem:

If X *is a separable metric space with* $\dim \leq n$, *then there is a topological imbedding* $f: X \to E^{2n+1}$ *such that* $\dim(f(X) \cap Y) \leq 0$ *for every* $(n+1)$*-dimensional plane* Y *of* E^{2n+1}.

The first theorem of Janos was extended by H. Martin [1] to more general metric spaces, and the second was extended by L. Janos - H. Martin [1] to separable metrizable spaces.

J. H. Roberts [1] showed that his theorem above practically implies the following stronger theorem:

If X *is a separable metric space with* $\dim \leq n$, *then there is a topological imbedding* $f: X \to E^{2n+1}$ *such that for every k-dimensional plane* Y *of* E^{2n+1} *we have* $\dim(f(X) \cap Y) \leq k-n-1$.

This theorem implies the following theorem which improves the corollary of Theorem IV.12:

A separable metric space R *has dimension* $\leq n$ *if and only if one can introduce a metric* ρ *in* R *such that for every* $r>0$ *the boundary* $B(S_r(x))$ *of the spherical neighbourhood* $S_r(x)$ *of each point* x *has* $\dim \leq n-1$.

Another interesting imbedding theorem for n-dimensional compact metric spaces can be found in M. A. Štanko [1].

IV. 8. Dimension and the ring of continuous functions

Let $C(R)$ denote the ring of all real-valued continuous functions defined on R. It is well-known that the topology of R is completely characterized by the ring structure of $C(R)$ if R is a compact Hausdorff space. In this respect it is an interesting problem to study relations between the dimension of R and the algebraic structure of $C(R)$.

M. Katětov [1] defined dimension of $C(R)$ and established a remarkable connection between the dimension of $C(R)$ and that of R. The purpose of this section is to give a quick review of his theory in the most interesting case in which R is a compact metric space.

Note that in the following discussion in the present section *all spaces are assumed to be compact metric,* although some propositions are true for more general spaces.

A) Let $\mathcal{U}$ be an open covering of a space R and T a closed set of R. Then $\tilde{\delta}(T) < \mathcal{U}$ if and only if the decomposition of T into its components is a refinement of $\mathcal{U}$. (Recall Definition III.8.)

Proof. The 'only if' follows directly from Definition III.8.

To prove the 'if' part, let us denote by $K(x)$ the component of T which contains $x \in T$. Then $K(x)$ will be proved to be an intersection of sets which are closed and open in T. Put $K' = \cap \{F \mid F$ is a closed and open set in T such that $F \supset K(x)\}$.

To prove that K' is connected, assume the contrary. Then $K' = K_1 \cup K_2$ where $K_1 \cap K_2 = \emptyset$ for some non-empty closed sets K_1 and K_2 in R. There are open sets U_1 and U_2 in R such that $U_1 \supset K_1$, $U_2 \supset K_2$, $U_1 \cap U_2 = \emptyset$. Since $U_1 \cup U_2 \supset K'$ and since R is compact, there is a set F closed and open in T satisfying $K(x) \subset K' \subset F \subset U_1 \cup U_2$.

Assume e.g. $x \in F \cap U_1 = F \cap (R - U_2)$. Then $F \cap U_1$ is closed and open in T and accordingly contains $K(x)$, because $K(x)$ is connected. Hence $F \cap U_1 \supset K'$, which implies $K_2 \cap K' = \emptyset$, and thus $K_2 = K_2 \cap K' = \emptyset$, which is a contradiction. Therefore K' is connected, and hence $K' = K(x)$ follows because $K(x)$ is a component.

Now recall that we have assumed the condition $\{K(x) \mid x \in T\} < \mathcal{U}$. Because of the above argument we can select a closed and open set $F(x)$ in T and $U(x) \in \mathcal{U}$ such that $K(x) \subset F(x) \subset U(x)$ for each $x \in T$. Since T is compact we can cover it by finitely many of the $F(x)$'s, say $T = \cup_{j=1}^{k} F(x_j)$. Put

$$F_j = F(x_j) - F(x_1) \cup \ldots \cup F(x_{j-1});$$

then each F_j is a closed set in R satisfying $F_j \subset U(x_j)$, $T = \bigcup_{j=1}^{k} F_j$, and $F_j \cap F_l = \emptyset$ if $j \neq l$.

We can find open sets V_j, $j = 1, \dots, k$ in R such that $F_j \subset V_j \subset U(x_j)$. Then $\{ V_j \mid j = 1, \dots, k \}$ is an open collection which refines U, covers T and has order 1. Thus $\tilde{\delta}(T) < U$.

B) A space R is totally disconnected (i.e. each component is a singleton) if and only if $\dim R \leqq 0$. (This proposition is generally true for every compact Hausdorff space).

Proof. The proof easily follows from the fact that each component of R is an intersection of closed and open sets, which was proved in the proof of A), but the detail is left to the reader.

C) Let U be an open covering of a space R and f a continuous mapping from R into another space S such that $\tilde{\delta}(f^{-1}(y)) < U$ for $y \in Y$. Then there is an open neighbourhood W of y in S such that $\tilde{\delta}(f^{-1}(W)) < U$.

Proof. Let V be an open collection which satisfies $V < U$, $f^{-1}(y) \subset \bigcup V$, and ord $V \leqq 1$. Then $W = Y - f(X - \bigcup V)$ satisfies the desired condition.

Definition IV. 7. *A subset C_1 of $C(R)$ is called an analytical subring if it satisfies*

i) *C_1 is a subring of $C(R)$,*

ii) *every constant belongs to C_1,*

iii) *if $f \in C(R)$ and $f^2 \in C_1$, then $f \in C_1$,*

iv) *C_1 is closed in the metric space $C(R)$ with the metric*
$\rho(f,g) = \sup\{ |f(x) - g(x)| \mid x \in R \}$.

A subset D of $C(R)$ is called an analytical base if every analytical subring containing D is equal to $C(R)$. Finally, the analytical dimension of $C(R)$, a - dim $C(R)$*, is defined as the least cardinality of analytical bases of $C(R)$.*

Note that a - dim $C(R) = 0$ means that $\emptyset$ is an analytical base of $C(R)$, i.e. $C(R)$ is the only analytical subring of $C(R)$.

D) Let C_1 be an analytical subring of $C(R)$. If $g \in C(R)$ takes on only finitely many distinct values, then $g \in C_1$.

Proof. Assume that g takes on ℓ distinct values.

1) In case of $\ell = 1$ the assertion is obviously true.

2) Let $\ell = 2$, and $g(R) = \{ a, b \}$. Then

$$\left(g - \frac{a+b}{2}\right)^2 = \left(\frac{a-b}{2}\right)^2 \in C_1 .$$

Hence from condition iii) it follows that $g-(a+b)/2 \in C_1$. Therefore $g \in C_1$ by the condition ii).

3) To prove the general case, assume that the assertion is true for every function which takes on at most $\ell-1$ distinct values. Let $g \in C(R)$ take on ℓ values $a_1,\dots,a_\ell$. Then define $h \in C(R)$ by

$$h(x) = a_i \text{ if } x \in g^{-1}(a_i) \text{ for some } i \text{ satisfying } 1 \leq i \leq \ell - 1 ,$$
$$h(x) = a_{\ell-1} \text{ if } x \in g^{-1}(a_\ell) .$$

Then $h \in C_1$ follows from the induction hypothesis. Note that $g-h$ is identical to 0 or $a_\ell - a_{\ell-1}$. Thus from the case 2) it follows that $g-h \in C_1$. This yields $g \in C_1$.

E) Let R be a non-empty space; then $\text{a-dim}\, C(R) = 0$ if and only if $\dim R = 0$.

Proof. Assume $\dim R = 0$. We consider an arbitrary analytical subring C_1 of $C(R)$, $f \in C(R)$, and $\varepsilon > 0$. Since R is compact, $\{f^{-1}(S_{\frac{\varepsilon}{2}}(y)) \mid y \in R\}$ has a finite subcovering, say $U = \{f^{-1}(S_{\frac{\varepsilon}{2}}(y_i)) \mid i = 1,\dots,k\}$. Choose an open covering $W = \{W_1,\dots,W_k\}$ such that $W < U$ and $\operatorname{ord} W = 1$. Select $x_j \in W_j$, $j = 1,\dots,k$ to define $g \in C(R)$ by $g(x) = f(x_j)$ if $x \in W_j$. Then it is easy to see that $\rho(f,g) < \varepsilon$ in the metric space $C(R)$. Besides $g \in C_1$ follows from D). Thus $f \in \bar{C}_1 = C$ is obtained by the condition iv). This proves that $C_1 = C(R)$. Therefore $\text{a-dim}\, C(R) = 0$.

Conversely, let us assume that $\text{a-dim}\, C(R) = 0$, and let K be a component of R. Now $C_1 = \{f \in C(R) \mid f \text{ is constant on } K\}$ is obviously an analytical subring of $C(R)$. Since $\text{a-dim}\, C(R) = 0$ implies $C(R) = C_1$, K is a singleton. Hence R is totally disconnected. Thus by B) we conclude that $\dim R = 0$.

F) Let $\dim R \leq n$ $(1 \leq n \leq \infty)$. Then $\text{a-dim}\, C(R) \leq n$.

Proof. By Theorem III.10 there is a uniformly 0-dimensional continuous mapping f from R into I^n. Put $f = (f_1,\dots,f_n)$, where $f_i \in C(R)$, $i = 1,\dots,n$. Then we claim that $\{f_1,\dots,f_n\}$ is an analytical base for $C(R)$. As will be seen in the following, we can prove our claim by use of Theorem I.8. Let C be an analytical subring which contains $f_1,\dots,f_n$. Suppose p and q are distinct points of R. Then we can find $h \in C_1$ satisfying $h(p) \neq h(q)$, as follows. Let

(1) $\rho(p,q) > \varepsilon > 0$.

Then we choose $\eta > 0$ such that for every subset T of I^n with $\delta(T) \leq \eta$, the following holds:

(2) $\tilde{\delta}(f^{-1}(T)) < \mathcal{U}_\varepsilon = \{U \mid U$ is an open set of R with $\delta(U) \leq \varepsilon\}$.

Now we choose $\xi > 0$ such that

(3) $$\delta[\prod_{i=1}^{n} (r_i - \xi, r_i + \xi)] \leq \eta \text{ for every } (r_1, \dots, r_n) \in I^n ,$$

where $(r_i - \xi, r_i + \xi)$ denotes the open segment $\{x \mid r_i - \xi < x < r_i + \xi\}$. Put $f(p) = (r_1, \dots, r_n)$, i.e. $f_i(p) = r_i$, $i = 1, \dots, n$, $J = \prod_{i=1}^{n} (r_i - \xi, r_i + \xi)$, and $U = f^{-1}(J)$. Then U is an open neighbourhood of p.

Now we define $g_i \in C(R)$, $i = 1, \dots, n$ by

$$g_i(x) = \frac{1}{\xi} (\xi - | f_i(x) - r_i |) , \; x \in R .$$

Since $|f_i - r_i|^2 = (f_i - r_i)^2 \in C_1$, it follows from iii) of Definition IV.7 that $|f_i - r_i| \in C_1$. Therefore $g_i \in C_1$. Define $g \in C(R)$ by $g = \pi_{i=1}^{n} g_i$. Then $g \in C_1$. It follows from the definitions of g and g_i that

(4) $g(p) = 1$,

(5) $g(x) = 0$ for every $x \in B(U)$.

Note that $\delta(J) \leq \eta$ is implied by (3). Hence we can choose, in view of (2), an open covering $\mathcal{V}$ of $U = f^{-1}(J)$ with mesh $\mathcal{V} \leq \varepsilon$ and ord $\mathcal{V} \leq 1$. Put $V = S(p, \mathcal{V})$; then V is an open neighbourhood of p satisfying

(6) $p \in V$ and $q \notin V$ (see (1)).

Note that $B(V) \subset B(U)$. Finally, we define $h \in C(R)$ by

(7) $h(x) = g(x)$ for $x \in V$, and $h(x) = 0$ for $x \notin V$.

(Recall (5) to verify $h \in C(R)$.) Now it is obvious that $(h - (g/2))^2 = (g/2)^2 \in C_1$, which implies $h - (g/2) \in C_1$ and hence $h \in C$. By (4), (6) and (7) we obtain $h(p) = g(p) = 1$, and by (6) and (7) $h(q) = 0$. Thus we can apply Theorem I.8 to conclude that $\bar{C}_1 = C(R)$, which implies $C_1 = C(R)$ because of iv) of Definition IV.7. This proves that $\{f_1, \dots, f_n\}$ is an analytical base for $C(R)$. Therefore a-dim $C(R) \leq n$.

G) Let $\{f_1, \dots, f_n\}$ be an analytical base for $C(R)$. Then the mapping $f = (f_1, \dots, f_n): R \to E^n$ satisfies: For every binary open covering $\mathcal{U}$ of R and for every $y \in f(R)$, $\tilde{\delta}(f^{-1}(y)) < \mathcal{U}$.

Proof. Note that $C_1 = \{g \in C(R) \mid g$ is constant on each component of $f^{-1}(y)\}$ is an analytical subring of $C(R)$ such that $f_1, \dots, f_n \in C_1$. Hence $C_1 = C(R)$.

Let $\mathcal{U} = \{U_1, U_2\}$ and put $F_1 = R - U_1$ and $F_2 = R - U_2$. Then there is a continuous function $h: R \to [0,1]$ such that $h(F_1) = 0$ and $h(F_2) = 1$. Since $h \in C_1$ follows from the above note, each component of $f^{-1}(y)$ cannot hit both F_1 and F_2 at the same time, i.e. it is contained either in U_1 or in U_2. Hence by A) we obtain $\tilde{\delta}(f^{-1}(y)) < \mathcal{U}$.

H) If a-dim $C(R) \leqq n$, then $\dim R \leqq n$. $(1 \leqq n \leqq \infty)$.

Proof. Assume that $\{f_1, \ldots, f_n\}$ is an analytical base for $C(R)$. Then $f = (f_1, \ldots, f_n)$ is a function satisfying the condition of G). Since $f(R)$ is a compact subset of E^n, we may assume $f(R) \subset I^n$. Decompose I^n by use of the decomposition theorem as $I^n = Y_0 \cup \ldots \cup Y_n$ with $\dim Y_i = 0$. Put $X_i = f^{-1}(Y_i)$, $i = 0, \ldots, n$; then $R = \bigcup_{i=0}^{n} X_i$.

Consider $x \in X_i$ and an open neighbourhood U of x in X_i. Find an open set V in R such that $V \cap X_i = U$. Then put $\mathcal{U} = \{R - \{x\}, V\}$. By the above mentioned property of f we obtain $\tilde{\delta}(f^{-1}(y)) < \mathcal{U}$, where $y = f(x)$. By C) there is an open neighbourhood W of y in I^n such that

$$(1) \qquad \tilde{\delta}(f^{-1}(W)) < \mathcal{U} .$$

Since $\dim Y_i = 0$, we may assume that $W \cap Y_i$ is closed and open in Y_i. Hence $f^{-1}(W) \cap X_i$ is closed and open in X_i. By (1) there is an open covering $\mathcal{P}$ of $f^{-1}(W)$ such that

$$(2) \qquad \mathcal{P} < \mathcal{U} \text{ and } \operatorname{ord} \mathcal{P} \leqq 1 .$$

Let P be the element of $\mathcal{P}$ which contains x. Then P is a closed and open neighbourhood of x in X_i such that $P \subset V \cap X_i = U$ because of (2). This proves ind $X_i \leqq 0$, and accordingly $\dim X_i \leqq 0$ follows. Hence again by use of the decomposition theorem we conclude that $\dim R \leqq n$.

Theorem IV. 13. *If R is a non-empty compact metric space, then either* $\dim R$ *and* a-dim $C(R)$ *are both finite and equal, or they are both infinite* [12].

Proof. Combine E), F) and H).

[12] Extensions to more general spaces of this theorem of Katětov were proposed in a number of articles. We just mention J. Hejcman [1] who extended it to uniform spaces. We shall discuss later (Chapter VI) a completely different definition of dim $C(R)$ to characterize the dimension of non-metrizable spaces. N. T. Peck [1] found a partial but interesting characterization of $\dim R$ of a compact metric space R in terms of a Banach space property of $C(R, E^n)$.