CHAPTER III

MAPPINGS AND DIMENSION

In the preceding chapter our investigations were chiefly conducted on the basis of the theory of coverings. In this chapter we shall concern ourselves with mappings, another powerful tool for dimension theory. In Section 7 we shall study the theory of uniformly 0-dimensional mappings which has been recently developed by M. Katětov while in the other sections the main object will be the extension of classical theorems to general metric spaces.

III. 1. Stable value

Definition III. 1. *Let f be a continuous mapping of a space R into a space S. A point q of $f(R)$ is called an unstable value of f if for every $\varepsilon > 0$ there exists a continuous mapping g of R into S such that $\rho(f(p), g(p)) < \varepsilon$ for every $p \in R$, $g(R) \subset S - \{q\}$, where we denote by ρ the metric of S. A point q of $f(R)$ is called a stable value if it is not unstable.*

Example III.1. We consider a continuous mapping $f(x) = x^2$ of the 1-dimensional Euclidean space E^1 into itself. Then all the values $y > 0$ are stable, while 0 is unstable. On the other hand the mapping $F(x) = (x, f(x))$ of E^1 in E^2 has only unstable values.

A) Let f be a continuous mapping of a (metric) space R into the $n+1$-dimensional cube $I^{n+1} = \{(x_1, \dots, x_{n+1}) \mid |x_i| \leq 1,\ i = 1, \dots, n+1\}$. If $\dim R \leq n$, then all values of f are unstable.

Proof. Let $f(p)$ have the coordinates $f_1(p),\dots,f_{n+1}(p)$ in I^{n+1}. To begin with, let us show that every point of $B(I^{n+1})$ is unstable. For any given $\varepsilon > 0$, the mapping

$$g(p) = (g_1(p),\dots,g_{n+1}(p)),\ g_i(p) = (1-\varepsilon)f_i(p),\ i = 1,\dots,n+1$$

is a mapping whose image contains no boundary point of I^{n+1}. Since we can easily see that $\rho(f(p),\ g(p)) \leqq \varepsilon\sqrt{n+1}$ for every $p \in R$, our assertion is true.

Now, let us prove that any interior point q of I^{n+1} is unstable. To this end we may assume q to be the origin $(0,\dots,0)$. Let ε be a given positive number. We put

$$F_i = \{p \mid f_i(p) \geqq \varepsilon\},\ G_i = \{p \mid f_i(p) \leqq -\varepsilon\},\ i = 1,\dots,n+1 .$$

Since F_i is a closed set contained in the open set $R - G_i$, by the corollary of Theorem II.8 there exist open sets V_i, $i = 1,\dots,n+1$ such that

(1) $$F_i \subset V_i \subset R - G_i \quad\text{and}\quad \bigcap_{i=1}^{n+1} B(V_i) = \emptyset .$$

By repeated application of the corollary to Theorem I.6, we can construct a continuous real valued function φ_i such that $|\varphi_i| \leqq \varepsilon$, $\{p \mid \varphi_i(p) = 0\} = B(V_i)$, $\{p \mid \varphi_i(p) = \varepsilon\} = F_i$, $\{p \mid \varphi_i(p) = -\varepsilon\} = G_i$. Defining g_i by $g_i(p) = f_i(p)$ for $p \in F_i \cup G_i$, and $g_i(p) = \varphi_i(p)$ for $p \in R - F_i \cup G_i$, we obtain a continuous function $g_i(p)$, which, as easily seen, satisfies

(2) $$|f_i(p) - g_i(p)| < 2\varepsilon \quad\text{for every } p \in R\text{, and}$$

(3) $$\{p \mid g_i(p) = 0\} = B(V_i) .$$

Let $g(p) = (g_1(p),\dots,g_{n+1}(p))$; then g is a continuous mapping of R in I^{n+1} and satisfies $\rho(f(p),\ g(p)) < 2\varepsilon\sqrt{n+1}$ for every $p \in R$ by virtue of (2). From (1) and (3) we obtain $(0,\dots,0) \notin g(R)$. This means that the origin $(0,\dots,0)$ is an unstable value.

B) If all values of every continuous mapping f of a space R into I^{n+1} are unstable, then $\dim R \leqq n$.

Proof. Let $\{U_i \mid i = 1,\dots,n+1\}$ be an open collection and $\{F_i \mid i = 1,\dots,n+1\}$ a closed collection such that $F_i \subset U_i$, $i = 1,\dots,n+1$. By Theorem I.6 we construct continuous functions f_i, $i = 1,\dots,k$ defined over R such that

(1) $$\begin{cases} -1 \leqq f_i \leqq 1 , \\ f_i(F_i) = -1 , \\ f_i(R - U_i) = 1 ,\ i = 1,\dots,n+1 . \end{cases}$$

Since $f = (f_1, \dots, f_{n+1})$ is a continuous mapping of R into I^{n+1}, the origin $q = (0, \dots, 0)$ of I^{n+1} is unstable. This means that there exists a continuous mapping $g = (g_1, \dots, g_{n+1})$ of R into I^{n+1} such that

(2) $q \notin g(R)$, and

(3) $\rho(f(p), g(p)) < \frac{1}{2}$ for every $p \in R$.

Then we define open sets V_i by $V_i = \{ p \mid g_i(p) < 0 \}$. From (1) and (3) it follows that $F_i \subset V_i \subset U_i$ and $B(V_i) \subset \{ p \mid g_i(p) = 0 \}$. Since (2) implies $\bigcap_{i=1}^{n+1} \{ p \mid g_i(p) = 0 \} = \emptyset$, we get $\bigcap_{i=1}^{n+1} B(V_i) = \emptyset$. Hence from the corollary to Theorem II.8 we obtain $\dim R \leqq n$.

By combining B) with A) we get

Theorem III. 1. *A space R has dimension $\leqq n$ if and only if all values of every continuous mapping of R into I^{n+1} are unstable.*

III. 2. Extensions of mappings

A) Let f be a continuous mapping of a space R into I^{n+1} such that the center q of I^{n+1} is an unstable value of f. Then there exists a continuous mapping g of R into I^{n+1} such that $q \notin g(R)$ and $g(p) = f(p)$ for every point p whose image is contained in $B(I^{n+1})$.

Proof. We may assume the boundary $B(I^{n+1})$ of I^{n+1} to be the n-sphere $S^n = \{ (x_1, \dots, x_{n+1}) \mid x_1^2 + \dots + x_{n+1}^2 = 1 \}$. Since the center q is an unstable value of $f(R)$, there exists a continuous mapping g' of R into I^{n+1} such that, in vectorial notation,

(1) $\rho(f(p), g'(p)) = | f(p) - g'(p) | < \frac{1}{3}$ for every $p \in R$, and $q \notin g'(R)$.

Then, let $F = \{ p \mid | f(p) | \leqq \frac{1}{3} \}$ and $G = \{ p \mid | f(p) | \geqq \frac{2}{3} \}$ to construct by Theorem I.6 a real valued continuous function φ over R such that $\varphi(F) = 0$, $\varphi(G) = 1$, and $0 \leqq \varphi \leqq 1$. Using this function we define a mapping g of R by

$$g(p) = g'(p) \quad \text{for} \quad p \in F,$$
$$g(p) = \varphi(p) f(p) + (1 - \varphi(p)) g'(p) \quad \text{for} \quad p \in R - F \cup G,$$
$$g(p) = f(p) \quad \text{for} \quad p \in G.$$

We can easily see that g is a continuous mapping of R into I^{n+1} and that $g(F) \cup g(G) \subset I^{n+1} - \{ q \}$, and $g(p) = f(p)$ if $f(p) \in S^n (= B(I^{n+1}))$. There only remains to prove

(2) $g(R - F \cup G) \subset I^{n+1} - \{ q \}$.

Let p be a given point of $R - F \cup G$; then $\frac{1}{3} < | f(p) | < \frac{2}{3}$, and hence, from (1) and

$$| g(p) - f(p) | = (1 - \varphi(p)) \, | g'(p) - f(p) | ,$$

we obtain

$$| g(p) | \leqq (1 - \varphi(p)) \, | g'(p) - f(p) | + | f(p) | < 1 ,$$
$$| g(p) | \geqq | f(p) | - (1 - \varphi(p)) \, | g'(p) - f(p) | > 0 ,$$

which proves (2). Thus g is the desired mapping.

Theorem III. 2. *A space R has dimension $\leqq n$ if and only if for every closed set C of R and every continuous mapping f of C into S^n there exists a continuous extension of f over R.*

Proof. To prove the necessity we regard S^n as the boundary of I^{n+1}. Then we can regard f as a mapping of C into I^{n+1}. Hence by Theorem I.7 there exists a continuous mapping f' of R into I^{n+1} such that

(1) $f'(p) = f(p)$ for $p \in C$.

Since $\dim R \leqq n$, by Theorem III.1 the center q of I^{n+1} is an unstable value of f'. Hence in view of A) we can construct a continuous mapping g of R into I^{n+1} such that

(2) $q \notin g(R)$, and $g(p) = f'(p)$ if $f'(p) \in S^n$.

Since $p \in C$ implies by (1) $f'(p) = f(p) \in S^n$, we obtain $g(p) = f(p)$ if $p \in C$. Let $\varphi(p)$ be the projection of the point $g(p)$ on the boundary S^n of I^{n+1} from the center. Such a projection is possible by virtue of (2). Now it is clear that φ is the desired extension of $f \,|\, C \to S^n$ over R.

To show the sufficiency it is enough, as Theorem III.1 asserts, to prove that every continuous mapping f of R into I^{n+1} has unstable values only. As seen in the proof of I A), any boundary point of I^{n+1} is unstable, and hence we shall prove only the unstability of the center q of I^{n+1}. For a given $\varepsilon > 0$ we consider a spherical neighbourhood U of q with radius $\frac{1}{2}\varepsilon$ and with boundary S^n. Let $C = f^{-1}(S^n)$; then by the hypothesis there exists a continuous mapping φ of R into S^n such that $\varphi(p) = f(p)$ for $p \in C$.

We define a new continuous mapping g of R into I^{n+1} as follows:

$$g(p) = f(p) \quad \text{if} \quad f(p) \notin U ,$$
$$g(p) = \varphi(p) \quad \text{if} \quad f(p) \in U .$$

Then we can easily see that g is a continuous mapping of R into $I^{n+1} - U$ such that $\rho(f(p), g(p)) < \varepsilon$, for every $p \in R$. This proves the unstability of q. Therefore from Theorem III.1 it follows that $\dim R \leqq n$.

Corollary. *Let C be a closed set of a space R. If $\dim (R - C) \leqq n$, then every continuous mapping f of C into S^n can be continuously extended over R.*

Proof. By the corollary to Theorem I.7 there is an open set U containing C and a continuous extension g of f over U. We consider an open set V such that $C \subset V \subset \bar{V} \subset U$. Then we denote by g_0 the restriction of g to $\bar{V} - C$. Since $\bar{V} - C$ is a closed subset of $R - C$ with $\dim (R - C) \leqq n$, g_0 can be continuously extended over $R - C$ by Theorem III.2. We denote this continuous extension by h. Letting $\varphi(p) = f(p)$ for $p \in C$, and $\varphi(p) = h(p)$ for $p \in R - C$, we obtain the desired continuous extension φ of f over R.

III. 3. Essential mappings

Definition III. 2. *A continuous mapping f of a space R into a space S is called homotopic to a continuous mapping g of R into S if there exists a continuous mapping $f(p,t)$ of $R \times I$, the topological product of R and the unit segment $I = \{ x \mid 0 \leqq x \leqq 1 \}$, into S such that $f(p,0) = f(p)$ $f(p,1) = g(p)$.*

Roughly speaking, f is homotopic to g if f can be continuously deformed into g.

Definition III. 3. *A continuous mapping f of a space R into the n-sphere S^n is called essential if any continuous mapping g of R into S^n which is homotopic to f satisfies $g(R) = S^n$. If f is not essential, then it is called inessential.*

Example III. 2. Let f and g be continuous mappings of a space R into S^n with diameter d. If $\rho(f(p), g(p)) < d$ for every $p \in R$, then f is homotopic to g. To see this, for each $p \in R$ we consider the minor arc of the great circle joining $f(p)$ with $g(p)$. For $p \in R$, $0 \leqq t \leqq 1$, we denote by $f(p,t)$ the point dividing this arc in the ratio $t : 1 - t$. Then $f(p,t)$ is a mapping of $R \times I$ into S^n and satisfies the condition of Definition III.2. Hence f is homotopic to g. It follows from this assertion that

A) a continuous mapping f from R into S^n is inessential if and only if it is homotopic to a constant mapping.

B) Let A be a subset of a space R and I the unit segment. If U is an open set of the topological product $R \times I$ such that $U \supset A \times I$, then there exists an open set V of R such that $A \subset V$ and $V \times I \subset U$.

Proof. Let p be a given point of A and $V_i(p)$, $i = 1,2,\ldots$ an open neighbourhood basis of p. If we suppose $V_i(p) \times I \not\subset U$, $i = 1,2,\ldots$, then we get a sequence $\{ (p_i, x_i) \mid i = 1,2,\ldots \}$ of points of $R \times I$ such that $(p_i, x_i) \in (V_i(p) \times I) - U$, $i = 1,2,\ldots$. Since I is compact, we can choose a converging subsequence $\{ x_{i_k} \mid k =$

$= 1,2,\dots\}$ of $\{x_i\}$. Let $x_{i_k} \to x$; then

$$(R \times I) - U \ni (p_{i_k}, x_{i_k}) \to (p,x) \in A \times I ,$$

which contradicts the fact that U is an open set containing $A \times I$. Hence $V(p) \times I \subset U$ for some open neighbourhood $V(p)$ of p . Now for each point p of A we choose such an open neighbourhood $V(p)$, and put $V = \cup\{V(p) \mid p \in A\}$. Then V is the desired open set.

Theorem III. 3 [1]. *Let C be a closed subset of a space R and f and g homotopic continuous mappings of C into S^n . If f admits a continuous extension φ over R , then g also admits a continuous extension ψ over R which is homotopic to φ .*

Proof. Since f and g are homotopic, there exists a mapping $f(p,t)$ of $C \times I$ into S^n such that

$$(1) \qquad \begin{cases} f(p,0) = f(p) \\ f(p,1) = g(p) , \end{cases}$$

where we denote by I the unit segment. Let us define a closed set F of $R \times I$ by

$$F = \{(p,0) \mid p \in R\} \cup \{(p,t) \mid p \in C , \ t \in I\} .$$

We define a continuous mapping $\varphi(p,t)$ of F into S^n by

$$(2) \qquad \begin{cases} \varphi(p,0) = \varphi(p) & \text{for } p \in R \\ \varphi(p,t) = f(p,t) & \text{for } p \in C , \quad t \in I . \end{cases}$$

By the corollary to Theorem I.7 we can continuously extend $\varphi(p,t)$ over an open set U containing F in $R \times I$. We denote this extension by $\varphi_1(p,t)$. By B) there exists an open set V of R such that $C \subset V$, $V \times I \subset U$. Using Theorem I.6 we can construct a real valued continuous function $u(x)$ over R satisfying

$$(3) \qquad u(C) = 1 ,$$

$$(4) \qquad u(R - V) = 0 , \text{ and } 0 \leqq u \leqq 1 .$$

Now, we set

$$(5) \qquad \psi(p,t) = \varphi_1(p, tu(p)), \ p \in R , \ t \in I .$$

Then ψ is a continuous mapping of $R \times I$ into S^n . For, if $p \in R - V$, then by (4) $u(p) = 0$, and hence $\psi(p,t) = \varphi_1(p,0) = \varphi(p,0)$ is defined for every t . If $p \in V$, then $(p, tu(p)) \in V \times I$, and hence $\varphi_1(p, tu(p))$ is defined for every t . Thus $\psi(p,t)$ is defined for every $(p,t) \in R \times I$. We define a continuous mapping ψ of R into S^n by $\psi(p) = \psi(p,1)$.

[1] This theorem is due to K. Borsuk [1].

Let p be a given point of C ; then by (5), (3), (2), (1) $\psi(p)=\psi(p,1)=\varphi_1(p,1)=f(p,1)=g(p)$. Hence ψ is a continuous extension of g over R . On the other hand from (5) and (2) it follows that $\psi(p,0)=\varphi_1(p,0)=\varphi(p)$. Therefore φ and ψ are homotopic.

C) Let f and g be two continuous mappings of a space R into S^n such that $\dim \{p \mid f(p)\neq g(p)\} \leqq n-1$. Then f and g are homotopic.

Proof. Let $D=\{p \mid f(p)\neq g(p)\}$; then D is obviously an open set of R . We define a closed set F of the topological product $R\times I$ of R with the unit segment I by

$$F = R\times\{0\}\cup R\times\{1\}\cup(R-D)\times I .$$

Then we define a continuous mapping $f(p,t)$ of F into S^n by

$$f(p,t)=f(p)=g(p) \quad \text{for} \quad p\in R-D ,$$
$$f(p,0)=f(p) ,$$
$$f(p,1)=g(p) .$$

Now, it suffices to show that $f(p,t)$ can be continuously extended over $R\times I$. It follows from the product theorem that $\dim D\times I\leqq n$, because $\dim I\leqq 1$ is clear. Since $(R\times I)-F\subset D\times I$, we get $\dim((R\times I)-F)\leqq n$.

Hence by the corollary to Theorem III.2 we can continuously extend $f(p,t)$ over $R\times I$. Thus f and g are homotopic.

The following is a direct consequence of C).

Theorem III. 4. *If* $\dim R\leqq n-1$ *, then all continuous mappings of* R *into* S^n *are homotopic, and hence inessential.*

D) Let a space R be the sum of two closed subsets F and G . Let f and g be continuous mappings of F and G respectively into S^n . If

$$\dim\{p \mid f(p)\neq g(p) , p\in F\cap G\}\leqq n-1 ,$$

then f can be continuously extended over R .

Proof. The restriction f_0 and g_0 of f and g respectively to $F\cap G$ are homotopic by virtue of C). Since g_0 can be continuously extended to the mapping g over G , by Theorem III.3 there exists a continuous extension f_1 of f_0 over G . Putting $\varphi(p)=f(p)$ for $p\in F$, and $\varphi(p)=f_1(p)$ for $p\in G$, we obtain a desired extension φ of f .

Definition III. 4. *Let* f *and* g *be continuous mappings of a space* R *into a space* S *. Let* A *be a subset of* R *; then* f *and* g *are called homotopic relative to* A *if there exists a continuous mapping* $f(p,t)$ *of* $R\times I$ *into* S *such that*

$f(p,0) = f(p)$, $f(p,1) = g(p)$, and $f(p,t) = f(p) = g(p)$ for $p \in A$, $t \in I$, where the unit segment is denoted by I .

Definition III. 5. *Let f be a continuous mapping of a space R into the n-dimensional simplex T^n or equivalently into the n-dimensional cube I^n . If any continuous mapping g of R into T^n which is homotopic to f relative to $f^{-1}(S^{n-1})$ satisfies $g(R) = I^n$, then f is called essential, where S^{n-1} denotes the $n-1$-sphere which is regarded as the boundary of T^n in the n-dimensional Euclidean space E^n . If f is not essential, then it is called inessential.*

E) A continuous mapping f of R into I^n is essential if and only if every continuous mapping g of R into I^n which coincides with f on $f^{-1}(S^{n-1})$ satisfies $g(R) = I^n$, where we regard S^{n-1} as the boundary of I^n .

Proof. Suppose f is a continuous mapping of R into I^n . If a continuous mapping g of R into I^n satisfies $f(p) = g(p)$ for every $p \in f^{-1}(S^{n-1})$, then for each point p of R we consider the segment joining $f(p)$ and $g(p)$ which lies in I^n . For $p \in R$, $0 \leqq t \leqq 1$ we denote by $f(p,t)$ the point dividing this segment in the ratio $t : 1 - t$. Then $f(p,t)$ is a continuous mapping of $R \times I$ into I^n satisfying the condition of Definition III.4 for $S = I^n$, $A = f^{-1}(S^{n-1})$. Hence g is homotopic to f relative to $f^{-1}(S^{n-1})$. Therefore E) is proved.

Theorem III. 5. *A space R has $\dim \leqq n$ if and only if every continuous mapping of R into I^{n+1} is inessential.*

Proof. Let $\dim R \geqq n+1$; then by Theorem III.1 there exists a continuous mapping f of R into I^{n+1} with a stable value $f(p) = p'$. Let us suppose that ε is a positive number for which

(1) $\rho(f(x), g(x)) < \varepsilon$ for every $x \in R$

implies $p' \in g(R)$. If $p' \in B(f(R))$, then we can choose a point

$$q \in S_{\frac{1}{2}\varepsilon}(p') \cap (I^{n+1} - f(R))$$

and consider the projection $\varphi(y)$ of $S_{\frac{1}{2}\varepsilon}(p') \cap f(R)$ on the boundary $B(S_{\frac{1}{2}\varepsilon}(p'))$ from q . We define a continuous mapping g of R into I^{n+1} by

$$g(x) = \varphi f(x) \quad \text{if} \quad f(x) \in S_{\frac{1}{2}\varepsilon}(p') \cap f(R) ,$$

$$g(x) = f(x) \quad \text{if} \quad f(x) \in f(R) - S_{\frac{1}{2}\varepsilon}(p') .$$

Then one can easily see that $p' \notin g(R)$, and $\rho(f(x), g(x)) < \varepsilon$ for every $x \in R$, which contradicts the fact that p' is a stable value. Hence p' must be an interior point of $f(R)$. Thus we can choose a positive number δ such that $\delta < \frac{1}{2}\varepsilon$,

$\overline{S_\delta(p')} \subset f(R)$. We denote by $\psi(y)$ the projection of $f(R)-S_\delta(p')$ on $B(S_\delta(p'))$ from p' and define a continuous mapping h of R onto $\overline{S_\delta(p')}$ by

$$h(x) = \psi f(x) \quad \text{if} \quad f(x) \in f(R) - S_\delta(p') ,$$
$$h(x) = f(x) \quad \text{if} \quad f(x) \in S_\delta(p') .$$

It easily follows from (1) that h is an essential mapping from R into $\overline{S_\delta(p')}$ which is homeomorphic to I^{n+1} .

Conversely, suppose f is an essential mapping of R onto I^{n+1} . Then, by E) and 2 A), the center of I^{n+1} is a stable value of f . Hence from Theorem III.1 it follows that $\dim R \geqq n+1$, which proves this theorem.

Remark. It is easy to see that the converse of Theorem III.4 is also true if $n>0$. Because if all continuous mappings from R to S^n are inessential, then for every continuous mapping $f: R\to I^n$ the composite mapping $gf: R\to S^n$ of f with the quotient mapping $g: I^n \to S^n$ is inessential. Thus f is inessential.

It is easy to construct examples of inessential mappings while the identity mapping from I^n (S^n) onto itself is essential as proved later on (IV. 3 A)).

III. 4. Some lemmas

In the present section we are going to prove some propositions which will be used in the next section and also later in the book.

A) Let $\{U_\alpha \mid \alpha<\tau\}$ be an open collection in a space R such that $\{U_\beta \mid \beta<\alpha\}$ is locally finite for every $\alpha<\tau$. Let $\{F_\alpha \mid \alpha<\tau\}$ be a closed collection in R such that $F_\alpha \subset U_\alpha$ for all α . If A_n , $n=1,2,\dots$ are subsets of R with $\dim A_n \leqq 0$, then there exists for every $\alpha<\tau$ an open set V_α such that $F_\alpha \subset V_\alpha \subset U_\alpha$ and $\operatorname{ord}_p \{B(V_\alpha) \mid \alpha<\tau\} \leqq n-1$ for every point $p\in A_n$.

Proof. We shall define, by induction on the ordinal α , V_α satisfying

a)$_\alpha$ $\quad F_\alpha \subset V_\alpha \subset U_\alpha$,

b)$_\alpha$ $\quad \operatorname{ord}_p \{B(V_\beta) \mid \beta \leqq \alpha\} \leqq n-1$ for every $p\in A_n$.

First, by virtue of the normality of R there exist open sets P and Q such that $P\supset F_0$, $Q\supset R-U_0$ and $\bar P \cap \bar Q = \emptyset$. Since $\dim A_1 \leqq 0$, there exists an open and closed set N of A_1 satisfying $\bar P \cap A_1 \subset N \subset (R-\bar Q)\cap A_1$. If we let

(1) $\quad B = N \cup F_0 , \quad C = (A_1 - N) \cup (R-U_0)$,

then $(\bar{B}\cap C)\cup(B\cap\bar{C})=\emptyset$. Hence by I B) there exists an open set V_0 such that

(2) $$B\subset V_0\subset\bar{V}_0\subset R-C\ .$$

Then from (1) and (2) it follows that $B(V_0)\cap A_1=\emptyset$, because H is open and closed in A_1 .

Thus V_0 satisfies $a)_\alpha$ and $b)_\alpha$ for $\alpha=0$. Suppose that V_β has been constructed for every $\beta<\alpha$ $(<\tau)$. Then we let $H_1=A_1$,

(3) $$H_n=\cup\{B(V_{\beta_1})\cap\ldots\cap B(V_{\beta_{n-1}})\cap A_n\mid\beta_1,\ldots,\beta_{n-1}<\alpha\ ,\ \beta_i\neq\beta_j\ \text{ if }\ i\neq j\}\ ,\ n=2,3,\ldots\ ,$$

(4) $$K_\alpha=\bigcup_{n=1}^{\infty}H_n\ .$$

Since $H_n\subset A_n$, it follows from $\dim A_n\leqq 0$ and Theorem II.3 that

(5) $$\dim H_n\leqq 0\ .$$

We can easily see that for every n $\bigcup_{i=1}^{n}H_i$ is open in K_α . For let $p\in\bigcup_{i=1}^{n}H_i$; then $p\in\bigcup_{i=1}^{n}A_i$. Hence from the induction hypothesis $b)_\beta$ for $\beta<\alpha$ we obtain $\operatorname{ord}_p\{B(V_\beta)\mid\beta<\alpha\}\leqq n-1$. Therefore we can choose, from every collection $\{B(V_{\beta_1}),\ldots,B(V_{\beta_j})\}$ which satisfies $\beta_1,\ldots,\beta_j<\alpha$, $j\geqq n$ and $\beta_i\neq\beta_j$ if $i\neq j$, $B(V_{\beta_i})$ such that $p\notin B(V_{\beta_i})$.

Hence, if we let $U(p)=\cap\{R-B(V_\beta)\mid\beta<\alpha\ ,\ p\notin B(V_\beta)\}$, then from (3) it follows that

$$U(p)\cap(\bigcup_{i=n+1}^{\infty}H_i)=\emptyset\ .$$

Since $\{B(V_\beta)\mid\beta<\alpha\}$ is locally finite by virtue of the local finiteness of $\{U_\beta\mid\beta<\alpha\}$ and $a)_\beta$ for $\beta<\alpha$, $U(p)$ is an open neighbourhood of p . Hence $\bigcup_{i=1}^{n}H_i$ is open in K_α . Thus, for every n

$$I_n=H_n-\bigcup_{i=1}^{n-1}H_i=\bigcup_{i=1}^{n}H_i-\bigcup_{i=1}^{n-1}H_i$$

is an F_σ-set in K_α . Hence $I_n=\bigcup_{m=1}^{\infty}I_{nm}$ for certain closed sets I_{nm} of K_α . Since $I_{nm}\subset I_n\subset H_n$, by (5) and Theorem II.3 we obtain $\dim I_{nm}\leqq 0$. Since

$$K_\alpha=\bigcup_{n=1}^{\infty}I_n=\bigcup_{n,m=1}^{\infty}I_{nm}\ ,$$

we have $\dim K_\alpha\leqq 0$ by the sum theorem. Therefore we can define, in the same way as for the case $\alpha=0$, an open set V_α such that

(6) $$F_\alpha\subset V_\alpha\subset U_\alpha\ ,$$

(7) $\qquad B(V_\alpha) \cap K_\alpha = \emptyset$.

The condition (6) is equal to $a)_\alpha$. In view of (3) and (4) we easily see that (7) implies $b)_\alpha$. Thus the proof of this proposition is complete.

In view of the nature of the proof of this proposition we can modify it as follows.

B) Let $\{ U_\alpha \mid \alpha < \tau \}$, $\{ F_\alpha \mid \alpha < \tau \}$ and $\{ A_n \mid n = 1, 2, \dots \}$ satisfy the same conditions as in A). Let $\mathcal{U}_1$ be a locally finite open collection such that $\operatorname{ord}_p B(\mathcal{U}_1) \leqq n-1$ for every $p \in A_n$. Then there exists an open collection $\mathcal{U}_2 = \{ U'_\alpha \mid \alpha < \tau \}$ such that $F_\alpha \subset U'_\alpha \subset U_\alpha$, and $\operatorname{ord}_p B(\mathcal{U}_1 \cup \mathcal{U}_2) \leqq n-1$ for every $p \in A_n$.

C) Let $\{ U_\gamma \mid \gamma \in \Gamma \}$ be an open collection and $F = \{ F_\gamma \mid \gamma \in \Gamma \}$ a locally finite closed collection such that $F_\gamma \subset U_\gamma$ for all $\gamma \in \Gamma$ and $\operatorname{ord} F \leqq n$. Then there exists a locally finite open collection $V = \{ V_\gamma \mid \gamma \in \Gamma \}$ such that $F_\gamma \subset V_\gamma \subset U_\gamma$ for all $\gamma \in \Gamma$ and $\operatorname{ord} V \leqq n$. Thus $\dim R \leqq n$ holds if every finite open covering of R has a locally finite closed refinement of order $\leqq n+1$.

Proof. We, of course, restrict ourselves to a (metric) space R . Since F is a locally finite closed collection of order $\leqq n$, there exists an open covering P such that each element of P intersects at most n elements of F . Since R is fully normal, we can choose an open covering Q such that $Q^* < P$. Let $V_\gamma = S(F_\gamma, Q) \cap U_\gamma$. Then as is easily seen, $V = \{ V_\gamma \mid \gamma \in \Gamma \}$ is the desired open collection.

D) Let F be a closed set of R with $\dim F \leqq n$. Let F_α and U_α , $\alpha < \tau$, be closed and open sets, respectively such that $F_\alpha \subset U_\alpha$, and $\{ U_\alpha \mid \alpha < \tau \}$ is locally finite. Then there exist open sets V_α satisfying $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, $\dim B_k \leqq n-k$, $k = 1, \dots, n+1$, where $B_k = \{ p \mid p \in F ,\ \operatorname{ord}_p B(V) \geqq k \}$, and $V = \{ V_\alpha \mid \alpha < \tau \}$.

Proof. By the decomposition theorem we can decompose F into the sum of $n+1$ subsets A_k , $k = 1, \dots, n+1$ with $\dim A_k \leqq 0$. Hence by II.7 A) we can construct open sets V_α for $\alpha < \tau$ such that $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, and $\operatorname{ord}_p B(V) \leqq k-1$ for $p \in A_k$, where we put $V = \{ V_\alpha \mid \alpha < \tau \}$.

Hence $B_k \subset A_{k+1} \cup \dots \cup A_{n+1}$, which implies $\dim B_k \leqq n-k$ by virtue of the decomposition theorem.

E) Let F , F_α and U_α satisfy the same condition as in D). Then there exist open sets V_α , W_α , $\alpha < \tau$ satisfying $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset W_\alpha \subset U_\alpha$, and $\operatorname{ord}_p \{ W_\alpha - \bar{V}_\alpha \mid \alpha < \tau \} \leqq n$ for every $p \in F$.

Proof. By II.5 B) there exist open sets V'_α and W'_α of the subspace F such that $F_\alpha \cap F \subset V'_\alpha \subset \bar{V}'_\alpha \subset W'_\alpha \subset U_\alpha \cap F$ and $\operatorname{ord}\{ W'_\alpha - \bar{V}'_\alpha \mid \alpha < \tau \} \leqq n$.

We can easily construct open sets V_α and W_α of R such that $V_\alpha \cap F = V'_\alpha$, $\bar{V}_\alpha \cap F = \bar{V}'_\alpha$, $W_\alpha \cap F = W'_\alpha$, $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset W_\alpha \subset U_\alpha$. Then $(W_\alpha - \bar{V}_\alpha) \cap F = W'_\alpha - \bar{V}'_\alpha$. Hence $\operatorname{ord}_p \{ W_\alpha - \bar{V}_\alpha \mid \alpha < \tau \} \leqq n$ for every $p \in F$.

F) Let G_k, $k = 0,\dots,n$ be closed sets with $\dim G_k \leqq n-k$ of a space R. Let $\{ F_\alpha \mid \alpha < \tau \}$ be a closed collection and $\{ U_\alpha \mid \alpha < \tau \}$ a locally finite open collection such that $F_\alpha \subset U_\alpha$. Then there exists an open collection $V = \{ V_\alpha \mid \alpha < \tau \}$ such that $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$ and $\operatorname{ord}_p B(V) \leqq n-k$ for every $p \in G_k$.

Proof. We can easily prove this proposition by using E). In fact, by E) we can construct open sets V^0_α, W^0_α such that $F_\alpha \subset V^0_\alpha \subset \overline{V^0_\alpha} \subset W^0_\alpha \subset U_\alpha$ and $\operatorname{ord}_p \{ W^0_\alpha - \overline{V^0_\alpha} \mid \alpha < \tau \} \leqq n$ for every $p \in G_0$. Now, again by use of E), we can construct open sets V^1_α, W^1_α such that $\overline{V^0_\alpha} \subset V^1_\alpha \subset \overline{V^1_\alpha} \subset W^1_\alpha \subset W^0_\alpha$, and $\operatorname{ord}_p \{ W^1_\alpha - \overline{V^1_\alpha} \mid \alpha < \tau \} \leqq n-1$ for every $p \in G_1$.

By repeating this process we get open sets V^k_α, W^k_α, $k = 0,\dots,n$, such that $\overline{V^{k-1}_\alpha} \subset V^k_\alpha \subset \overline{V^k_\alpha} \subset W^k_\alpha \subset W^{k-1}_\alpha$, and $\operatorname{ord}_p \{ W^k_\alpha - \overline{V^k_\alpha} \mid \alpha < \tau \} \leqq n-k$ for every $p \in G_k$. Letting $V_\alpha = \bigcup_{k=0}^n V^k_\alpha$ we obtain the desired open sets V_α for $\alpha < \tau$.

III. 5. Continuous mappings which lower dimension

The projection of E^2 on E^1 lowers the dimension of E^2 by one, the dimension of the inverse image of each point of E^1. (Precisely speaking, $\dim E^2 = 2$ is not yet proved while $\dim E^1 = 1$ is easy to see.) Generally we can prove the following Theorem III.6 as a result of investigations in this direction.

Definition III. 6. *A mapping f of a space R into a space S is called a closed (open) mapping if the image of every closed (open) set of R is closed (open) in S. A closed continuous mapping f is called a perfect mapping if $f^{-1}(q)$ is compact for every $q \in S$.*

Theorem III. 6 [2]. *Let f be a closed continuous mapping of a space R onto a space S such that $\dim f^{-1}(q) \leqq k$ for every $q \in S$. Then $\dim R \leqq \dim S + k$.*

Proof. Since the validity of this proposition is clear if $\dim S = -1$, we shall assume this theorem for every S with $\dim S < n$ $(n \geqq 0)$ to prove it by induction for S with $\dim S = n$. Let $U = \{ U_i \mid i = 1,\dots,s \}$ be a given open covering of R.

[2] This theorem was first proved for separable metric spaces by W. Hurewicz and extended by K. Morita [6] and K. Nagami [1] to general types of spaces which include metric spaces as a special case. See Theorem VII. 7.

Since $\dim f^{-1}(q) \leqq k$ for each point $q \in S$, by I.1 A) and III.4 C) there exist open sets V_i, $i = 1,\dots,s$ such that $f^{-1}(q) \subset \cup_{i=1}^{s} V_i$, $V_i \subset U_i$, $i = 1,\dots,s$, and

(1) $$\operatorname{ord}\{V_i \mid i = 1,\dots,s\} \leqq k+1 .$$

Let

(2) $$V'(q) = \bigcup_{i=1}^{s} V_i ;$$

then $\{V'(q) \mid q \in S\}$ is an open covering of R satisfying $V'(q) \supset f^{-1}(q)$. We choose for each q an open set $V(q)$ satisfying

(3) $$V'(q) \supset \overline{V(q)} \supset V(q) \supset f^{-1}(q) .$$

It follows from $q \in S - f(R - V(q))$ and the closedness of f, that $\mathcal{V} = \{S - f(R - V(q)) \mid q \in S\}$ is an open covering of S. Since $\dim S \leqq n$, by use of III. 4 D) combined with the paracompactness of S we can construct a locally finite open covering $\mathcal{W}$ of S such that $\mathcal{W} < \mathcal{V}$ and

(4) $$\dim B_m \leqq n - m , \quad m = 1,\dots,n+1 ,$$

where $B_m = \{p \mid \operatorname{ord}_p B(\mathcal{W}) \geqq m\}$. Let $W_i \in \mathcal{W}$, $i = 1,\dots,m$; then since

$$ff^{-1}(B(W_1) \cap \dots \cap B(W_m)) = B(W_1) \cap \dots \cap B(W_m) ,$$

f is clearly a closed continuous mapping of $f^{-1}(B(W_1) \cap \dots \cap B(W_m))$ onto $B(W_1) \cap \dots \cap B(W_m)$. Since $B(W_1) \cap \dots \cap B(W_m) \subset B_m$, it follows from (4) that $\dim (B(W_1) \cap \dots \cap B(W_m)) \leqq n - m$.

Hence we obtain from the induction hypothesis

$$\dim f^{-1}(B(W_1) \cap \dots \cap B(W_m)) \leqq n - m + k , \quad m = 1,\dots,n+1 .$$

Hence, in view of

$$B(f^{-1}(W_1)) \cap \dots \cap B(f^{-1}(W_m)) \subset f^{-1}(B(W_1) \cap \dots \cap B(W_m)) ,$$

we conclude that

$$\dim B(f^{-1}(W_1)) \cap \dots \cap B(f^{-1}(W_m)) \leqq n - m + k .$$

Thus, by virtue of the sum theorem, $f^{-1}(\mathcal{W}) = \{f^{-1}(W) \mid W \in \mathcal{W}\}$ is a locally finite open covering of R such that

(5) $$f^{-1}(\mathcal{W}) < \{V(q) \mid q \in S\} , \text{ and}$$

(5') $$\dim A_m \leqq n - m + k , \quad m = 1,\dots,n+1 ,$$

for $A_m = \{p \mid \operatorname{ord}_p B(f^{-1}(\mathcal{W})) \geqq m\}$, because A_m is a locally finite sum of closed sets like $B(f^{-1}(W_1)) \cap \dots \cap B(f^{-1}(W_m))$.

We well-order the elements of $f^{-1}(\mathcal{W})$ as $f^{-1}(\mathcal{W}) = \{M_\alpha \mid \alpha < \tau\}$ to define a new locally finite open collection $\mathcal{N} = \{N_\alpha \mid \alpha < \tau\}$ by

$$N_0 = M_0 \;,\quad N_\alpha = M_\alpha - \cup\{\bar{M}_\beta \mid \beta < \alpha\} \;,\; 0 < \alpha < \tau \;.$$

Then $C_m = \{p \mid \mathrm{ord}_p \bar{\mathcal{N}} \geqq m\} \subset A_{m-1}$, $m = 2,\dots,n+1$, and hence, from (5)', it follows that

(6) $\dim C_m \leqq n - m + k + 1 \;,\; m = 2,\dots,n+1 \;.$

Since by (3) and (5) $\bar{N}_\alpha \subset V'(q)$ for every α and for some q , from (1), (2) and I.1 A) it follows that there exists a closed covering $\{K_i^\alpha \mid i = 1,\dots,s\}$ of $\bar{N}_\alpha$ satisfying $K_i^\alpha \subset U_i$, $i = 1,\dots,s$, and $\mathrm{ord}\,\{K_i^\alpha \mid i = 1,\dots,s\} \leqq k+1$.

By use of 4 C) we construct open sets P_i^α such that $K_i^\alpha \subset P_i^\alpha \subset \bar{P}_i^\alpha \subset U_i$, and

(7) $\mathrm{ord}\,\{P_i^\alpha \mid i = 1,\dots,s\} \leqq k+1$.

Note that we choose P_i^α such that $\{P_i^\alpha \mid \alpha < \tau\,,\; i = 1,\dots,s\}$ is locally finite, because $\{K_i^\alpha \mid \alpha < \tau,\; i = 1,\dots,s\}$ is locally finite.

Now, in view of (6), we apply F) to K_i^α, P_i^α to get open sets Q_i^α, $\alpha < \tau$, $i = 1,\dots,s$ such that $K_i^\alpha \subset Q_i^\alpha \subset \bar{Q}_i^\alpha \subset P_i^\alpha$, and

(8) $\mathrm{ord}_p B(\mathcal{Q}) \leqq n - m + k + 1$ for $p \in C_m$,

where $\mathcal{Q} = \{Q_i^\alpha \mid \alpha < \tau\,,\; i = 1,\dots,s\}$. Let

(9) $S_i^\alpha = \bar{N}_\alpha \cap \overline{Q_i^\alpha - (Q_1^\alpha \cup \dots \cup Q_{i-1}^\alpha)}$.

We shall show that $\mathcal{S} = \{S_i^\alpha \mid \alpha < \tau,\; i = 1,\dots,s\}$ is a closed covering of order $\leqq n+k+1$ and a refinement of $\mathcal{U}$. Suppose

(10) $p \in \bigcap_{\nu=1}^{s_1} S_{i(1,\nu)}^{\alpha_1} \bigcap_{\nu=1}^{s_2} S_{i(2,\nu)}^{\alpha_2} \cdots \bigcap_{\nu=1}^{s_m} S_{i(m,\nu)}^{\alpha_m} \neq \emptyset$ [3] .

If $m = 1$, then from the definition of S_i^α it follows that

$$\bigcap_{\nu=1}^{s_1} \overline{Q_{i(1,\nu)}^{\alpha_1}} \neq \emptyset \;,$$

which combined with the relation $\bar{Q}_i^\alpha \subset P_i^\alpha$ implies

$$\bigcap_{\nu=1}^{s_1} P_{i(1,\nu)}^{\alpha_1} \neq \emptyset \;,$$

[3] For brevity we abridge some symbols $\cap$ in this formula, but the reader will easily understand the real meaning.

and hence by (7) $s_1 \leqq k+1 \leqq n+k+1$. If $m \geqq 2$, then by assuming $i(l,1) < i(l,2) < \dots < i(l,s_l)$, $l = 1,\dots,m$, we get from (9) and (10) that

$$p \in \bigcap_{\nu=1}^{s_1} S^{\alpha_1}_{i(1,\nu)} \cap \dots \bigcap_{\nu=1}^{s_m} S^{\alpha_m}_{i(m,\nu)}$$
$$\subset \bigcap_{\nu=1}^{s_1-1} B(Q^{\alpha_1}_{i(1,\nu)}) \cap \dots \bigcap_{\nu=1}^{s_m-1} B(Q^{\alpha_m}_{i(m,\nu)}) \bigcap_{i=1}^{m} \bar{N}_{\alpha_i} .$$

Since $p \in \bigcap_{i=1}^{m} \bar{N}_{\alpha_i} \subset C_m$, it follows from (8) that $(s_1-1)+(s_2-1)+\dots+(s_m-1) \leqq n-m+k+1$, and hence $s_1+s_2+\dots+s_m \leqq n+k+1$. Thus in any case we get $\mathrm{ord}_p S \leqq n+k+1$.

Now we note that $\{M_\alpha \mid \alpha < \tau\}$ covers R , and hence by the definition of N , $\bar{N}$ also covers R . On the other hand, since $\{K^\alpha_i \mid i=1,\dots,s\}$ and accordingly $\{Q^\alpha_i \mid i=1,\dots,s\}$ covers $\bar{N}_\alpha$, we have $\bar{N}_\alpha = \bigcup_{i=1}^{s} S^\alpha_i$. Therefore S covers R .

Since by (9) and the definitions of Q^α_i and P^α_i we have $S^\alpha_i \subset \bar{Q}^\alpha_i \subset \bar{P}^\alpha_i \subset U_i$, S is a locally finite closed refinement of U . Thus by 4 C) $\dim R \leqq n+k$.

III. 6. Continuous mappings which raise dimension

As shown by Peano, a closed segment can be continuously mapped on a closed square. This example shows that a closed continuous mapping can raise dimension. As for the raising of dimension by a closed continuous mapping we have the following theorem.

Theorem III. 7 [*] . *Let f be a closed continuous mapping of a space R onto a space S such that for each point q of S , $B(f^{-1}(q))$ contains at most $m+1$ points $(m \geqq 0)$; then* $\dim S \leqq \dim R + m$.

Proof. We shall begin the proof with the special case that every $f^{-1}(q)$ contains at most $m+1$ points. Let $\dim R \leqq n$; then we shall carry out the proof of $\dim S \leqq n+m$ by induction on the numbers $n \geqq -1$ and $m \geqq 0$.

i) This assertion is clearly true for every $m \geqq 0$ if $n = -1$.

ii) Let us show this theorem for every $n > -1$ in case $m = 0$. To use induction on n , assume that the theorem is true if $\dim R \leqq n-1$. Assume G_1 and G_2 are

[*] W. Hurewicz [3] first proved this theorem for separable metric spaces considering $f^{-1}(q)$ in place of $B(f^{-1}(q))$. With respect to this theorem K. Nagami [2] solved a problem posed by W. Hurewicz as follows: Let S be an n-dimensional space and $0 < m < n$; then there exists an m-dimensional space R and a closed continuous mapping f of R onto S such that $f^{-1}(q)$ contains at most $n-m+1$ points for each $q \in S$.

given closed sets of S such that $G_1 \cap G_2 = \emptyset$. Then $F_1 = f^{-1}(G_1)$ and $F_2 = f^{-1}(G_2)$ are disjoint closed sets of R . Since $\dim R \leqq n$, there exists an open set U satisfying $F_1 \subset U \subset R - F_2$, and

(1) $\dim B(U) \leqq n-1$.

Since f is a closed mapping, $V = S - f(R-U)$ is an open set of S . It is obvious that

(2) $G_1 \subset V \subset S - G_2$.

Then

(3) $B(V) \subset f(B(U))$.

Because for each point $q \in B(V)$, $f^{-1}(q) = \{p\}$. If $p \in U$, then $q \in V$. If $p \notin \bar{U}$, then $q \notin \bar{V}$. Since both are impossible, $p \in B(U)$, and hence $q = f(p) \in f(B(U))$ proving the assertion. Since f is a one-to-one closed continuous mapping even if restricted to $B(V)$, by the induction hypothesis and (1) we obtain $\dim f(B(U)) \leqq n-1$. Thus $\dim B(V) \leqq n-1$ follows from (3). Hence $\dim S \leqq n$ is proved.

iii) Now we assume the proposition for the case in which $\dim R \leqq n-1$, and every $f^{-1}(q)$ contains at most $m+1$ points, and for the case in which $\dim R \leqq n$, and every $f^{-1}(q)$ contains at most m points.

Then we shall prove it for the case in which $\dim R \leqq n$, and every $f^{-1}(q)$ contains at most $m+1$ points. Let G_1 and G_2 be given disjoint closed sets of S . Then we can define F_1 , F_2 , U and V in the same way as in the proof ii). Let $H = f(B(U))$. The restriction of f to $B(U)$ is a closed continuous mapping of $B(U)$ onto H . Therefore it follows from $\dim B(U) \leqq n-1$ and the induction hypothesis that

(4) $\dim H \leqq n+m-1$.

$B(V) - H$ is a G_δ-set of S . Therefore, let

(5) $B(V) - H = \bigcup_{k=1}^{\infty} H_k$ for closed sets H_k , $k = 1, 2, \ldots$.

Note that for each $q \in B(V) - H$, $f^{-1}(q) \cap B(U) = \emptyset$ holds. Then put

$$f^{-1}(q) \cap U = f^{-1}(q)_1 , \text{ and } f^{-1}(q) \cap (R - \bar{U}) = f^{-1}(q)_2 .$$

Further put

$$E_k = f^{-1}(H_k) \cap (R-U) = f^{-1}(H_k) \cap (R - \bar{U}) = \cup \{ f^{-1}(q)_2 \mid q \in H_k \} .$$

Then E_k is a closed set of R . Note that for each $q \in H_k$, $f^{-1}(q)_2 \neq \emptyset$ holds, because otherwise $q \in V$ would follow, contradicting $q \in B(V)$. Thus $f(E_k) = H_k$. Namely $f \mid E_k = f_k$ is a closed continuous mapping from E_k onto H_k .

On the other hand $f^{-1}(q)_1 \neq \emptyset$ for each $q \in H_k$, because otherwise $q \notin \bar{V}$ would follow. Hence $f_k^{-1}(q) = f^{-1}(q)_2$ consists of at most m points for each $q \in H_k$.

Since $\dim E_k \leq n$ is obvious we obtain from the induction hypothesis $\dim H_k \leq n+m-1$. Thus by (4) and the sum theorem we can conclude that

$$\dim (H \cup (\bigcup_{k=1}^{\infty} H_k)) \leq n+m-1 ,$$

because H and H_k, $k=1,2,\ldots$ are closed sets.

Note that (5) implies $B(V) \subset H \cup (\bigcup_{k=1}^{\infty} H_k)$. Therefore $\dim B(V) \leq n+m-1$, which proves $\dim S \leq n+m$.

Now, let us turn to the most general case in which each $B(f^{-1}(q))$ contains at most $m+1$ points. To each point $q \in S$ with $B(f^{-1}(q)) = \emptyset$ we assign a point $p(q) \in f^{-1}(q)$ and put $C = \{ p(q) \mid q \in S , B(f^{-1}(q)) = \emptyset \}$. Let

$$B = \bigcup \{ B(f^{-1}(q)) \mid q \in S \} \cup C .$$

Then B is a closed set of R with $\dim B \leq n$, and $f \mid B$ is a closed continuous mapping from B onto S such that each inverse image of a point of S contains at most $m+1$ points. Hence from the above argument of the special case it follows that $\dim S \leq n+m$ [5].

III. 7. Baire's zero-dimensional spaces

Definition III. 7. *Let Ω be a given set. We denote by $N(\Omega)$ the set of all sequences of elements of Ω. Defining the distance of two points $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ of $N(\Omega)$ by*

$$\rho(\alpha, \beta) = \frac{1}{\min \{ k \mid \alpha_k \neq \beta_k \}} ,$$

we get a 0-dimensional metric space $N(\Omega)$. We call this space a generalized Baire's 0-dimensional space.

As easily seen, $N(\Omega)$ is topologically the countable product of discrete spaces Ω. We characterize n-dimensional metric space in terms of $N(\Omega)$, that is to say we can get every n-dimensional metric space as the image of a subset of $N(\Omega)$ by a closed continuous mapping which satisfies the condition of Theorem III.7 for $m=n$.

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[5] In relation to this theorem J. Suzuki [1] proved that if f is a closed continuous mapping of a space R onto a space S such that each $f^{-1}(q)$ consists of exactly k points, then $\dim R = \dim S$. As for open mappings K. Nagami [2] proved that if f is an open continuous mapping of R onto S such that each $f^{-1}(q)$ consists of finitely many points, then $\dim R = \dim S$. R. Hodel [1] proved the same for a countable-to-one open mapping but assuming that R is locally compact.
Yu. A. Rozanskaya [1] proved that if $m < n$, then there exists no open continuous mapping which maps I^m onto I^n. See also J. Keesling [1] for results in this aspect.

A) Let A_n, $n=1,2,\ldots$ be a sequence of 0-dimensional sets of a space R. Let $U=\{U_\alpha \mid \alpha<\tau\}$ be a locally finite open covering. Then there exists a closed covering $F=\{F_\alpha \mid \alpha<\tau\}$ such that $F_\alpha \subset U_\alpha$ and $\mathrm{ord}_p F \leqq n$ for every $p\in A_n$.

Proof. By 4.A) combined with I.1 A) there exists a locally finite open covering $V=\{V_\alpha \mid \alpha<\tau\}$ such that $\bar{V}_\alpha \subset U_\alpha$ and $\mathrm{ord}_p B(V)\leqq n-1$ for $p\in A_n$. Let

$$F_0=\bar{V}_0 \ , \text{ and } F_\alpha=\overline{V_\alpha-\cup\{\bar{V}_\beta \mid \beta<\alpha\}} \text{ for } \alpha<\tau .$$

Then it is easy to see that $F=\{F_\alpha \mid \alpha<\tau\}$ is the desired covering. For, let p be a given point of A_n. We suppose $p\in V_\alpha$ and $p\notin V_\beta$ for every $\beta<\alpha$. Then from the definition of F_α we have $p\notin F_\gamma$ for every $\gamma>\alpha$. If $p\in F_\beta$ for some $\beta<\alpha$, then from $p\notin V_\beta$ it follows that $p\in \bar{V}_\beta-V_\beta=B(V_\beta)$. Hence from $\mathrm{ord}_p B(V)\leqq n-1$ we obtain $\mathrm{ord}_p F\leqq n$.

B) Let A_n, $n=1,2,\ldots$ be a countable number of 0-dimensional sets of a space R. Then there exists a sequence $\{F_i \mid i=1,2,\ldots\}$ of locally finite closed coverings of R satisfying

i) for every neighbourhood $U(p)$ of every point p of R there exists some i with $S(p,F_i)\subset U(p)$,

ii) $F_i=\{F(\alpha_1,\ldots,\alpha_i) \mid \alpha_k\in\Omega,\ k=1,2,\ldots,i\}$, where $F(\alpha_1,\ldots,\alpha_i)$ may be empty,

iii) $F(\alpha_1,\ldots,\alpha_{i-1})=\cup\{F(\alpha_1,\ldots,\alpha_{i-1},\beta) \mid \beta\in\Omega\}$,

iv) $\mathrm{ord}_p F_i\leqq n$ for each point $p\in A_n$ and for each $i\in\{1,2,\ldots\}$.

Proof. First we define $S_i=\{S_{1/i}(p) \mid p\in R\}$, $i=1,2,\ldots$. Then we shall define F_i satisfying ii), iii), iv) and $F_i<S_i$. For $i=1$ we can define, by use of A) and the corollary to Theorem I.1, a locally finite closed covering F_1 such that $F_1<S_1$ and $\mathrm{ord}_p F_1\leqq n$ for every $p\in A_n$. We shall use induction and accordingly assume that F_k has been defined for every $k<i$. Now we shall define F_i.

Let us well-order the elements of F_{i-1} as $F_{i-1}=\{F_\alpha \mid \alpha<\tau\}$. To obtain F_i we shall define closed sets $F_{\alpha\beta}$ for $\alpha<\tau$ and $\beta\in\Omega$ such that

a)$_\alpha$ $\quad F_\alpha=\cup\{F_{\alpha\beta} \mid \beta\in\Omega\}$, $\{F_{\alpha\beta} \mid \beta\in\Omega\}<S_i$,

b)$_\alpha$ $\quad G_\alpha=\{F_{\alpha'\beta} \mid \alpha'\leqq\alpha,\ \beta\in\Omega\}\cup\{F_{\alpha'} \mid \alpha'>\alpha\}$

is locally finite for every $\alpha<\tau$,

c)$_\alpha$ $\quad \mathrm{ord}_p G_\alpha\leqq n$ for every $\alpha<\tau$ and for every point $p\in A_n$.

First we define $F_{0\beta}$ for $\beta\in\Omega$ as follows. We let

$$H_{10} = \{ p \mid p \in F_0 \cap A_1 ,\ \operatorname{ord}_p \{ F_\alpha \mid 0<\alpha<\tau \} = 0 \},$$
$$H_{21} = \{ p \mid p \in F_0 \cap A_2 ,\ \operatorname{ord}_p \{ F_\alpha \mid 0<\alpha<\tau \} = 1 \},$$
$$H_{32} = \{ p \mid p \in F_0 \cap A_3 ,\ \operatorname{ord}_p \{ F_\alpha \mid 0<\alpha<\tau \} = 2 \},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$K_1 = H_{10} \cup H_{21} \cup H_{32} \cup \dots$$

and

$$H_{20} = \{ p \mid p \in F_0 \cap A_2 ,\ \operatorname{ord}_p \{ F_\alpha \mid 0<\alpha<\tau \} = 0 \},$$
$$H_{31} = \{ p \mid p \in F_0 \cap A_3 ,\ \operatorname{ord}_p \{ F_\alpha \mid 0<\alpha<\tau \} = 1 \},$$
$$H_{42} = \{ p \mid p \in F_0 \cap A_4 ,\ \operatorname{ord}_p \{ F_\alpha \mid 0<\alpha<\tau \} = 2 \},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$K_2 = H_{20} \cup H_{31} \cup H_{42} \cup \dots$$

and generally

(1) $$H_{r+ss} = \{ p \mid p \in F_0 \cap A_{r+s} ,\ \operatorname{ord}_p \{ F_\alpha \mid 0<\alpha<\tau \} = s \},$$

(2) $$K_r = \bigcup_{s=0}^{\infty} H_{r+ss} .$$

If $p \in H_{r+ss}$, then $p \in F_{\alpha_1} \cap \dots \cap F_{\alpha_s}$ for some different $\alpha_1,\dots,\alpha_s > 0$, and $p \notin F_{\alpha_1} \cap \dots \cap F_{\alpha_{s+1}}$ for any different $\alpha_1,\dots,\alpha_{s+1} > 0$. Hence the open neighbourhood $U(p) = \bigcap \{ R - F_\alpha \mid p \notin F_\alpha ,\ 0<\alpha<\tau \}$ of p satisfies $U(p) \cap H_{r+tt} = \emptyset$ for every $t > s$. Hence for every s', $\bigcup_{s=0}^{s'} H_{r+ss}$ is open in K_r. Since (1) implies $H_{r+ss} \subset A_{r+s}$, from $\dim A_{r+s} \leqq 0$ it follows that $\dim H_{r+ss} \leqq 0$ for every $s \geqq 0$.

Since any open set is an F_σ-set, each H_{r+ss} is an F_σ-set in K_r. Hence from the sum theorem we obtain $\dim K_r \leqq 0$. Therefore by A) we can define a locally finite closed covering $\mathcal{G}_0' = \{ F_{0\beta} \mid \beta \in \Omega \}$ of F_0 such that $\mathcal{G}_0' < \mathcal{S}_i$ and

(3) $$\operatorname{ord}_p \mathcal{G}_0' \leqq n \quad \text{for every } p \in K_n .$$

We put $\mathcal{G}_0 = \mathcal{G}_0' \cup \{ F_\alpha \mid \alpha > 0 \}$. To show $\operatorname{ord}_p \mathcal{G}_0 \leqq n$ for every $p \in A_n$ we suppose $\operatorname{ord}_p \mathcal{F}_{i-1} = s+1$ and

(4) $$p \in F_0 \cap F_{\alpha_1} \cap \dots \cap F_{\alpha_s}$$

for some different $\alpha_1,\dots,\alpha_s > 0$. Then $0 \leqq s \leqq n-1$ because by the induction hypothesis $\operatorname{ord}_p \mathcal{F}_{i-1} \leqq n$. Thus from (1) and (2) we deduce that $p \in H_{ns} \subset K_{n-s}$. Hence by (3) $\operatorname{ord}_p \mathcal{G}_0' \leqq n-s$, which combined with (4) proves c)$_0$ $\operatorname{ord}_p \mathcal{G}_0 \leqq n$.

Furthermore a)$_0$ and b)$_0$ are clearly satisfied by $F_{0\beta}$ and $\mathcal{G}_0$.

Suppose we have defined $F_{\alpha'\beta}$ for every $\alpha' < \alpha$; then we can define $F_{\alpha\beta}$ for every $\beta \in \Omega$ satisfying a)$_\alpha$, b)$_\alpha$, c)$_\alpha$. The definition of $F_{\alpha\beta}$ is quite analogous to that of $F_{0\beta}$ except that we adopt F_α and $\{ F_{\alpha'\beta} \mid \alpha' < \alpha ,\ \beta \in \Omega \} \cup \{ F_{\alpha'} \mid \alpha' > \alpha \}$ in place of F_0 and $\{ F_\alpha \mid \alpha > 0 \}$ respectively, so the proof is left to the reader.

Thus we can define the desired covering $\mathrm{F}_i = \{ F_{\alpha\beta} \mid \alpha < \tau, \beta \in \Omega \}$. It easily follows from $a)_\alpha$ and $c)_\alpha$ that F_i satisfies ii), iii), iv) and $\mathrm{F}_i < \mathrm{S}_i$. From $b)_\alpha$ combined with the local finiteness of F_{i-1} it follows that F_i is locally finite. (To assure that F_i is subindexed as in ii), we select a sufficiently large Ω at the beginning of the induction and at the end of each stage of the induction we put $F(\alpha_1,\dots,\alpha_i) = \emptyset$ for those $(\alpha_1,\dots,\alpha_i)$ for which F has not been defined.)

C) If a space R has dimension $\leq n$, then there exists a sequence $\{ \mathrm{F}_i \mid i = 1,2,\dots \}$ of locally finite closed coverings of R which satisfies

i) for every neighbourhood $U(p)$ of every point p of R there exists some i for which $S(p, \mathrm{F}_i) \subset U(p)$,

ii) $\mathrm{F}_i = \{ F(\alpha_1,\dots,\alpha_i) \mid \alpha_k \in \Omega,\ k = 1,\dots,i \}$, where $F(\alpha_1,\dots,\alpha_i)$ may be empty,

iii) $F(\alpha_1,\dots,\alpha_{i-1}) = \bigcup \{ F(\alpha_1,\dots,\alpha_{i-1},\beta) \mid \beta \in \Omega \}$,

iv) $\operatorname{ord} \mathrm{F}_i \leq n+1$.

Proof. We can easily prove this proposition by applying to R the decomposition theorem and B), putting $A_i = \emptyset$ for $i > n+1$. (Note that the condition iv) should be understood in the strong sense that each point of R is contained in $F(\alpha_1,\dots,\alpha_i)$ for at most $n+1$ distinct subindices $(\alpha_1,\dots,\alpha_i)$.)

Theorem III. 8 [6]. *A space R has dimension $\leq n$ if and only if there exists a subspace P of $N(\Omega)$ for suitable Ω and a closed continuous mapping f of P onto R such that for each point q of R, $f^{-1}(q)$ consists of at most $n+1$ points.*

Proof. Since $\dim N(\Omega) = 0$, the "if" part follows from Theorem III.7.
To show the "only if" part we construct closed coverings F_i, $i = 1,2,\dots$ satisfying the conditions of C). Then we define a subset P of $N(\Omega)$ by

$$P = \{ \alpha \mid \alpha = (\alpha_1,\alpha_2,\dots) \in N(\Omega),\ \bigcap_{i=1}^{\infty} F(\alpha_1,\dots,\alpha_i) \neq \emptyset \} .$$

Furthermore, we define a mapping f of P onto R by

$$f(\alpha) = \bigcap_{i=1}^{\infty} F(\alpha_1,\dots,\alpha_i) \quad \text{for } \alpha = (\alpha_1,\alpha_2,\dots) \in P .$$

[6] First proved by W. Hurewicz [2] and C. Kuratowski [1] in the separable case. This theorem for general metric space and Theorem III.9 are due to K. Morita [5]. Yu. Smirnov [4], V. Ponomarev [1] and K. Nagami [3] obtained a wider category of topological spaces as images of zero-dimensional metric spaces by continuous mappings satisfying some conditions. For example, the last author showed that every non-empty metric space S is the image of a metric space R with $\dim R = 0$ by an open continuous mapping f such that $f^{-1}(q)$ is compact for every point q of S.

By virtue of the condition i) of C) f is continuous. From iv) it follows that for every $q \in R$, $f^{-1}(q)$ consists of at most $n+1$ points. (Recall the note in the proof of C).)

To show that f is a closed mapping, for each collection $(\alpha_1,\ldots,\alpha_i)$ of elements of Ω we define an open set $N(\alpha_1,\ldots,\alpha_i)$ of P by

$$N(\alpha_1,\ldots,\alpha_i) = \{ (\alpha_1',\alpha_2',\ldots) \mid (\alpha_1',\alpha_2',\ldots) \in P;\ \alpha_k' = \alpha_k\ ,\ k=1,\ldots,i \}$$

and an open covering N_i of P by

$$N_i = \{ N(\alpha_1,\ldots,\alpha_i) \mid \alpha_k \in \Omega,\ k=1,\ldots,i \}.$$

Let K be a closed subset of P and choose $q \notin f(K)$ in R. Then it follows from the finiteness of $f^{-1}(q)$ that $S(K,N_i) \cap f^{-1}(q) = \emptyset$ for some i. Hence $H = \cup \{ f(N) \mid N \in N_i,\ N \cap K \neq \emptyset \}$ satisfies $q \notin H \supset f(K)$. Since by iii) $f(N(\alpha_1,\ldots,\alpha_i)) = F(\alpha_1,\ldots,\alpha_i)$, the local finiteness of F_i implies that H is a closed set of R. Hence putting $U(q) = R - H$ we obtain a neighbourhood of q satisfying $U(q) \cap f(K) = \emptyset$, which shows that $f(K)$ is a closed set of R. Therefore f is a closed mapping.

Thus the proof of this theorem is complete.

Theorem III. 9. *A space R has dimension $\leq n$ if and only if there exists a sequence $\{ F_i \mid i = 1,2,\ldots \}$ of locally finite closed coverings of R which satisfies the conditions i) - iv) of C)* [7].

Proof. The "only if" part is proved by C). Conversely, if there exists such a sequence $\{ F_i \mid i = 1,2,\ldots \}$, then by use of the proof of Theorem III.8 we can construct a subspace P of $N(\Omega)$ and a closed continuous mapping f of P onto R such that $f^{-1}(q)$ consists of at most $n+1$ points for each $q \in R$. Hence $\dim R \leq n$ is deduced from Theorem III.8.

III. 8. Uniformly zero-dimensional mappings

Closely related to the diameter $\delta(S)$ of a subset S of a space R is the following notion.

Definition III.3. *Let U and S be an open covering and a subset respectively of a space R. We denote by $\tilde{\delta}(S) < U$ the fact that there exists an open collection V satisfying $V < U$, $\cup \{ V \mid V \in V \} \supset S$, and* ord $V \leq 1$.

[7] Here iv) should be understood in the strong sense as noted in the proof of C).

Definition III. 9. *Let* f *be a mapping of a space* R *into a space* S *. Suppose for every* $\varepsilon>0$ *there exists* $\eta>0$ *such that* $\delta(T)\leq\eta$ *for* $T\subset S$ *implies*

$$\tilde{\delta}(f^{-1}(T))<\mathcal{U}_\varepsilon=\{U\mid U \text{ is an open set of } R \text{ with } \delta(U)\leq\varepsilon\}.$$

Then we call f *a uniformly 0-dimensional mapping.*

A) If there is a uniformly 0-dimensional continuous mapping f of a space R onto S, then $\dim R\leq\dim S$.

Proof. Suppose $\dim S\leq n$. By use of the decomposition theorem we decompose S into $n+1$ 0-dimensional spaces S_i, $i=1,\dots,n+1$. The mapping f is clearly a uniformly 0-dimensional continuous mapping of $R_i=f^{-1}(S_i)$ onto S_i if it is restricted to R_i . Hence, again in view of the decomposition theorem it suffices to prove $\dim R_i\leq 0$. Namely, all we have to do is to prove this proposition in case $\dim S=0$.

We define $\mathcal{S}_i=\{S_{1/i}(p)\mid p\in R\}$. Since f is uniformly 0-dimensional, there exists $\eta>0$ such that if $\delta(T)\leq\eta$ for a set T of S, then $\tilde{\delta}(f^{-1}(T))<\mathcal{S}_i$.

Since S has dimension ≤ 0, there exists an open covering $\mathcal{V}_i$ of S such that mesh $\mathcal{V}_i\leq\eta$, i.e. $\delta(V)\leq\eta$ for every $V\in\mathcal{V}_i$, and ord $\mathcal{V}_i\leq 1$. Then every element V of $\mathcal{V}_i$ satisfies $\tilde{\delta}(f^{-1}(V))<\mathcal{S}_i$. Hence for every $V\in\mathcal{V}_i$ there exists an open covering $\mathcal{U}_i(V)$ of $f^{-1}(V)$ with ord $\mathcal{U}_i(V)\leq 1$ and $\mathcal{U}_i(V)<\mathcal{S}_i$. Thus $\mathcal{W}_i=\bigcup\{\mathcal{U}_i(V)\mid V\in\mathcal{V}_i\}$ is an open covering of R such that ord $\mathcal{W}_i\leq 1$ and $\mathcal{W}_i<\mathcal{S}_i$.

Therefore we can easily see that $\bigcup_{i=1}^\infty\mathcal{W}_i$ is a σ-locally finite open basis of R consisting of sets which are both open and closed. Hence by Theorem II.2 we conclude $\dim R\leq 0$.

B) We denote by $C(R,I^n)$ the set of all continuous mappings of a space R into a Euclidean n-cube I^n . We regard $C(R,I^n)$ as a metric space by assigning a metric ρ to it by

$$\rho(f,g)=\sup\{d(f(x),g(x))\mid x\in R\} \quad \text{for } f,g\in C(R,I^n),$$

where $d(a,b)$ denotes the metric of I^n . Moreover, for $\varepsilon>0$ we put $D_\varepsilon(R,I^n)=\{f\mid f\in C(R,I^n)$, there exists $\eta>0$ such that for every $T\subset I^n$ with $\delta(T)\leq\eta$, $\tilde{\delta}(f^{-1}(T))<\mathcal{U}_\varepsilon$ holds $\}$, where $\mathcal{U}_\varepsilon$ denotes the same covering as in Definition III.9. Then $D_\varepsilon(R,I^n)$ is an open subset of $C(R,I^n)$.

Proof. Let f be a point of $D_\varepsilon(R,I^n)$ and η a positive number such that

(1) $\delta(T)\leq\eta$ implies $\tilde{\delta}(f^{-1}(T))<\mathcal{U}_\varepsilon$ for all $T\subset I^n$.

Suppose g to be a point of $C(R,I^n)$ satisfying $\rho(f,g)<\frac{1}{3}\eta$. Consider $T\subset I^n$ with $\delta(T)\leq\frac{1}{3}\eta$. Then a', $b'\in fg^{-1}(T)$ implies $a'=f(a)$ and $b'=f(b)$ for some $a,b\in R$ for which $g(a)\in T$ and $g(b)\in T$.

Therefore

$$d(a',b') = d(f(a),\ f(b))$$
$$\leqq d(f(a),\ g(a)) + d(g(a),\ g(b)) + d(g(b),\ f(b)) < \eta\ .$$

Hence $\delta(fg^{-1}(T)) \leqq \eta$. Thus it follows from (1) that $\tilde{\delta}(f^{-1}fg^{-1}(T)) < U_\varepsilon$, which combined with $f^{-1}fg^{-1}(T) \supset g^{-1}(T)$ implies $\tilde{\delta}(g^{-1}(T)) < U_\varepsilon$. This means $g \in D_\varepsilon(R, I^n)$. In consequence the neighbourhood $S_{\frac{1}{3}\eta}(f)$ of f is contained in $D_\varepsilon(R, I^n)$, which proves that $D_\varepsilon(R, I^n)$ is an open set of $C(R, I^n)$.

C) Using the same notation as in B), for any $\varepsilon > 0$ and any 0-dimensional space R, $D_\varepsilon(R, I^0)$ is dense in $C(R, I^0)$, where I^0 is the space of just one point p_0.

Proof. As a matter of fact, $D_\varepsilon(R, I^0)$ coincides with $C(R, I^0)$ itself which consists of just one point $f\colon f(R) = p_0$.

D) Let F be a closed set of a space R. If $h' \in D_\varepsilon(F, I^k)$ and h is a continuous extension of h' over R, then there exists an open set U containing F such that the restriction of h to U belongs to $D_\varepsilon(U, I^k)$, i.e. for some $\eta' > 0$, $\delta(T) \leqq \eta'$ and $T \subset I^k$ imply $\tilde{\delta}(h^{-1}(T) \cap U) < U_\varepsilon$, where U_ε is the same covering as in Definition III.9.

Proof. Suppose $\delta(T) \leqq \eta$ and $T \subset F$ imply $\tilde{\delta}(h'^{-1}(T)) < U_\varepsilon$. Since I^k is compact, we can cover it with finitely many open balls with radius $\frac{1}{4}\eta$, which we denote by $V_i = S_{\frac{1}{4}\eta}(q_i)$, $i = 1, \dots, s$. Define

(1) $$S_i = S_{\frac{1}{2}\eta}(q_i)\ ,\quad i = 1, \dots, s\ .$$

Then there exists an open covering N_i of $h'^{-1}(S_i)$ with $\operatorname{ord} N_i \leqq 1$ and $N_i < U_\varepsilon$. Let $N_i = \{ N_\gamma \mid \gamma \in \Gamma_i \}$; then we can assign to every N_γ an element $A_\gamma \in U_\varepsilon$ such that $N_\gamma \subset A_\gamma$.

For each $p \in h'^{-1}(S_i)$ there exists a unique $N_{\gamma(p)} \in N_i$ such that $p \in N_{\gamma(p)}$. Hence to each point p of $h'^{-1}(S_i)$ we can assign a positive number $\varepsilon_i(p)$ satisfying

i) $F \cap S_{\varepsilon_i(p)}(p) \subset N_{\gamma(p)} \in N_i$,

ii) $S_{\varepsilon_i(p)}(p) \subset A_{\gamma(p)} \in U_\varepsilon$,

iii) $d(h(q),\ h(p)) < \frac{1}{4}\eta$ for every $q \in S_{\varepsilon_i(p)}(p)$.

We let

$$(2)\quad \begin{cases} \varepsilon(p) = \min\{\frac{1}{2}\varepsilon_i(p) \mid h'^{-1}(S_i) \ni p\}, \\ U_\gamma = \cup\{S_{\varepsilon(p)}(p) \mid p \in H_\gamma\} \quad \text{for every } \gamma \in \Gamma_i, \\ U_i = \cup\{U_\gamma \mid \gamma \in \Gamma_i\}, \quad i = 1, \dots, s, \\ U = \bigcup_{i=1}^{s} U_i . \end{cases}$$

Then U is an open set containing F.

Now, to show

$$(3)\qquad h^{-1}(V_i) \cap U \subset U_i$$

we assume the contrary, i.e. $x \in h^{-1}(V_i) \cap U - U_i \neq \emptyset$. Then by (2) $x \notin S_{\varepsilon(p)}(p)$ for every $p \in \cup\{H_\gamma \mid \gamma \in \Gamma_i\}$. On the other hand, since $x \in U$, then necessarily, by (2) $x \in S_{\varepsilon(p)}(p)$ for some $p \in F$. Hence we know that $x \in S_{\varepsilon(p)}(p)$ for some p with

$$p \in F - \cup\{H_\gamma \mid \gamma \in \Gamma_i\} = F - h'^{-1}(S_i) = F - h^{-1}(S_i) .$$

This implies, by virtue of (1) and iii) $d(h(p),q_i) \geq \frac{1}{2}\eta$ and then $d(h(x),q_i) > \frac{1}{4}\eta$. On the other hand, it follows from $x \in h^{-1}(V_i)$ that $h(x) \in V_i$, i.e. $d(h(x),q_i) < \frac{1}{4}\eta$.

This contradiction proves (3).

By (2) $U_i' = \{U_\gamma \mid \gamma \in \Gamma_i\}$ is an open covering of U_i. ii) and (2) imply that $U_\gamma \subset A_\gamma$ for every $\gamma \in \Gamma_i$ and accordingly $U_i' < U_\varepsilon$. Since N_i has order ≤ 1, i) and (2) imply that ord $U_i' \leq 1$. Hence $\delta(U_i) < U_\varepsilon$. By virtue of (3) this proves $\tilde{\delta}(h^{-1}(V_i) \cap U) < U_\varepsilon$.

Since I^k is compact, by I.2 A) we can find $\eta' > 0$ such that if a subset T of I^k has diameter $\leq \eta'$, then $T \subset V_i$ for some i. Then $\delta(T) \leq \eta'$ implies $\tilde{\delta}(h^{-1}(T) \cap U) < U_\varepsilon$.

Thus the proof of this assertion is complete.

E) For any $\varepsilon > 0$ and any space R with $\dim R \leq n$, $D_\varepsilon(R,I^n)$ is dense in $C(R,I^n)$.

Proof. Since for $n = 0$ this assertion is proved in C), to use induction on n we assume the assertion valid for $n-1$.

Let Φ be a given point of $C(R,I^n)$ and ε' a given positive number. Suppose

$$\Phi(p) = (g(p),\varphi(p)), \quad g \in C(R,I^{n-1}), \quad \varphi \in C(R,I^1) .$$

Then, since $\dim R \leq n$, in view of the proof of 7 A), we can easily construct an open collection $M = \{M_\gamma \mid \gamma \in \Gamma\}$ in R such that

i) M is a locally finite open collection with ord $M \leq 1$,

ii) $\overline{\cup\{M_\gamma \mid \gamma \in \Gamma\}} = R$,

iii) $M < \mathfrak{U}_\varepsilon$ for the covering $\mathfrak{U}_\varepsilon$ in Definition III.9,

iv) $\dim \cup \{ B(M_\gamma) \mid \gamma \in \Gamma \} \leq n-1$,

v) $| \varphi(a) - \varphi(b) | < \frac{1}{3}\varepsilon'$ if $a,b \in \bar{M}_\gamma$.

(Choose a sufficiently small $\mathfrak{U}$ in the said proof and $\mathfrak{V}$ satisfying $\dim B(V_\alpha) \leq n-1$ for every α ; then put $M_\alpha = V_\alpha - \cup \{ \bar{V}_\beta \mid \beta < \alpha \}$.) By applying the induction hypothesis to $F = \cup \{ (B(M_\gamma) \mid \gamma \in \Gamma \}$ we get a mapping $h' \in D_\varepsilon(F, I^{n-1})$ such that $d(h'(p), g(p)) \leq \delta < \frac{1}{6}\varepsilon'$ for every $p \in F$. By Theorem I.7 we can continuously extend h' to a mapping h_0 of R into I^{n-1} . Then, for each $p \in F$ we choose a neighbourhood $N(p)$ such that $q \in N(p)$ implies $d(h_0(p), h_0(q)) < \delta$ and $d(g(p), g(q)) < \delta$.

Define $N = \cup \{ N(p) \mid p \in F \}$. Then we construct a real valued continuous function α on R satisfying $\alpha(F) = 0$, $\alpha(R-N) = 1$, and $0 \leq \alpha \leq 1$.

Putting $h = \alpha g + (1-\alpha)h_0$ in vectorial notation, we have a continuous extension h of h' over R .

Since $d(h_0(p), g(p)) \leq \delta$ for every $p \in F$, it follows from the definition of N that $d(h(q), g(q)) \leq 3\delta$ for every $q \in R$, and hence

(1) $\qquad \rho(h,g) \leq 3\delta < \frac{1}{2}\varepsilon'$

where ρ denotes the metric in $C(R, I^{n-1})$. Then, by use of D) we can construct an open set U such that $U \supset F$, and

(2) $\qquad$ for some $\eta' > 0$, $\delta(T) \leq \eta'$ and $T \subset I^{n-1}$ imply $\tilde{\delta}(U \cap h^{-1}(T)) < \mathfrak{U}_\varepsilon$.

Now, we consider a real valued continuous function σ on R such that $\sigma(R-U) = 0$, $\sigma(F) = 1$ and $0 \leq \sigma \leq 1$, and put $\psi = \varphi + \frac{1}{2}\varepsilon'\sigma$. Then $\Psi(p) = (h(p), \psi(p))$ is a mapping which satisfies $\Psi \in C(R, I^n)$ and $\rho(\Psi, \Phi) < \varepsilon'$.

We can easily prove this assertion by use of (1) and the definition of ψ . Thus to complete this proof it remains only to prove that $\Psi \in D_\varepsilon(R, I^n)$.

For this purpose we shall show that if

(3) $\qquad \delta(T) \leq \min (\eta', \frac{1}{6}\varepsilon')$

for a closed set T of I^n , then

(4) $\qquad$ either $\bar{M}_\gamma \cap \Psi^{-1}(T) \subset R - F$ or $\bar{M}_\gamma \cap \Psi^{-1}(T) \subset U$

for each $\gamma \in \Gamma$.

Let us assume the contrary; then for some γ we get two points a and b such that $a \in F \cap \bar{M}_\gamma \cap \Psi^{-1}(T) \neq \emptyset$, which implies $\psi(a) = \varphi(a) + \frac{1}{2}\varepsilon'\sigma(a) = \varphi(a) + \frac{1}{2}\varepsilon'$ and $b \in (R-U) \cap \bar{M}_\gamma \cap \Psi^{-1}(T) \neq \emptyset$, which implies $\psi(b) = \varphi(b) + \frac{1}{2}\varepsilon'\sigma(b) = \varphi(b)$.

Since $\psi(a)$ and $\psi(b) \in T$, in view of (3) we know that

$$| \varphi(a) - \varphi(b) + \frac{1}{2}\varepsilon' | = | \psi(a) - \psi(b) | \leq \frac{1}{6}\varepsilon' .$$

Hence $| \varphi(a) - \varphi(b) | \geq \frac{1}{3}\varepsilon'$. This, however, contradicts v) and accordingly proves (4).

We let

$$\Gamma' = \{\gamma \mid \gamma \in \Gamma,\ \bar{M}_\gamma \cap \Psi^{-1}(T) \subset R - F\}\ , \text{ and } \Gamma'' = \{\gamma \mid \gamma \in \Gamma,\ \bar{M}_\gamma \cap \Psi^{-1}(T) \subset U\}\ ;$$

then by (4) $\Gamma = \Gamma' \cup \Gamma''$. Suppose T is a given closed set of I^n which satisfies (3). If $\gamma \in \Gamma'$, then by the definition of F

$$\bar{M}_\gamma \cap \Psi^{-1}(T) \subset \bar{M}_\gamma \cap \Psi^{-1}(T) \cap (R - F)$$

$$\subset \bar{M}_\gamma \cap \Psi^{-1}(T) \cap (R - B(M_\gamma)) \subset M_\gamma \ , \text{ i.e.}$$

(5) $\bar{M}_\gamma \cap \Psi^{-1}(T) \subset M_\gamma$ for every $\gamma \in \Gamma'$.

If $\gamma \in \Gamma''$, then note that $\Psi^{-1}(T) \subset h^{-1}(T')$, where T' denotes the projection of T on I^{n-1} . Therefore

$$\bar{M}_\gamma \cap \Psi^{-1}(T) \subset U \cap \bar{M}_\gamma \cap \Psi^{-1}(T) \subset U \cap h^{-1}(T) \quad \text{for every } \gamma \in \Gamma''\ .$$

Hence

(6) $A = \cup\{\bar{M}_\gamma \mid \gamma \in \Gamma''\} \cap \Psi^{-1}(T) \subset U \cap h^{-1}(T')$.

On the other hand, we can deduce from (2), (3) and $\delta(T') \leqq \delta(T)$ that $\tilde{\delta}(U \cap h^{-1}(T')) < \mathcal{U}_\varepsilon$. In consequence, (6) implies

(7) $\delta(A) < \mathcal{U}_\varepsilon$.

From the closedness of T and i) it follows that A is closed. Therefore

$$K = \{A, \bar{M}_\gamma \cap \Psi^{-1}(T) \mid \gamma \in \Gamma' - \Gamma''\}$$

is a discrete closed collection by virtue of (5) and i).

Now we choose an open set W of R such that $W \supset A$ and $\bar{W} \cap \bar{M}_\gamma \cap \Psi^{-1}(T) = \emptyset$ for all $\gamma \in \Gamma' - \Gamma''$.

Because of (7) there is an open collection $\mathcal{W}$ of order $\leqq 1$ which covers A and refines $\mathcal{U}_\varepsilon$. We may assume that each member of $\mathcal{W}$ is contained in W . Then $\mathcal{W} \cup \{M_\gamma \cap (R - \bar{W}) \mid \gamma \in \Gamma' - \Gamma''\}$ is an open collection of order $\leqq 1$ which refines $\mathcal{U}_\varepsilon$ and covers $\cup\{K \mid K \in K\} = \Psi^{-1}(T)$. (Recall i), ii), iii) and (5).)

This implies $\tilde{\delta}(\Psi^{-1}(T)) < \mathcal{U}_\varepsilon$, and thus $\Psi \in D_\varepsilon(R, I^n)$ is proved, and hence $D_\varepsilon(R, I^n)$ is dense in $C(R, I^n)$.

Theorem III. 10 [a] . *A space* R *has* $\dim \leqq n$ $(n \geqq 0)$ *if and only if it satisfies*

1) for any metrization of R *there exists a uniformly 0-dimensional continuous mapping of* R *into* I^n *or*

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[a] Proved first by M. Katětov [2].

2) *for a suitable metrization of* R *there exists a uniformly 0-dimensional continuous mapping of* R *into* I^n .

Proof. Let $\dim R \leq n$; then from B) and E) it follows that $D_{1/i}(R,I^n)$, $i = 1,2,\ldots$ are open and dense in $C(R,I^n)$, for any metric of R . Since $C(R,I^n)$ is a complete metric space, by Theorem I.5 we know that $\bigcap_{i=1}^{\infty} D_{1/i}(R,I^n) \neq \emptyset$. Thus $f \in \bigcap_{i=1}^{\infty} D_{1/i}(R,I^n)$ is the desired mapping in 1).

1) obviously implies 2). Finally, by A) we can deduce $\dim R \leq n$ from 2) because $\dim I^n \leq n$.