CHAPTER II

DIMENSION OF METRIC SPACES

Throughout chapter II – chapter VI all spaces are metric unless the contrary is explicitly stated. We shall begin this chapter with the sum theorem and shall deduce further fundamental theorems from it.

II. 1. Lemmas to sum theorem

A) Let F be a closed subset of a space R . Then $\operatorname{Ind} F \leqq \operatorname{Ind} R$.

Proof. The proof is by induction on the dimension of R . If $\operatorname{Ind} R = -1$, then $R = \emptyset$, which implies $F = \emptyset$; hence $\operatorname{Ind} F = -1$.

Assume A) for R with $\operatorname{Ind} R \leqq n-1$. If $\operatorname{Ind} R \leqq n$, then for given disjoint closed sets G and H of the subspace F there exists an open set U of R such that

$$G \subset U \subset R - H \text{ , and } \operatorname{Ind} B(U) \leqq n-1 \text{ ,}$$

because G and H are also closed in R . Then $V = U \cap F$ is an open set of the subspace F and satisfies

$$G \subset V \subset F - H \text{ , and } B_F(V) \subset B(U) \text{ ,}$$

where $B_F(V)$ denotes the boundary of V in the subspace F . Hence $\operatorname{Ind} B_F(V) \leqq n-1$ follows from $\operatorname{Ind} B(U) \leqq n-1$ and the induction hypothesis. Thus we get $\operatorname{Ind} F \leqq n$.

B) Let C and D be subsets of a space R . If $(\bar{C} \cap D) \cup (C \cap \bar{D}) = \emptyset$, then there exist open sets M and N satisfying $C \subset M$, $D \subset N$, $M \cap N = \emptyset$. This assertion can be deduced from I.1 B) and the hereditary normality of R, but we shall directly prove it as follows:

Proof. We assign to each point $p \in C$ a positive number $\varepsilon(p)$ satisfying $S_{\varepsilon(p)}(p) \cap D = \emptyset$ and to each point $q \in D$ a positive number $\varepsilon(q)$ satisfying $S_{\varepsilon(q)}(q) \cap C = \emptyset$.

Putting

$$M = \cup \{ S_{\frac{1}{2}\varepsilon(p)}(p) \mid p \in C \} , \text{ and } N = \cup \{ S_{\frac{1}{2}\varepsilon(q)}(q) \mid q \in D \}$$

we get the desired open sets.

C) Let A and B be subsets of a space R such that

$$R = A \cup B , \quad \operatorname{Ind} A \leqq n , \quad \operatorname{Ind} B \leqq 0 ;$$

then $\operatorname{Ind} R \leqq n + 1$.

Proof. Let F and G be given disjoint closed sets of R; then by virtue of the normality of R there exist open sets V and W satisfying

$$F \subset V , \quad G \subset W , \quad \text{and} \quad \bar{V} \cap \bar{W} = \emptyset .$$

Since $\operatorname{Ind} B \leqq 0$, we can find an open and closed set U of the subspace B such that

$$\bar{V} \cap B \subset U \subset B - (\bar{W} \cap B) .$$

Since $F \cup U$ and $G \cup (B - U)$ satisfy the condition of B)[1], we can find open sets M and N such that

$$F \cup U \subset M , \quad G \cup (B - U) \subset N , \quad \text{and} \quad M \cap N = \emptyset .$$

$B(M) \subset A$ is obvious by the above property of N and by the fact that U is open and closed in B.

Hence $\operatorname{Ind} B(M) \leqq \operatorname{Ind} A \leqq n$ follows from A). Since M is an open set satisfying $F \subset M \subset R - G$, we have $\operatorname{Ind} R \leqq n + 1$.

D) If $\operatorname{Ind} R \leqq n$, then there exists a σ-locally finite open basis $\mathcal{V}$ of R such that $\operatorname{Ind} B(V) \leqq n - 1$ for every $V \in \mathcal{V}$.

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[1] We mean by B) Proposition B) of the same section and chapter, by 1 B) Proposition B) of Section 1 of the same chapter and by I.1 B) Proposition B) of Section 1 of Chapter I.

Proof. By Theorem I.1 or by Theorem I.3 there exists a σ - locally finite open basis $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ of R, where we may suppose each $\mathcal{U}_i$ to be a locally finite open covering of R such that $\lim_{i\to\infty} \text{mesh}\, \mathcal{U}_i = 0$. Let $\mathcal{U}_i = \{ U_\alpha \mid \alpha \in A_i \}$; then by I.1 A) there exists an open covering $\mathcal{W}_i = \{ W_\alpha \mid \alpha \in A_i \}$ satisfying $\bar{W}_\alpha \subset U_\alpha$. Since $\text{Ind}\, R \leqq n$, we can construct open sets V_α for $\alpha \in A_i$ such that

$$\bar{W}_\alpha \subset V_\alpha \subset U_\alpha, \quad \text{and} \quad \text{Ind}\, B(V_\alpha) \leqq n - 1 .$$

Putting $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, and $\mathcal{V}_i = \{ V_\alpha \mid \alpha \in A_i \}$ we get the desired open basis.

E) If a space R has a σ - locally finite open basis $\mathcal{V}$ such that $B(V) = \emptyset$ for every $V \in \mathcal{V}$, then $\text{Ind}\, R \leqq 0$.

Proof. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ for locally finite open collections $\mathcal{V}_i$. Let F and G be disjoint closed sets of R. Then for every i

$$U_i = R - \cup \{ V \mid V \in \bigcup_{j=1}^{i} \mathcal{V}_j ,\ V \cap F = \emptyset \}$$

is an open and closed set containing F because $\bigcup_{j=1}^{i} \mathcal{V}_j$ is a locally finite collection of open and closed sets V. Thus we get a sequence $\{ U_i \mid i = 1,2,\dots \}$ of open and closed sets such that

$$U_1 \supset U_2 \supset \dots \supset F , \quad \text{and} \quad \bigcap_{i=1}^{\infty} U_i = F .$$

In a similar way we can construct a sequence $\{ W_i \mid i = 1,2,\dots \}$ of open and closed sets such that

$$W_1 \supset W_2 \supset \dots \supset G , \quad \text{and} \quad \bigcap_{i=1}^{\infty} W_i = G .$$

Now

$$U = \bigcup_{i=1}^{\infty} (U_i - W_i)$$

is an open and closed set and satisfies $F \subset U \subset R - G$. Therefore $\text{Ind}\, R \leqq 0$.

F) Let $A \subset R$, $\text{Ind}\, R \leqq 0$; then $\text{Ind}\, A \leqq 0$.

Proof. F) is a direct consequence of D) and E).

II. 2. Sum theorem

A) Let $\{ F_i \mid i = 1,2,\dots \}$ be a closed covering of a space R such that

$\operatorname{Ind} F_i \leqq 0$ for $i = 1, 2, \ldots$.

Then $\operatorname{Ind} R \leqq 0$.

Proof. Let G and H be given disjoint closed sets of R . Since $\operatorname{Ind} F_1 \leqq 0$, there exists an open and closed subset U_1 of F_1 such that

$$F_1 \cap H \subset U_1 \subset F_1 - G \,.$$

Since $H \cup U_1$ and $G \cup (F_1 - U_1)$ are disjoint closed sets, by the normality of R we can find open sets V_1 and W_1 such that

(1) $$H \cup U_1 \subset V_1 \,, \quad G \cup (F_1 - U_1) \subset W_1 \,, \quad \text{and} \quad \bar{V}_1 \cap \bar{W}_1 = \emptyset \,.$$

Since $\operatorname{Ind} F_2 \leqq 0$, there exists an open and closed subset U_2 of F_2 such that

$$F_2 \cap H \subset F_2 \cap \bar{V}_1 \subset U_2 \subset F_2 - \bar{W}_1 \subset F_2 - G \,.$$

Since $\bar{V}_1 \cup U_2$ and $\bar{W}_1 \cup (F_2 - U_2)$ are disjoint closed sets, we can find open sets V_2 and W_2 such that

$$\bar{V}_1 \cup U_2 \subset V_2 \,, \ \bar{W}_1 \cup (F_2 - U_2) \subset W_2 \,, \ \text{and} \quad \bar{V}_2 \cap \bar{W}_2 = \emptyset \,.$$

By repeating this process we get sequences

$$\{ U_i \mid i = 1,2,\ldots \} \,, \quad \{ V_i \mid i = 1,2,\ldots \} \,, \quad \{ W_i \mid i = 1,2,\ldots \}$$

of sets such that

(2) $$F_i \cap H \subset U_i \subset F_i - G \,,$$

each U_i is an open and closed subset of F_i ,

(3) $$\bar{V}_{i-1} \cup U_i \subset V_i \,, \quad \bar{W}_{i-1} \cup (F_i - U_i) \subset W_i \,, \ i = 2,3,\ldots \,,$$

(4) $$\bar{V}_i \cap \bar{W}_i = \emptyset \,,$$

V_i and W_i are open sets.

Now we put

$$V = \bigcup_{i=1}^{\infty} V_i \,, \quad W = \bigcup_{i=1}^{\infty} W_i \,;$$

then from (1) it follows that V and W contain H and G respectively. Furthermore it follows from (3) and (4) that V and W are disjoint open sets. Since (1) and (3) imply

$$V_i \cup W_i \supset U_i \cup (F_i - U_i) = F_i \,, \ i = 1,2,\ldots \,,$$

we get

$$V \cup W \supset \bigcup_{i=1}^{\infty} F_i = R .$$

Thus V is an open and closed set satisfying $H \subset V \subset R - G$, which yields $\operatorname{Ind} R \leqq 0$.

B) Let $\{ F_\gamma \mid \gamma \in \Gamma \}$ be a locally finite closed covering of a space R such that $\operatorname{Ind} F_\gamma \leqq 0$ for every $\gamma \in \Gamma$.
Then $\operatorname{Ind} R \leqq 0$.

Proof. For each positive integer i we take, using the corollary to Theorem I.1, a locally finite open covering $\mathcal{U}_i = \{ U_\delta \mid \delta \in \Delta_i \}$ such that mesh $\mathcal{U}_i < 1/i$ and such that the closure $\bar{U}_\delta$ of each element of $\mathcal{U}_i$ intersects only finitely many elements of $\{ F_\gamma \mid \gamma \in \Gamma \}$. Then

(1) $\operatorname{Ind} \bar{U}_\delta \leqq 0$

follows from A). On the other hand, by I.1 A) there exists an open covering $\mathcal{V}_i = \{ V_\delta \mid \delta \in \Delta_i \}$ with $\bar{V}_\delta \subset U_\delta$ for every $\delta \in \Delta_i$.
Hence, by virtue of (1) we can find an open and closed set W_δ of $\bar{U}_\delta$ such that

$$\bar{V}_\delta \subset W_\delta \subset \bar{U}_\delta - (R - U_\delta) .$$

Since this relation implies $W_\delta \subset U_\delta$, W_δ is clearly also open and closed in R.
Thus, putting

$$\mathcal{W}_i = \{ W_\delta \mid \delta \in \Delta_i \} , \quad \mathcal{W} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$$

we get a σ-locally finite open basis $\mathcal{W}$ consisting of open and closed sets. Hence by 1 E) $\operatorname{Ind} R \leqq 0$.

C) Let $\{ F_i \mid i = 1,2,\dots \}$ be a closed covering of a space R such that $\operatorname{Ind} F_i \leqq n$, $i = 1,2,\dots$.
Then $\operatorname{Ind} R \leqq n$.

D) Let $\{ F_\gamma \mid \gamma \in \Gamma \}$ be a locally finite closed covering of a space R such that $\operatorname{Ind} F_\gamma \leqq n$ for every $\gamma \in \Gamma$.
Then $\operatorname{Ind} R \leqq n$.

Proof. The validity of C) and D) is clear for $n = -1$. To prove C) and D) at the same time by induction we assume C) and D) for $n = m - 1$ $(m \geqq 0)$. First let us show C) for $n = m$.

Since $\operatorname{Ind} F_i \leqq m$, by 1 D) there exists a σ-locally finite open basis $\mathcal{V}_i = \bigcup_{k=1}^{\infty} \mathcal{V}_{i,k}$ of the subspace F_i such that

(1) $\operatorname{Ind} B_{F_i}(V) \leq m - 1$ for $V \in \mathcal{V}_i$.

Since the σ - local finiteness of $\mathcal{V}_i$ implies the σ - local finiteness of

$$\mathcal{V}_i' = \{ B_{F_i}(V) \mid V \in \mathcal{V}_i \} ;$$

the closed collection $\bigcup_{i=1}^{\infty} \mathcal{V}_i'$ is also σ - locally finite.

Let

$$H_i = \bigcup \{ B_{F_i}(V) \mid V \in \mathcal{V}_i \} ;$$

then $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a σ - locally finite closed covering of $\bigcup_{i=1}^{\infty} H_i$. Hence from (1) and the induction hypothesis it follows that

(2) $\operatorname{Ind} \bigcup_{i=1}^{\infty} H_i \leq m - 1$.

On the other hand $\mathcal{V}_i = \bigcup_{k=1}^{\infty} \mathcal{V}_{i,k}$ restricted to $F_i - H_i$ constitutes an open basis of $F_i - H_i$ satisfying the condition of I E). Hence we have $\operatorname{Ind} (F_i - H_i) \leq 0$. Hence, if we define G_i by $G_i = F_i - \bigcup_{j=1}^{\infty} H_j$, then by I F) $\operatorname{Ind} G_i \leq 0$. Since each G_i is closed in $\bigcup_{i=1}^{\infty} G_i$,

(3) $\operatorname{Ind} \bigcup_{i=1}^{\infty} G_i \leq 0$

follows from A).

Since

$$R = (\bigcup_{i=1}^{\infty} G_i) \cup (\bigcup_{i=1}^{\infty} H_i) ,$$

it follows from I C) combined with (2) and (3) that $\operatorname{Ind} R \leq (m - 1) + 1 = m$. This proves C) for $n = m$.

We can prove D) in a similar way by using I C), I D), I E), I F) and B) .

Theorem II. 1 (Sum Theorem) [2] . *Let* $\{ F_\gamma \mid \gamma \in \Gamma \}$ *be a locally countable closed covering of a space* R *such that* $\operatorname{Ind} F_\gamma \leq n$ *for every* $\gamma \in \Gamma$. *Then* $\operatorname{Ind} R \leq n$.

Proof. Let $\mathcal{V}$ be an open covering each of whose elements meets at most countably many members of $\{ F_\gamma \mid \gamma \in \Gamma \}$. By virtue of the paracompactness and regularity of R we can find a locally finite open covering $\mathcal{U}$ such that $\bar{\mathcal{U}} < \mathcal{V}$. Suppose $\mathcal{U} = \{ U_\delta \mid \delta \in \Delta \}$. Then each $\bar{U}_\delta$ meets at most countably many members of

[2] The sum theorem in the present form is due to K. Morita [4] and essentially to K. Morita [2] and M. Katětov [2].

$\{F_\gamma \mid \gamma \in \Gamma\}$, and $\operatorname{Ind} F_\gamma \cap \bar{U}_\delta \leqq n$ follows from $\operatorname{Ind} F_\gamma \leqq n$ and I A). Hence by C) we have $\operatorname{Ind} \bar{U}_\delta \leqq n$. This implies $\operatorname{Ind} R \leqq n$ by virtue of D).

Corollary [3] . *Let* $\{F_\gamma \mid \gamma < \tau\}$ *be a covering of a space* R *such that* $\operatorname{Ind} F_\gamma \leqq n$ *for every* $\gamma < \tau$ *and such that* $\cup\{F_\gamma \mid \gamma < \delta\}$ *is closed for any* $\delta < \tau$. *Then* $\operatorname{Ind} R \leqq n$.

Proof. Let

$$G_{\delta,i} = F_\delta - S_{1/i}(\cup\{F_\gamma \mid \gamma < \delta\}) \quad \text{for } \delta < \tau,\ i = 1,2,\ldots,$$

where $S_{1/i}(F)$ for a set F denotes the $1/i$-neighbourhood of F , i.e. the set $\{y \mid \rho(x,y) < 1/i$ for some $x \in F\}$. Then $\{G_{\delta,i} \mid \delta < \tau\}$ is discrete, and hence $\{G_{\delta,i} \mid \delta < \tau,\ i = 1,2,\ldots\}$ is a σ-discrete closed covering of R . Since by I A) $\operatorname{Ind} G_{\delta,i} \leqq n$, we have $\operatorname{Ind} R \leqq n$ by the sum theorem.

II. 3 Decomposition Theorem

Theorem II. 2. $\operatorname{Ind} R \leqq n$ *for* $n \geqq 0$ *if and only if there exists a* σ-*locally finite open basis* $\mathcal{V}$ *such that* $\operatorname{Ind} B(V) \leqq n-1$ *for every* $V \in \mathcal{V}$.

Proof. The "only if" part is proved in I D). The "if" part for $n=0$ is proved in I E). To prove the "if" part for any integer $n \geqq 0$ we put

$$A = \cup\{B(V) \mid V \in \mathcal{V}\}, \text{ and } B = R - A .$$

Since $\{B(V) \mid V \in \mathcal{V}\}$ is a σ-locally finite closed collection, from the sum theorem and $\operatorname{Ind} B(V) \leqq n-1$ for $V \in \mathcal{V}$ it follows that $\operatorname{Ind} A \leqq n-1$. Since $\mathcal{V}$ restricted to B is an open basis of B satisfying the condition of I E), we have $\operatorname{Ind} B \leqq 0$. Hence by virtue of I C)

$$\operatorname{Ind} R = \operatorname{Ind} A \cup B \leqq (n-1) + 1 = n .$$

Theorem II. 3. (Subspace Theorem). *For every subset* A *of a space* R $\operatorname{Ind} A \leqq \operatorname{Ind} R$.

Proof. This is an immediate consequence of Theorem II.2.

Theorem II. 4. (Decomposition Theorem). $\operatorname{Ind} R \leqq n$ *for* $n \geqq 0$ *if and only if* $R = \cup_{i=1}^{n+1} A_i$ *for some* $n+1$ *subsets* A_i *with* $\operatorname{Ind} A_i \leqq 0$, $i = 1,\ldots,n+1$.

Proof. The "if" part is directly deduced from I C). As for the "only if" part we saw in the proof of Theorem II.2 that R can be decomposed into two subsets A_1 and B_1 with $\operatorname{Ind} A_1 \leqq 0$, and $\operatorname{Ind} B_1 \leqq n-1$.

[3] Proved first by K. Nagami [1]. γ, τ, δ denote ordinal numbers.

We can decompose B_1 into two subsets A_2 and B_2 with $\text{Ind } A_2 \leqq 0$, and $\text{Ind } B_2 \leqq n-2$.

By repeating this process R can be decomposed into $n+1$ 0-dimensional subsets $A_1, \ldots, A_{n+1}$.

Corollary. $\text{Ind } A \cup B \leqq \text{Ind } A + \text{Ind } B + 1$ *for any subsets* A, B *of a space* R.

Proof. We can easily prove this corollary by decomposing A and B into 0-dimensional subsets by use of Theorem II.4.

II. 4. Product theorem

A) If V and W are subsets of spaces R and S respectively, then

$$B_{R \times S}(V \times W) = B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W).$$

Proof. To prove

$$B_{R \times S}(V \times W) \subset B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W),$$

let

$$(p,q) \notin B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W),$$

where $p \in R$, $q \in S$. Then the following holds.

(1) $p \notin B_R(V)$ or $q \notin \bar{W}$

(2) $p \notin \bar{V}$ or $q \notin B_S(W)$.

Since $q \notin \bar{W}$ or $p \notin \bar{V}$ obviously implies $(p,q) \notin B_{R \times S}(V \times W)$, suppose $q \in \bar{W}$, $p \in \bar{V}$, which implies by (1) and (2) that $p \notin B_R(V)$, $q \notin B_S(W)$. Therefore $p \in \text{Int}_R V$, $q \in \text{Int}_S W$, where $\text{Int}_R V$ for a subset V of R denotes the interior of V in R. This means $(p,q) \in \text{Int}_{R \times S}(V \times W)$, and hence $(p,q) \notin B_{R \times S}(V \times W)$. Thus we have

$$B_{R \times S}(V \times W) \subset B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W).$$

Since

$$B_{R \times S}(V \times W) \supset B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W)$$

is clear, we can conclude A).

Theorem II. 5. (Product Theorem)[4]. *Let* R *and* S *be spaces at least one of which is non-empty. Then* $\text{Ind } R \times S \leqq \text{Ind } R + \text{Ind } S$.

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[4] Proved first by M.Katětov [2] and K.Morita [4]. Recently E.Pol [2] proved the following theorem for infinite products: Let $\{R_\alpha \mid \alpha \in A\}$ be a collection of complete metric spaces such that $\dim R_{\alpha_1} \times \ldots \times R_{\alpha_k} \leqq n$ for any $\alpha_1, \ldots, \alpha_k \in A$; then $\dim \prod_{\alpha \in A} R_\alpha \leqq n$.

Proof. Let $\operatorname{Ind} R = n$, $\operatorname{Ind} S = m$. We shall prove this theorem by induction on the number $n + m$. If $n + m = -1$, this inequality clearly holds. Generally the theorem is obvious if either R or S is empty, so assume $n \geqq 0$ and $m \geqq 0$.

To complete the induction, note that by Theorem II.2 there exists a σ-locally finite open basis $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ of R such that

(1) $\operatorname{Ind} B_R(V) \leqq n - 1$ for every $V \in \mathcal{V}$

and a σ-locally finite open basis $\mathcal{W} = \bigcup_{j=1}^{\infty} \mathcal{W}_j$ of S such that

(2) $\operatorname{Ind} B_S(W) \leqq m - 1$ for every $W \in \mathcal{W}$,

where $\mathcal{V}_i$ and $\mathcal{W}_j$ are locally finite open collections of R and S respectively. Put

$$\mathcal{V}_i \times \mathcal{W}_j = \{ V \times W \mid V \in \mathcal{V}_i ,\ W \in \mathcal{W}_j \} ;$$

then $\mathcal{V}_i \times \mathcal{W}_j$ is a locally finite open collection of $R \times S$. Hence $\bigcup_{i,j=1}^{\infty} \mathcal{V}_i \times \mathcal{W}_j$ is a σ-locally finite open basis of $R \times S$.

On the other hand, it follows from (1) and (2) combined with the induction hypothesis, that

$$\operatorname{Ind} B_R(V) \times \bar{W} \leqq n + m - 1 , \quad \text{and} \quad \operatorname{Ind} \bar{V} \times B_S(W) \leqq n + m - 1$$

for every $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Since

$$B_{R \times S}(V \times W) = B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W)$$

follows from A), we get, from the sum theorem,

$$\operatorname{Ind} B_{R \times S}(V \times W) \leqq n + m - 1$$

for every $V \times W \in \bigcup_{i,j=1}^{\infty} \mathcal{V}_i \times \mathcal{W}_j$.

Hence by Theorem II.2 we conclude $\operatorname{Ind} R \times S \leqq n + m$.

As the following examples shows [5], the equality $\operatorname{Ind} R \times S = \operatorname{Ind} R + \operatorname{Ind} S$ is not true in general even if R and S are separable.

Example II. 1 (P. Erdös [1]). Let R^{ω} be the set of points in Hilbert space all of whose coordinates are rational. Then $\operatorname{Ind} R^{\omega} = 1$ can be proved as follows. Since $R^{\omega} = R^{\omega} \times R^{\omega}$,

$$\operatorname{Ind} R^{\omega} \times R^{\omega} < \operatorname{Ind} R^{\omega} + \operatorname{Ind} R^{\omega} .$$

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[5] W. Hurewicz [5] proved that the equality holds if R is a compact metric space with $\operatorname{Ind} R \geq 0$ and S is a separable metric space with $\operatorname{Ind} S = 1$. Recently K. Morita [7], [8] proved the equality $\dim R \times S = \dim R + \dim S$ under a much more general condition, which implies e.g. that $\operatorname{Ind} R \times S = \operatorname{Ind} R + \operatorname{Ind} S$ holds if R and S are nonempty metric spaces such that R is a countable sum of locally compact closed subsets and $\operatorname{Ind} S = 1$.

Now, let us show $\operatorname{Ind} R^{\omega} = 1$. Since R^{ω} is a separable metric space, and, as will be proved later (Theorem IV.4), ind = Ind = dim holds for every separable metric space, it suffices to show $\operatorname{ind} R^{\omega} = 1$. To prove $\operatorname{ind} R^{\omega} \leq 1$, we shall show that $\operatorname{ind} B(S_{\varepsilon}(p_0)) \leqq 0$ for the point $p_0 = (0,0,\dots)$ of R^{ω} and every positive rational number ε , where $S_{\varepsilon}(p_0) = \{ p \in R^{\omega} \mid \rho(p,p_0) < \varepsilon \}$. Since each point of R^{ω} can be mapped to p_0 by a homeomorphism of R^{ω} onto itself, this would be sufficient. Let $p = (p_1,p_2,\dots) \in B(S_{\varepsilon}(p_0))$ and δ a given positive number such that δ^2 is irrational. Select n such that $\sum_{i=n+1}^{\infty} p_i^2 < \frac{\delta^2}{2}$. Put

$$U = \{ (x_1,x_2,\dots) \in R^{\omega} \mid \sum_{i=1}^{n} x_i p_i > \varepsilon^2 - \frac{\delta^2}{2} \} .$$

Then U is an open neighbourhood of p because $\sum_{i=1}^{\infty} p_i^2 = \varepsilon^2$ and accordingly $\sum_{i=1}^{n} p_i^2 > \varepsilon^2 - \frac{\delta^2}{2}$. Let $y = (y_1,y_2,\dots) \in B(U) \cap B(S_{\varepsilon}(p_0))$; then $\sum_{i=1}^{n} y_i p_i = \varepsilon^2 - \frac{\delta^2}{2}$, which cannot happen because y_i and p_i are all rational, and $\varepsilon^2 - \frac{\delta^2}{2}$ is irrational. Thus $B(U) \cap B(S_{\varepsilon}(p_0))$ and accordingly the boundary of $U \cap B(S_{\varepsilon}(p_0))$ in $B(S_{\varepsilon}(p_0))$ is empty.

To prove $U \cap B(S_{\varepsilon}(p_0)) \subset S_{\delta}(p)$, let $x = (x_1,x_2,\dots) \in U \cap B(S_{\varepsilon}(p_0))$. Then

$$\sum_{i=1}^{n} x_i p_i > \varepsilon^2 - \frac{\delta^2}{2} , \qquad \sum_{i=1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} p_i^2 = \varepsilon^2 .$$

Thus we obtain $\sum_{i=1}^{\infty} (x_i - p_i)^2 < \delta^2$, proving our claim. Hence $\operatorname{ind} B(S_{\varepsilon}(p_0)) \leqq 0$, i.e. $\operatorname{ind} R^{\omega} \leqq 1$.

To prove $\operatorname{ind} R^{\omega} \geqq 1$, let U be a given open neighborhood of p_0 such that $U \subset S_1(p_0)$. Select a fixed point $q_0 = \left(\frac{1}{n_1} , \frac{1}{n_2} , \dots \right) \in U$ such that $n_1,n_2,\dots$ are natural numbers. Then, since $\left(1, \frac{1}{n_2} , \frac{1}{n_3} , \dots \right) \notin S_1(p_0) \supset U$, we can choose a natural number m_1 such that

$$a_1 = \left(\frac{1}{m_1} , \frac{1}{n_2} , \frac{1}{n_3} , \dots \right) \notin U , \text{ and } b_1 = \left(\frac{1}{m_1+1} , \frac{1}{n_2} , \frac{1}{n_3} , \dots \right) \in U .$$

Since $\left(\frac{1}{m_1+1} , 1, \frac{1}{n_3} , \dots \right) \notin S_1(p_0) \supset U$, looking at b_1 , we can choose a natural number m_2 such that

$$a_2 = \left(\frac{1}{m_1+1} , \frac{1}{m_2} , \frac{1}{n_3} , \dots \right) \notin U , \text{ and } b_2 = \left(\frac{1}{m_1+1} , \frac{1}{m_2+1} , \frac{1}{n_3} , \dots \right) \in U .$$

Repeating the same process we get natural numbers $m_1,m_2,m_3,\dots$ such that

$$a_k = \left(\frac{1}{m_1+1} , \dots , \frac{1}{m_{k-1}+1} , \frac{1}{m_k} , \frac{1}{n_{k+1}} , \dots \right) \notin U , \text{ and}$$

$$b_k = \left(\frac{1}{m_1 + 1}, \ldots, \frac{1}{m_{k-1} + 1}, \frac{1}{m_k + 1}, \frac{1}{n_{k+1}}, \ldots \right) \in U .$$

Then both $\{a_k\}$ and $\{b_k\}$ converge to the point $q = \left(\frac{1}{m_1 + 1}, \frac{1}{m_2 + 1}, \ldots \right) \in R^{\omega}$, from outside and inside of U, respectively. (Note that $\sum_{i=1}^{\infty} \left(\frac{1}{m_i + 1} \right)^2 \leqq 1$ follows from $b_k \in S_1(p_0)$, $k = 1, 2, \ldots$.) Thus $q \in B(U)$, proving that $B(U) \neq \emptyset$. Hence ind $R^{\omega} \geqq 1$.

II. 5. Strong inductive dimension and covering dimension

To prove the equivalence of the strong inductive dimension and the covering dimension, we shall prove the following theorem which was first proved by C. H. Dowker [1] for normal spaces and plays an important role in dimension theory for general metric spaces.

A) If F is a closed set of a space R, then dim $F \leqq$ dim R .

Theorem II. 6 [6]. *Let R be a space; then* dim $R \leqq n$ *if and only if for any locally finite open covering* $\mathfrak{U}$ *of* R *there exists a locally finite open covering* $\mathfrak{V}$ *with* ord $\mathfrak{V} \leqq n + 1$, $\mathfrak{V} < \mathfrak{U}$.

Proof. Since the "if" part is obvious, we shall prove the "only if" part. Let $\mathfrak{U} = \{ U_\gamma \mid \gamma \in \Gamma \}$. Since $\mathfrak{U}$ is locally finite, there exists an open covering $\mathfrak{A}$ each of whose elements meets at most finitely many elements of $\mathfrak{U}$.

By choosing a locally finite open covering $\mathfrak{B}$ with $\bar{\mathfrak{B}} < \mathfrak{A}$ we get a locally finite closed covering $\mathfrak{F} = \bar{\mathfrak{B}} = \{ F_\nu \mid \nu < \tau \}$ such that

(1) each F_ν meets at most finitely many elements of $\mathfrak{U}$.

Then in view of A) we have

(2) $\quad$ dim $F_\nu \leqq n$.

We assume that $F_0 = \emptyset$ and $F_\mu \neq F_\nu$ whenever $\mu \neq \nu$

Now let us construct a transfinite sequence of open coverings $\{ U_{\nu,\gamma} \mid \gamma \in \Gamma \}$, $\nu < \tau$ such that $U_{0,\gamma} = U_\gamma$ and $U_{\mu,\gamma} \supset U_{\nu,\gamma}$ for $\mu < \nu$, and such that

(3) each point of F_ν is contained in at most $n + 1$ members of $\{ U_{\nu,\gamma} \mid \gamma \in \Gamma \}$, and

(4) $\quad \cap \{ U_{\mu,\gamma} \mid \mu < \nu \} \cap (R - F_\nu) = U_{\nu,\gamma} \cap (R - F_\nu)$.

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[6] This theorem was originally proved for every normal space by C. H. Dowker [1]; we owe the present version of the proof to H. de Vries.

The construction can be carried out by use of the induction on ν . Let $U_{\mu,\gamma}$ be determined for all $\mu < \nu$. Put

$$U^*_{\nu,\gamma} = \cap \{ U_{\mu,\gamma} \mid \mu < \nu \} .$$

Then we claim that $\{ U^*_{\nu,\gamma} \mid \gamma \in \Gamma \}$ is an open covering of R . First to prove that $U^*_{\nu,\gamma}$ is open, let $x \in U^*_{\nu,\gamma}$. Assume that

(5) $x \in F_{\mu_i}$, $\mu_i < \nu$, $i = 1,\dots,k$, and $x \notin F_\mu$ for all μ such that $\mu_k < \mu < \nu$.

Put $N = (R - \cup \{ F_\mu \mid \mu_k < \mu < \nu \}) \cap U_{\mu_k,\gamma}$. Then N is a nbd of x . Let y be an arbitrary point of N ; then $y \in U_{\mu,\gamma}$ for all $\mu \leq \mu_k$. On the other hand $y \in U_{\mu,\gamma}$ for all $\mu > \mu_k$ with $\mu < \nu$ can be proved by induction on μ as follows. Assume $y \in U_{\mu',\gamma}$ for all $\mu' < \mu$. Then

$$y \in (\bigcap_{\mu' < \mu} U_{\mu',\gamma}) \cap (R - F_\mu) = U_{\mu,\gamma} \cap (R - F_\mu)$$

follows from the definition of N and the induction hypothesis (4). Hence $y \in U_{\mu,\gamma}$ for all $\mu < \nu$, i.e. $N \subset \bigcap_{\mu<\nu} U_{\mu,\nu} = U^*_{\nu,\gamma}$, which proves that $U^*_{\nu,\gamma}$ is an open set. To prove that $\{ U^*_{\nu,\gamma} \}$ covers R , let $x \in R$. Assume (5) again. Then $x \in U_{\mu_k,\gamma}$ for some γ . Then in view of the above argument on y we know $x \in U^*_{\nu,\gamma}$.

Now, restrict $\{ U^*_{\nu,\gamma} \mid \gamma \in \Gamma \}$ to F_ν and choose an open covering $\{ W_\gamma \mid \gamma \in \Gamma \}$ of F_ν with order $\leq n + 1$ such that $W_\gamma \subset U^*_{\nu,\gamma}$, $\gamma \in \Gamma$. (This is possible because of (1) and (2). Generally note that for a given finite open covering $\{ U_i \mid i = 1,\dots,k \}$ of a space with $\dim \leq n$, there is a finite open covering $\{ W_i \mid i = 1,\dots,k \}$ of order $\leq n + 1$ such that $W_i \subset U_i$, $i = 1,\dots,k$.) Put $U_{\nu,\gamma} = (U^*_{\nu,\gamma} - F_\nu) \cup W_\gamma$. At last $V = \{ V_\gamma \mid \gamma \in \Gamma \}$ meets our requirements if we put $V_\gamma = \cap \{ U_{\nu,\gamma} \mid \nu < \tau \}$. (The last claim is easy to check by use of an argument quite similar to the one used for $\{ U^*_{\nu,\gamma} \}$. Note that at any stage ν in the subsequent induction step, only pieces of F_ν were cut away from the elements of the preceding coverings.)

Corollary. *Let* R *be a space; then* $\dim R \leq n$ *if and only if for any open covering* $\mathcal{U}$ *of* R *there exists a locally finite open refinement* $\mathcal{V}$ *of* $\mathcal{U}$ *with* $\operatorname{ord} \mathcal{V} \leq n+1$.

Proof. Use the corollary to Theorem I.1 and Theorem II.6.

B) Let $\{ U_\gamma \mid \gamma \in \Gamma \}$ be a locally finite open collection in a space R with $\dim R \leq n$ and $\{ F_\gamma \mid \gamma \in \Gamma \}$ a closed collection such that $F_\gamma \subset U_\gamma$. Then there exist open collections $\{ V_\gamma \mid \gamma \in \Gamma \}$ and $\{ W_\gamma \mid \gamma \in \Gamma \}$ such that

$$F_\gamma \subset V_\gamma \subset \bar{V}_\gamma \subset W_\gamma \subset U_\gamma , \quad \text{and} \quad \operatorname{ord} \{ W_\gamma - \bar{V}_\gamma \mid \gamma \in \Gamma \} \leq n .$$

Proof. We denote the binary covering $\{ U_\gamma , R - F_\gamma \}$ by $\mathcal{U}_\gamma$. Then, since $\{ U_\gamma \mid \gamma \in \Gamma \}$ is locally finite, $\wedge \{ \mathcal{U}_\gamma \mid \gamma \in \Gamma \}$ is a locally finite open covering. Hence we can find, by Theorem II.6, a locally finite open covering $\mathcal{N} = \{ N_\delta \mid \delta \in \Delta \}$

such that

$$N < \wedge \{ U_\gamma \mid \gamma \in \Gamma \}, \text{ and ord } N \leqq n+1,$$

and such that every N_δ intersects at most finitely many members of $\{F_\gamma \mid \gamma \in \Gamma\}$ (actually $N < \wedge \{U_\gamma \mid \gamma \in \Gamma\}$ and the local finiteness of $\{U_\gamma\}$ assures the property of N_δ). By I.1 A) we construct a closed covering $\{K_\delta \mid \delta \in \Delta\}$ with $K_\delta \subset N_\delta$. To each N_δ and every F_γ intersecting N_δ, we assign an open set $N_\delta(F_\gamma)$ such that

(1) $$K_\delta \subset N_\delta(F_\gamma) \subset \overline{N_\delta(F_\gamma)} \subset N_\delta$$

and such that

(2) $$N_\delta(F_\gamma) \subset \overline{N_\delta(F_\gamma)} \subset N_\delta(F_{\gamma'})$$

or the inverse inclusion relation holds for $\gamma \neq \gamma'$. This is possible because there are at most finitely many F_γ's intersecting N_δ. Now, let

$$V_\gamma = \cup \{ N_\delta(F_\gamma) \mid \delta \in \Delta \};$$

then from $N < U_\gamma$ it follows that

$$F_\gamma \subset V_\gamma \subset \bar{V}_\gamma \subset U_\gamma,$$

because for every $N_\delta(F_\gamma)$ constituting V_γ, $N_\delta \cap F_\gamma \neq \emptyset$ which implies $\overline{N_\delta(F_\gamma)} \subset N_\delta \subset U_\gamma$, and thus $\bar{V}_\gamma \subset U_\gamma$ follow since V_γ is a locally finite sum of $N_\delta(F_\gamma)$'s. On the other hand $F_\gamma \subset V_\gamma$ easily follows from $K_\delta \cap F_\gamma \subset N_\delta(F_\gamma)$. To show

$$\text{ord } \{ \bar{V}_\gamma - V_\gamma \mid \gamma \in \Gamma \} \leqq n$$

we assume the contrary, i.e.

$$\bigcap_{i=1}^{n+1} (\bar{V}_{\gamma_i} - V_{\gamma_i}) \neq \emptyset$$

for distinct γ_i, $i = 1,\dots,n+1$. Then since each V_γ is a locally finite sum of $N_\delta(F_\gamma)$'s, there must be some $N_{\delta_i}(F_{\gamma_i})$, $i = 1,\dots,n+1$ such that

(3) $$\bigcap_{i=1}^{n+1} (\overline{N_{\delta_i}(F_{\gamma_i})} - N_{\delta_i}(F_{\gamma_i})) \neq \emptyset .$$

If $\delta_i = \delta_j$ and $i \neq j$, then from (2) we obtain

$$(\overline{N_{\delta_i}(F_{\gamma_i})} - N_{\delta_i}(F_{\gamma_i})) \cap (\overline{N_{\delta_j}(F_{\gamma_j})} - N_{\delta_j}(F_{\gamma_j})) = \emptyset$$

contradicting (3).

Therefore (3) implies that δ_i, $i = 1,\dots,n+1$ are distinct from each other. Let

(4) $$p \in \bigcap_{i=1}^{n+1} (\overline{N_{\delta_i}(F_{\gamma_i})} - N_{\delta_i}(F_{\gamma_i}));$$

then from (1) it follows that

(5) $p \notin \bigcup_{i=1}^{n+1} K_{\delta_i}$.

Since $\{K_\delta \mid \delta \in \Delta\}$ covers R, there exists K_δ containing p. Then (5) implies $\delta \neq \delta_i$, $i = 1,\dots,n+1$. On the other hand, by (1) and (4) we have $p \in \bigcap_{i=1}^{n+1} N_{\delta_i}$. Hence

$$p \in K_\delta \cap [\bigcap_{i=1}^{n+1} N_{\delta_i}] \subset N_\delta \cap [\bigcap_{i=1}^{n+1} N_{\delta_i}] ,$$

which contradicts that ord $N \leqq n+1$. Thus we can conclude

(6) ord $\{\bar{V}_\gamma - V_\gamma \mid \gamma \in \Gamma\} \leqq n$.

Since $\{U_\gamma \mid \gamma \in \Gamma\}$ is locally finite, $\wedge\{U'_\gamma \mid \gamma \in \Gamma\}$ is a locally finite open covering, where $U'_\gamma = \{U_\gamma, R - B(V_\gamma)\}$. Hence there is an open covering M such that $M < \wedge\{U'_\gamma \mid \gamma \in \Gamma\}$.

Since, by virtue of the local finiteness of $\{U_\gamma\}$, $\{\bar{V}_\gamma - V_\gamma \mid \gamma \in \Gamma\}$ is a locally finite collection satisfying (6), we can construct M so that for each point p of R, $S(p, M)$ intersects at most n members of $\{\bar{V}_\gamma - V_\gamma \mid \gamma \in \Gamma\}$. Now, it is easy to see that putting

$$W_\gamma = V_\gamma \cup S(B(V_\gamma), M)$$

we get the desired open sets W_γ, because $W_\gamma - \bar{V}_\gamma \subset S(B(V_\gamma), M)$.

C) Let $\{U_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\dots$ be locally finite open collections in a space R with $\dim R \leqq n$, and $\{F_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\dots$ closed collections such that $F_\gamma \subset U_\gamma$. Then there exist open sets V_γ for $\gamma \in \bigcup_{i=1}^\infty \Gamma_i$ satisfying

$$F_\gamma \subset V_\gamma \subset \bar{V}_\gamma \subset U_\gamma , \quad \text{and} \quad \text{ord}\{\bar{V}_\gamma - V_\gamma \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\} \leqq n .$$

Proof. By use of B) we define open sets

V^1_γ, W^1_γ for every $\gamma \in \Gamma_1$,

V^2_γ, W^2_γ for every $\gamma \in \Gamma_1 \cup \Gamma_2$,

....

V^j_γ, W^j_γ for every $\gamma \in \Gamma_1 \cup \dots \cup \Gamma_j$,

....

in such a way that

$$F_\gamma \subset \bar{V}^j_\gamma \subset W^j_\gamma \subset U_\gamma ,$$

$$\bar{V}^j_\gamma \subset V^{j+1}_\gamma , \qquad \bar{W}^{j+1}_\gamma \subset W^j_\gamma ,$$

$$\text{ord}\{W^j_\gamma - \bar{V}^j_\gamma \mid \gamma \in \Gamma_1 \cup \dots \cup \Gamma_j\} \leqq n .$$

It can be seen without difficulty that $V_\gamma = \bigcup_{j=i}^{\infty} V_\gamma^j$ for $\gamma \in \Gamma_i$ are the desired open sets, because $\bar{V}_\gamma - V_\gamma \subset \bigcap \{ W_\gamma^j - \bar{V}_\gamma^j \mid j = i, i+1, \dots \}$.

D) If there exists a σ-locally finite open basis V of a space R such that ord $B(V) \leqq n$ $(n \geqq 0)$, then Ind $R \leqq n$.

Proof. The proof will be carried out by use of the induction on n. In case $n = 0$ this proposition is implied by I E).

In case $n > 0$ let $V = \{ V_\gamma \mid \gamma \in \Gamma \}$. Then

$$V_\gamma = \{ B(V_\gamma) \cap V_{\gamma'} \mid \gamma' \in \Gamma \}$$

is a σ-locally finite open basis of $B(V_\gamma)$ such that

$$\text{ord} \{ B_{B(V_\gamma)}(V) \mid V \in V_\gamma \} \leqq n - 1 .$$

Hence by use of the induction hypothesis we have Ind $B(V_\gamma) \leqq n - 1$. Now Theorem II.2 implies the assertion.

Theorem II. 7 (Coincidence Theorem) [7]. *For every space* R, dim R = Ind R.

Proof. Let Ind $R \leqq n$; then by the decomposition theorem we can decompose R as $R = \bigcup_{i=1}^{n+1} A_i$, Ind $A_i \leqq 0$.
Let $U = \{ U_j \mid j = 1,\dots,k \}$ be a given open covering of R. We consider A_i for a fixed i. Then one can easily see that there exists an open covering $\{ V_j \mid j = 1,\dots,k \}$ of A_i such that

(1) $$\text{ord} \{ V_j \mid j = 1,\dots,k \} \leqq 1 , \quad V_j \subset U_j .$$

For that purpose we select a covering $\{ W_j \mid j = 1,\dots,k \}$ of A_i by open and closed sets of A_i such that $W_j \subset U_j$. Putting

$$V_j = W_j - \bigcup_{k=1}^{j-1} W_k ,$$

we get the desired covering $\{ V_j \mid j = 1,\dots,k \}$ of A_i. For every point $x \in V_j$ we determine $\varepsilon(x) > 0$ such that

$$S_{\varepsilon(x)}(x) \cap A_i \subset V_j , \quad \text{and} \quad S_{\varepsilon(x)}(x) \subset U_j .$$

Let $W_j = \bigcup \{ S_{\frac{1}{2}\varepsilon(x)}(x) \mid x \in V_j \}$. Then $W_i = \{ W_j \mid j = 1,\dots,k \}$ is an open collection which covers A_i.

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[7] This important theorem was first proved for general metric spaces by M. Katětov [2] and K. Morita [4] independently. C. H. Dowker and W. Hurewicz [1] gave another proof.

We can easily see that $W_j \subset U_j$ and from (1) it follows that $\operatorname{ord} W_i \leqq 1$. Hence $W = \bigcup_{i=1}^{n+1} W_i$ is an open refinement of U and has order $\leqq n+1$. Therefore $\dim R \leqq n$.

Conversely if $\dim R \leqq n$, then we choose, by use of the corollary to Theorem I.1, a sequence $U_i = \{U_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\ldots,$ of locally finite open coverings such that $\lim_{i\to\infty} \operatorname{mesh} U_i = 0$. Then by I.1 A) we construct closed coverings $F_i = \{F_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\ldots,$ satisfying $F_\gamma \subset U_\gamma$.

Now, by virtue of C) we obtain open sets

$$V_\gamma \quad \text{for} \quad \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i$$

such that

$$F_\gamma \subset V_\gamma \subset \bar{V}_\gamma \subset U_\gamma, \quad \text{and } \operatorname{ord}\{\bar{V}_\gamma - V_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\} \leqq n .$$

Since $V = \{V_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\}$ is clearly a σ-locally finite open basis, by D) we can conclude $\operatorname{Ind} R \leqq n$. Thus the theorem is established. In view of this theorem we are entitled to use notations $\operatorname{Ind} R$ and $\dim R$ indiscriminately for every (metric) space R.

II. 6. Some theorems characterizing dimension

We can intuitively see that the dimensions of Euclidean spaces E^n, $n = 1,2,3$, are characterized by the following property:

Let $\{U_i \mid i = 1,\ldots,k\}$ and $\{F_i \mid i = 1,\ldots,k\}$ be an open collection and a closed collection in E^n respectively such that $F_i \subset U_i$; then there exists an open collection $\{V_i \mid i = 1,\ldots,k\}$ such that

$$F_i \subset V_i \subset U_i, \quad \text{and} \quad \operatorname{ord}\{B(V_i) \mid i = 1,\ldots,k\} \leqq n .$$

The possibility of such a characterization for general metric spaces is intimated by 5 B), C) too. The purpose of this section is to carry out an investigation along this idea.

Theorem II. 8. *A space R has dimension $\leqq n$ if and only if for every open collection $\{U_i \mid i = 1,\ldots,k\}$ and closed collection $\{F_i \mid i = 1,\ldots,k\}$ with $F_i \subset U_i$, there exists an open collection $\{V_i \mid i = 1,\ldots,k\}$ such that*

$$F_i \subset V_i \subset U_i,\ i = 1,\ldots,k, \text{ and } \operatorname{ord}\{B(V_i) \mid i = 1,\ldots,k\} \leqq n .$$

Proof. The "only if" part is implied by 5 B). To show the "if" part we consider a given finite open covering $\{U_i \mid i = 1,\ldots,k\}$. Then by I.1 A) there exists a closed covering $\{F_i \mid i = 1,\ldots,k\}$ with $F_i \subset U_i$.

Let $\{V_i \mid i = 1,\dots,k\}$ be an open covering such that

(1) $\quad F_i \subset V_i \subset \bar{V}_i \subset U_i$, and $\operatorname{ord}\{B(V_i) \mid i = 1,\dots,k\} \leqq n$.

(Because of the normality of R we can choose V_i satisfying $\bar{V}_i \subset U_i$ besides the condition of this theorem.)

Let $G_i = \bar{V}_i - \bigcup_{j=1}^{i-1} V_j$; then $\operatorname{ord}\{G_i \mid i = 1,\dots,k\} \leqq n+1$. For, if p were a point such that

$$p \in G_{i_1} \cap G_{i_2} \cap \dots \cap G_{i_{n+2}},$$

where $i_1 < i_2 < \dots < i_{n+2}$, then we had

$$p \in (\bar{V}_{i_1} - V_{i_1}) \cap (\bar{V}_{i_2} - V_{i_2}) \cap \dots \cap (\bar{V}_{i_{n+1}} - V_{i_{n+1}})$$

contradicting (1). Hence $\{G_i \mid i = 1,\dots,k\}$ is a closed covering with order $\leqq n+1$ and $G_i \subset U_i$.

Therefore we can choose an open covering $\mathcal{U}$ such that for every point p of R, $S(p,\mathcal{U})$ intersects at most $n+1$ members of $\{G_i \mid i = 1,\dots,k\}$ and such that

(2) $\quad \mathcal{U} < \bigwedge_{i=1}^{k} \mathcal{U}_i$,

where $\mathcal{U}_i = \{U_i, R - G_i\}$. We let

(3) $\quad W_i = S(G_i, \mathcal{U})$.

Then, from (2) it follows that $W_i \subset U_i$, $i = 1,\dots,k$. Let $\mathcal{W} = \{W_i \mid i = 1,\dots,k\}$; then from (3) it follows that $\mathcal{W}$ is an open covering with $\operatorname{ord}\mathcal{W} \leqq n+1$. Since $\{U_i \mid i = 1,\dots,k\}$ is a given finite open covering of R, this proves $\dim R \leqq n$.

Corollary [8]. *A space R has dimension $\leqq n$ if and only if for every open collection $\{U_i \mid i = 1,\dots,n+1\}$ and closed collection $\{F_i \mid i = 1,\dots,n+1\}$ with $F_i \subset U_i$, there exists an open collection $\{V_i \mid i = 1,\dots,n+1\}$ such that*

$$F_i \subset V_i \subset U_i, \quad \text{and} \quad \bigcap_{i=1}^{n+1} B(V_i) = \emptyset .$$

Proof. Since the "only if" part is a direct consequence of the theorem, we shall prove only the "if" part. Let $\{U_i \mid i = 1,\dots,k\}$ be a given finite open collection and $\{F_i \mid i = 1,\dots,k\}$ a given finite closed collection such that $F_i \subset U_i$. Then we number all the combinations $C = \{i_1,\dots,i_{n+1}\}$ of $n+1$ numbers from

[8] This theorem was first proved for separable metric spaces by S. Eilenberg and E. Otto [1] and extended by E. Hemmingsen [1] and K. Morita [1] to normal spaces.

$\{1,\ldots,k\}$ as $C_1, C_2, \ldots, C_m$, where $m = \binom{k}{n+1}$. We can define open sets V_i^1 and W_i^1 for $i \in C_1$ such that

$$F_i \subset V_i^1 \subset \bar{V}_i^1 \subset W_i^1 \subset U_i, \quad \text{and} \quad \cap\{W_i^1 - \bar{V}_i^1 \mid i \in C_1\} = \emptyset .$$

Since C_1 consists of just $n+1$ numbers, we can choose V_i^1 to satisfy

$$F_i \subset V_i^1 \subset \bar{V}_i^1 \subset U_i \quad \text{and} \quad \cap\{B(V_i^1) \mid i \in C_1\} = \emptyset .$$

Then we obtain W_i^1 by use of the method in the proof of 5 B) or more precisely by the process which was applied there to construct W_γ from V_γ.

Suppose that open sets V_i^j and W_i^j, $i \in C_j$ have been defined for every $j < \ell$. Then we define open sets V_i^ℓ and W_i^ℓ for $i \in C_\ell$ so that

$$\text{if } i \notin \bigcup_{j=1}^{\ell-1} C_j, \quad \text{then } F_i \subset V_i^\ell \subset \bar{V}_i^\ell \subset W_i^\ell \subset U_i ,$$

$$\text{if } i \in \bigcup_{j=1}^{\ell-1} C_j, \quad \text{then } F_i \subset V_i^j \subset \overline{V_i^j} \subset V_i^\ell \subset \overline{V_i^\ell} \subset W_i^\ell \subset W_i^j \subset U_i$$

for every j for which $1 \leq j \leq \ell - 1$, $i \in C_j$, and so that

$$(1) \qquad \cap\{W_i^\ell - \overline{V_i^\ell} \mid i \in C_\ell\} = \emptyset .$$

We can construct V_i^ℓ and W_i^ℓ by a similar application of the proof in 5 B) as in the construction of V_i^1 and W_i^1. Now, let $V_i' = \cup\{V_i^\ell \mid i \in C_\ell\}$. (Note that ℓ denotes a variable in the above formula.) Select open sets V_i such that $\overline{V_i'} \subset V_i \subset \cap\{W_i^\ell \mid i \in C_\ell\}$. Then one can easily see that $F_i \subset V_i \subset U_i$, $i = 1,\ldots,k$, and $B(V_i) \subset W_i^\ell - \overline{V_i^\ell}$ for every ℓ satisfying $i \in C_\ell$. Hence for any combination C_ℓ of $n+1$ numbers from $\{1,\ldots,k\}$

$$\cap\{B(V_i) \mid i \in C_\ell\} \subset \cap\{W_i^\ell - \overline{V_i^\ell} \mid i \in C_\ell\} = \emptyset$$

follows from (1). This means $\operatorname{ord}\{B(V_i) \mid i = 1,\ldots,k\} \leq n$. Therefore by the theorem we conclude $\dim R \leq n$.

Theorem II. 9. *A space R has dimension $\leq n$ if and only if there exists a σ-locally finite open basis V such that* $\operatorname{ord} B(V) \leq n$.

Proof. The "if" part follows from 5 D) combined with Theorem II.7. To prove the "only if" part, we construct a σ-locally finite open basis $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ of R using Theorem I.3. We can suppose without loss of generality that each $\mathcal{U}_i = \{U_\gamma \mid \gamma \in \Gamma_i\}$ is a locally finite open covering of R such that $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$. Then by I.1 A) there exists a locally finite open covering

$\mathcal{W}_i = \{ W_\gamma \mid \gamma \in \Gamma_i \}$ such that $\bar{W}_\gamma \subset U_\gamma$. By 5 C) we can construct open sets V_γ for $\gamma \in \bigcup_{i=1}^{\infty} \Gamma_i$ satisfying

$$\bar{W}_\gamma \subset V_\gamma \subset U_\gamma, \quad \text{and} \quad \operatorname{ord} \{ B(V_\gamma) \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i \} \leqq n.$$

Thus we get a σ-locally finite open basis $\mathcal{V} = \{ V_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i \}$ with $\operatorname{ord} B(\mathcal{V}) \leqq n$.

Theorem II. 10 [9]. *Let B be a subset of a space R and let* $\dim B = n$. *Then there exists a G_δ-set A of R such that* $B \subset A$, $\dim A = n$.

Proof. In case $n = 0$, as in the proof of Theorem II.9, we choose a σ-locally finite open basis $\mathcal{U} = \{ U_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i \}$ and a σ-locally finite open collection $\mathcal{W} = \{ W_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i \}$ such that $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$ and $\bar{W}_\gamma \subset U_\gamma$.
Since $\bar{W}_\gamma$ and $R - U_\gamma$ are disjoint closed sets of R, in the same way as we constructed M for F and G in the proof of 1 C), we can construct an open set V_γ for $\bar{W}_\gamma$ and $R - U_\gamma$ such that $\bar{W}_\gamma \subset V_\gamma \subset U_\gamma$, and $B(V_\gamma) \cap B = \emptyset$. Let $A = R - \bigcup \{ B(V_\gamma) \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i \}$; then A is a G_δ-set containing B because each $\bigcup \{ B(V_\gamma) \mid \gamma \in \Gamma_i \}$ for a fixed i is a closed set by virtue of the local finiteness of $\{ U_\gamma \mid \gamma \in \Gamma_i \}$. Since for any γ $B_A(V_\gamma \cap A) = \emptyset$, $\mathcal{V} = \{ V_\gamma \cap A \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i \}$ is a σ-locally finite open basis of A which satisfies the condition of Theorem II.9 for $n = 0$. Hence $\dim A \leqq 0$, which combined with Theorem II.3 implies $\dim A = 0$.

In case $n > 0$, by use of the decomposition theorem we can decompose B into $n + 1$ 0-dimensional subsets B_i, $i = 1, \dots, n + 1$. By virtue of the preceding result we can find G_δ-sets A_i, $i = 1, \dots, n + 1$ such that $B_i \subset A_i$, and $\dim A_i \leqq 0$. Let $A = \bigcup_{i=1}^{n+1} A_i$; then again by the decomposition theorem and Theorem II.3 A is the desired G_δ-set.

II. 7. The rank of a covering

As implied by Theorem I.3, the σ-local finiteness of open bases of a metric space is important. In fact we cannot delete the σ-local finiteness from Theorem II.2 or Theorem II.9.

We can, however, establish a simple characterization theorem of dimension in terms of open bases without σ-local finiteness or any similar property. To this purpose we shall introduce a new notion "rank of a collection". Let us begin with some lemmas.

[9] This theorem was first proved for separable metric spaces by L. Tumarkin [2].

A) Let S be a space of $\dim S \leq 0$. Then there are locally finite open coverings $\mathcal{U}_i = \{ U(\alpha_1,\dots,\alpha_i) \mid \alpha_1,\dots,\alpha_i \in A \}$, $i = 1,2,\dots$ of S such that $U(\alpha_1,\dots,\alpha_i) \supset U(\alpha_1,\dots,\alpha_i, \alpha_{i+1})$, $U(\alpha_1,\dots,\alpha_i) \cap U(\alpha_1,\dots,\alpha_i') = \emptyset$ if $\alpha_i \neq \alpha_i'$, and $\lim_{i\to\infty} \text{mesh } \mathcal{U}_i = 0$.

Proof. Let $\mathcal{V}_i$ be a locally finite open covering of S such that mesh $\mathcal{V}_i < 1/i$. Find, by use of Theorem II.6, a locally finite open refinement $\mathcal{V}_i'$ of $\mathcal{V}_i$ satisfying ord $\mathcal{V}_i' \leq 1$. Put $\mathcal{U}_i = \mathcal{V}_1' \wedge \dots \wedge \mathcal{V}_i'$. Then $\mathcal{U}_i$ is a locally finite open covering of order ≤ 1 and with mesh $\leq 1/i$. Note that each (non-empty) element of $\mathcal{U}_{i+1}$ is contained in one and only one element of $\mathcal{U}_i$, and each element of $\mathcal{U}_i$ is a sum of elements of $\mathcal{U}_{i+1}$. Thus by choosing appropriate subindices and adding as many empty sets as necessary, we can let $\{\mathcal{U}_i\}$ satisfy the desired conditions.

B) Let S be a subspace of R with $\dim S \leq 0$. Assume that $\mathcal{U}_i$, $i = 1,2,\dots$ are open coverings of S satisfying the conditions of A). Then we can extend $\mathcal{U}_i$ to a collection $\mathcal{W}_i$ of open sets in R which satisfies the same conditions as $\mathcal{U}_i$, except that $\mathcal{W}_i$ covers only S (but not necessarily R) and $\mathcal{W}_i$ is not necessarily locally finite on R.

Proof. Let i be fixed. Then to each $x \in U(\alpha_1,\dots,\alpha_i)$ we assign $\varepsilon_i(x) > 0$ such that $\varepsilon_i(x) < 1/i$ and $S_{\varepsilon_i(x)}(x) \cap S \subset U(\alpha_1,\dots,\alpha_i)$. Now put

$$W(\alpha_1,\dots,\alpha_i) = \cup \{ S_{\varepsilon_i(x)/2}(x) \mid x \in U(\alpha_1,\dots,\alpha_i) \} .$$

Then $\mathcal{W}_i = \{ W(\alpha_1,\dots,\alpha_i) \mid \alpha_1,\dots,\alpha_i \in A \}$ satisfies the desired conditions if we choose $\{\varepsilon_i(x)\}$ satisfying $\varepsilon_i(x) \geq \varepsilon_{i+1}(x)$.

Definition II. 1. *Two subsets C and D of a space R are called independent if $C \not\subset D$ and $D \not\subset C$. A finite number of subsets $C_1,\dots,C_k$ are called independent if any two of them are independent.*

Definition II. 2. *Let $\mathcal{U}$ be a collection in a space R and p a point of R. Then we mean by the rank of $\mathcal{U}$ at p the largest integer n such that there are independent n members of $\mathcal{U}$ containing p, and denote it by* $\text{rank}_p\, \mathcal{U}$. *If there exist arbitrarily many independent members of $\mathcal{U}$ containing p, then* $\text{rank}_p\, \mathcal{U} = +\infty$.
We mean by the rank of $\mathcal{U}$ the supremum of $\text{rank}_p\, \mathcal{U}$ *for p ranging through R and denote it by* $\text{rank}\, \mathcal{U}$, *i.e.* $\text{rank}\, \mathcal{U} = \sup\{\text{rank}_p\, \mathcal{U} \mid p \in R\}$.

In view of this definition we clearly see $\text{rank}_p\, \mathcal{U} \leq \text{ord}_p\, \mathcal{U}$ for any point p and collection $\mathcal{U}$, and accordingly $\text{rank}\, \mathcal{U} \leq \text{ord}\, \mathcal{U}$.

Theorem II. 11 [10]. *A space* R *has dimension* $\leqq n$, *if and only if it has an open basis* $\mathfrak{U}$ *with* rank $\mathfrak{U} \leqq n+1$.

Proof. To begin with let us prove the "if" part by induction. Let $n = 0$, i.e. we suppose $\mathfrak{U}$ is an open basis with rank $\mathfrak{U} \leqq 1$. Let F and G be given disjoint closed sets of R . Then we put

$$U = \cup \{ U' \mid U' \in \mathfrak{U} , U' \cap F \neq \emptyset , U' \cap G = \emptyset \} .$$

Since $\mathfrak{U}$ is an open basis of R , U is an open set satisfying $F \subset U \subset R - G$. If $p \notin U$, then there exists $U' \in \mathfrak{U}$ such that $p \in U' \subset R - F$. If we assume $U' \cap U \neq 0$, then $U' \cap U'' = \emptyset$ for some $U'' \in \mathfrak{U}$ such that $U'' \cap F \neq \emptyset$ and $U'' \subset U$. Since U' and U'' are clearly independent, we reach a contradiction to rank $\mathfrak{U} \leqq 1$. Hence $U' \cap U = \emptyset$, which means that the open set U is closed in R . Thus we conclude $\dim R \leqq 0$.

Suppose we have proved that the existence of an open basis with rank $\leqq n$ implies $\dim R \leqq n-1$. Now we let $\mathfrak{U}$ be an open basis with rank $\leqq n+1$. Let F and G be given disjoint closed sets of R . Then we define an open set U by

$$U = \cup \{ U' \mid U' \in \mathfrak{U} , U' \cap F \neq \emptyset , U' \cap G = \emptyset \} .$$

U clearly satisfies $F \subset U \subset R - G$. We assert that $\mathfrak{U}' = \{ U' \mid U' \in \mathfrak{U} , U' \cap F = \emptyset \}$ restricted to $B(U)$ is an open basis of $B(U)$ satisfying rank $\mathfrak{U}' \leqq n$.

It is evident that $\mathfrak{U}'$ is an open basis of $B(U)$ if restricted to $B(U)$. Hence all we have to show is that $\text{rank}_p \mathfrak{U}' \leqq n$ for a given point $p \in B(U)$.

Assume the contrary, i.e. we suppose $U_1, \dots, U_{n+1}$ are independent sets of $\mathfrak{U}'$ which contain p . Since $p \in B(U)$, we get

$$q \in U_1 \cap \dots \cap U_{n+1} \cap U \neq \emptyset .$$

Then

$$q \in U_1 \cap \dots \cap U_{n+1} \cap U'$$

for some $U' \in \mathfrak{U}$ such that $U' \cap F \neq \emptyset$, $U' \subset U$. Since $p \in U_i \cap (R - U') \neq \emptyset$, and by the definition of $\mathfrak{U}'$ $U_i \cap F = \emptyset$, $U_1, \dots, U_{n+1}$ and U' are independent. Hence $\text{rank}_q \mathfrak{U} \geqq n+2$, which contradicts rank $\mathfrak{U} \leqq n+1$.

Thus we conclude that $\text{rank}_p \mathfrak{U}' \leqq n$ for any point $p \in B(U)$, and hence from the induction hypothesis it follows that $\dim B(U) \leqq n-1$. Therefore $\dim R \leqq n$ is proved.

To prove the "only if" part we suppose R is a space with $\dim R \leqq n$. Note that if $n = 0$, then the assertion follows directly from A), because $\cup_{i=1}^{\infty} \mathfrak{U}_i$ in A) is an open basis of rank $\leqq 1$. In the general case, decompose R , by use of Theorem II.4, as $R = \cup_{k=1}^{n+1} A_k$, where $\dim A_k \leqq 0$. Then by B), there is a sequence $\mathfrak{W}_i^k$,

[10] Proved first by J. Nagata [5].

$i = 1,2,\ldots$, of open collections which cover A_k and satisfy rank $\bigcup_{i=1}^{\infty} \mathcal{W}_i^k \leq 1$ and $\lim_{i\to\infty}$ mesh $\mathcal{W}_i^k = 0$. Now it is easy to see that $\mathcal{U} = \bigcup_{i=1}^{\infty}\bigcup_{k=1}^{n+1} \mathcal{W}_i^k$ is an open basis for R with rank $\mathcal{U} \leq n+1$ [11].

II. 8. Normal families

The theory of normal families was first established by W. Hurewicz [1] to deduce systematically fundamental theorems of dimension theory for separable metric spaces and was extended by K. Morita [4] to nonseparable metric spaces. The purpose of this section is to give a short skètch of this theory to re-establish the principal results of dimension theory.

Definition II. 3. *A family N of (metric) spaces is called a normal family if it satisfies.*

i) if $S \subset R$ and $R \in N$, *then* $S \in N$,

ii) if $\{F_\gamma \mid \gamma \in \Gamma\}$ *is a locally countable closed covering of a space* R *such that* $F_\gamma \in N$ *for every* $\gamma \in \Gamma$, *then* $R \in N$.

A) A family N of spaces is a normal family if and only if it satisfies i), iii) if $\{F_i \mid i = 1,2,\ldots\}$ is a closed covering of a space R such that $F_i \in N$, $i = 1,2,\ldots$, then $R \in N$, and iv) if $\{F_\gamma \mid \gamma \in \Gamma\}$ is a locally finite closed covering of a space R such that $F_\gamma \in N$ for every $\gamma \in \Gamma$, then $R \in N$.

Proof. The "only if" part is evident. Conversely, let N be a family satisfying i), iii) and iv). If $\{F_\gamma \mid \gamma \in \Gamma\}$ is a locally countable closed covering of R such that $F_\gamma \in N$, then we can construct a locally finite closed covering $\{G_\delta \mid \delta \in \Delta\}$ such that each G_δ intersects at most countably many elements of $\{F_\gamma \mid \gamma \in \Gamma\}$. It follows from i) and iii) that $G_\delta \in N$, and hence from iv) we obtain $R \in N$. This means that $\{F_\gamma \mid \gamma \in \Gamma\}$ satisfies ii), i.e. N is a normal family.

Definition II. 4. *For any normal family N we define a family N' of spaces by* $N' = \{R \mid$ *for any disjoint closed sets F and G there exists an open set U of R such that* $F \subset U \subset R - G$, $B(U) \in N\}$.

[11] Bases with finite rank were extensively studied not only in dimension theory but also in metrization theory by A. V. Arhangelskii [1], A. V. Arhangelskii - V. V. Filippov [1], G. Gruenhage-P. Nyikos [1] and others. For example, it was proved in the last paper that every compact Hausdorff space with an open basis of finite rank is metrizable. It should also be noted that A. V. Arhangelskii [1] proved that a normal space has $\dim \leq n$ if and only if for every finite open covering $\mathcal{U}$ there is an open refinement $\mathcal{V}$ of rank $\leq n+1$.

To re-establish the principal results of dimension theory by use of normal families we need the following four propositions B) - E) on 0-dimensional spaces which have been proved once in Sections 1 and 2 and are easily deduced by some elementary knowledge of general topology.

B) Ind $R \leq 0$ if and only if there exists a σ - locally finite open basis V of R consisting of open and closed sets.

C) If $S \subset R$ and Ind $R \leq 0$, then Ind $S \leq 0$.

D) Let $\{ F_i \mid i = 1,2,\dots \}$ be a closed covering of R such that Ind $F_i \leq 0$, $i = 1,2,\dots$; then Ind $R \leq 0$.

E) Let $\{ F_\gamma \mid \gamma \in \Gamma \}$ be a locally finite closed covering of R such that Ind $F_\gamma \leq 0$ for every $\gamma \in \Gamma$; then Ind $R \leq 0$.

By A), C), D) and E) we conclude that

F) All the (metric) spaces with Ind ≤ 0 form a normal family.

Theorem II. 12. *Let N be a normal family and R a space. Then $R \in N'$ if and only if there exist two subspaces A and B such that $R = A \cup B$, $A \in N$, and Ind $B \leq 0$.*

Proof. To see the "only if" part let $R \in N'$. By I.1 A) and the Corollary to Theorem I.1 we can construct σ - locally finite open bases $U = \{ U_\gamma \mid \gamma \in \Gamma \}$ and $W = \{ W_\gamma \mid \gamma \in \Gamma \}$ of R such that $\bar{W}_\gamma \subset U_\gamma$. Since $R \in N'$, there exist open sets V_γ for $\gamma \in \Gamma$ satisfying $\bar{W}_\gamma \subset V_\gamma \subset U_\gamma$, and $B(V_\gamma) \in N$. Put $A = \bigcup \{ B(V_\gamma) \mid \gamma \in \Gamma \}$ and $B = R - A$. It follows from the conditions iii) and iv) in A) that $A \in N$ because $\{ B(V_\gamma) \mid \gamma \in \Gamma \}$ is a σ - locally finite closed collection. Since $\{ V_\gamma \mid \gamma \in \Gamma \}$ restricted to B is a σ - locally finite open basis consisting of open and closed subsets of B, using B) we get Ind $B \leq 0$.

The proof of the "if" part is as follows. Let F and G be disjoint closed sets of R. Since Ind $B \leq 0$, we can construct an open set M with $F \subset M \subset R - G$, $B(M) \subset A$ in the same way as in the proof of 1 C). Since $A \in N$, from the condition i) for N it follows that $B(M) \in N$. This proves $R \in N'$.

Theorem II. 13. *If N is a normal family, then N' is also a normal family.*

Proof. We assert that N' satisfies i), iii) and iv). Let $S \subset R \in N'$. Then by Theorem II.12 $R = A \cup B$, $A \in N$, and Ind $B \leq 0$. This implies $S = (A \cap S) \cup (B \cap S)$. Since N is normal, from i) we obtain $A \cap S \in N$. Moreover, from C) we obtain

Ind $(B \cap S) \leq 0$. Thus, using Theorem II.12 once more, we conclude that $S \in N'$. Therefore N' satisfies i).

Let $\{ F_\nu \mid \nu < \tau \}$ be a locally finite closed covering of a space R such that

(1) $F_\nu \in N'$.

Then

(2) $G_\nu = F_\nu - \cup \{ F_\mu \mid \mu < \nu \}$

is an F_σ-set, because $\{ F_\mu \mid \mu < \nu \}$ is a locally finite closed collection, and in consequence its sum is a closed set. Since (1) implies $G_\nu \in N'$, by Theorem II.12 we can decompose G_ν as $G_\nu = A_\nu \cup B_\nu$, $A_\nu \in N$, with Ind $B_\nu \leq 0$. By (2) G_ν and G_μ are disjoint if $\nu \neq \mu$, and hence A_ν is an F_σ-set in $A = \cup \{ A_\nu \mid \nu < \tau \}$, and B_ν is also an F_σ-set in $B = \cup \{ B_\nu \mid \nu < \tau \}$. Thus $\{ A_\nu \mid \nu < \tau \}$ is a locally finite covering of A consisting of F_σ-sets A_ν with $A_\nu \in N$.

Since N is a normal family, it follows from i), ii) that $A \in N$. Moreover, from C), D) and E) it follows that Ind $B \leq 0$. Since $R = A \cup B$, by Theorem II.12 we can conclude that $R \in N'$. Therefore N' satisfies iv). In a similar way we can also show that N' satisfies iii). Thus N' is a normal family by A).

Denote by $N^{(-1)}$ the normal family consisting of the empty set alone. Then, let

$$N^{(k+1)} = (N^{(k)})', \quad k = -1,0,1,\dots .$$

Then $N^{(n)}$ is the family consisting of all the spaces with Ind $\leq n$. On the other hand, by Theorem II.13, $N^{(n)}$ is a normal family. Hence we can again deduce Theorem II.3, the sum theorem and the decomposition theorem from i), ii) and Theorem II.12 respectively.

We can apply the theory of normal families to dimension theory in other ways. For example we may let $N^{(0)} = \{ R \mid R$ is a countable sum of closed subspaces R_i , $i = 1,2,\dots$ such that any point p of R has a neighbourhood containing at most countably many points of $R_i \}$. One can easily see that $N^{(0)}$ is a normal family; hence for every positive integer n $N^{(n)}$ is also a normal family. A space R is said to have *rational dimension* $\leq n$ if and only if $R \in N^{(n)}$.

Thus we can apply the theory of normal families to establish another dimension theory on this new concept of dimension, but the details are left to the reader.

More generally, let P be a (not necessarily normal) family of metric spaces. Then we can define $P^{(k)}$, $k = -1,0,1,2,\dots$, in the same way as we did for a normal family. We can also modify Definition II.4 as follows: $'P = \{ R \mid$ for any point $p \in R$ and any neighbourhood U of p there is an open neighbourhood V of p such that $V \subset U$ and $B(V) \in P \}$. Then ${}^{(k)}P$ can be defined in a similar way as $P^{(k)}$.
J. de Groot defined *compactness degree* of a separable metric space R (notation: comp R) by comp $R \leq n$ if and only if $R \in {}^{(n)}P$, where P is the family of all

compact (metric) spaces. He also defined *deficiency* of a separable metric space R (notation: def R) by def R = the least n such that there is a compact (metric) space R' which contains R as a (topological) subspace satisfying $\dim (R' - R) = n$ and asked the question 'comp $R \leq n$ if and only if def $R \leq n$?' This question is still unsolved though it is partially answered in the positive by J. de Groot - T. Nishiura [1]. (The answer is positive e.g. in case of $n = 0$. It is generally true that comp $R \leq$ def $R \leq \dim R$.) J. M. Aarts [1] defined *completeness degree* and *complete deficiency* in a similar way, but by use of 'completely metrizable spaces' in place of 'compact spaces' and $P^{(n)}$ in place of ${}^{(n)}P$. Thus he proved that the completeness degree coincides with the complete deficiency for every metric space [12] .

[12] In a similar aspect of dimension theory Yu. Smirnov [9] studied $\dim (\beta X - X)$ for a normal space X .