CHAPTER I

INTRODUCTION

In this book we assume an elementary knowledge of general topology [1] . However, in this chapter we shall give a quick review, without proofs, of some results, especially from the theory of coverings. In recent years remarkable progress has been made in the last mentioned field, and the reader might not be so familiar with the terminologies and the theorems we shall need.

I. 1. Coverings

Let R be a topological space and $\mathcal{U}$ a collection of subsets of R . Throughout this book we shall merely call such a collection of subsets of R a *collection* in R . If $\mathcal{U}$ consists of finitely many (countably many, two) members, then $\mathcal{U}$ is called a *finite (countable, binary) collection* . If every point of R is covered by only finitely many members of $\mathcal{U}$, then $\mathcal{U}$ is called a *point-finite collection* . If every point p of R has a neighbourhood $U(p)$ which intersects only finitely many (countably many) members of $\mathcal{U}$, then $\mathcal{U}$ is called a *locally finite (locally countable) collection* . If every member U of $\mathcal{U}$ intersects only finitely many members of $\mathcal{U}$, then $\mathcal{U}$ is called a *star-finite collection* . If for every subcollection $\mathcal{V}$ of $\mathcal{U}$

$$\overline{\cup\{U \mid U \in \mathcal{V}\}} = \cup\{\bar{U} \mid U \in \mathcal{V}\}$$

[1] As books on general topology we may recommend, for example, P. Alexandroff - H. Hopf [1], C. Kuratowski [2], J. W. Tukey [1], J. L. Kelley [1], H. J. Kowalsky [2], W. Franz [1], D. Bushaw [1], J. Dugundji [1], R. Engelking [1], J. Nagata [8].

holds, then $\mathcal{U}$ is called a *closure-preserving collection*, where $\bar{U}$ denotes the closure of U. It is easy to see that every locally finite collection is closure-preserving. In this connection we often use the fact that if F is a locally finite collection of closed sets, then $\cup\{F \mid F \in F\}$ is closed. A closure-preserving collection $\mathcal{U}$ is called *discrete* if the closures of the members of $\mathcal{U}$ are disjoint, i.e. if any two of them do not intersect. In other words $\mathcal{U}$ is discrete if and only if each point of R has a neighbourhood which intersects at most one member of $\mathcal{U}$.

If $\mathcal{U}$ can be decomposed as

$$\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$$

for locally finite (star-finite, discrete, etc.) collections $\mathcal{U}_i$, then $\mathcal{U}$ is called a *σ-locally finite (σ-star-finite, σ-discrete, etc.) collection.*

$\mathcal{U}$ is called an *open (closed) collection* if every member of $\mathcal{U}$ is an open (closed) set. We may use the symbol $\cup\mathcal{U}$ to denote the set $\cup\{U \mid U \in \mathcal{U}\}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be two collections. If for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ for which $U \subset V$, then we denote this relation by $\mathcal{U} < \mathcal{V}$ and call $\mathcal{U}$ a *refinement* of $\mathcal{V}$.

A collection $\mathcal{U}$ is called a *covering* if $\cup\{U \mid U \in \mathcal{U}\} = R$. In this book we are often concerned with *open coverings*, coverings consisting of open sets. The other attributes for collections are of course applicable to coverings too.

Frequently a collection will be indexed, e.g. $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$. As far as possible it will be tacitly assumed that the indexing is faithful, i.e. that distinct indices denote distinct members of the collection. In construction starting from a faithfully indexed collection, however, new collections may arise for which the indexing is normally not faithful. For an indexed collection it is sometimes useful to look at the indices instead of considering the members of the collections themselves.

A) Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be a point-finite open covering of a normal space[2] R. Then there exists an open covering $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ such that $\bar{V}_\alpha \subset U_\alpha$ for every $\alpha \in A$. Generally a covering $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is said to *shrink* to (or to be shrunk to) a covering $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ if $V_\alpha \subset U_\alpha$ for all $\alpha \in A$. We may also say (less frequently) that $\mathcal{V}$ shrinks $\mathcal{U}$.

Definition I. 1. *Let R be a topological space. If for each open covering $\mathcal{U}$ of R there exists a locally finite open (covering) refinement, i.e. a locally finite open covering $\mathcal{V}$ satisfying $\mathcal{V} < \mathcal{U}$, then we call R a paracompact space.*

[2] In this book we mean by a normal space a normal T_1-space i.e. a normal space each of whose points is closed. Similarly every regular space will be a regular T_1-space in this book.

If each countable open covering of R *has a locally finite open refinement, then it is called a countably paracompact space.*

If each open covering of a topological space R *has a star-finite open (covering) refinement, then we shall say that* R *has the star-finite property.*

If every subspace of R *is paracompact (normal, etc.), then* R *is called hereditarily paracompact (normal, etc.).*

If every closed set of a normal space R *is a* G_δ*-set, then* R *is called perfectly normal.*

B) A T_1-space R is hereditarily normal if and only if every subset of R is normal. Equivalently, for any subsets A, B of R satisfying $A \cap \bar{B} = \emptyset$, $\bar{A} \cap B = \emptyset$ there exist open sets U and V such that $U \supset A$, $V \supset B$, $U \cap V = \emptyset$, where we can select U and V so that $\bar{U} \cap \bar{V} \subset \bar{A} \cap \bar{B}$.

Now, we shall fix various notations for collections $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{U}_\alpha$;

$$\bar{\mathfrak{U}} = \{\bar{U} \mid U \in \mathfrak{U}\}, \quad B(\mathfrak{U}) = \{B(U) \mid U \in \mathfrak{U}\},$$

where $B(U)$ denotes the boundary of U,

$$\mathfrak{U} \wedge \mathfrak{V} = \{U \cap V \mid U \in \mathfrak{U}, V \in \mathfrak{V}\},$$

$$\wedge \{\mathfrak{U}_\alpha \mid \alpha \in A\} = \{\cap \{U_\alpha \mid \alpha \in A\} \mid U_\alpha \in \mathfrak{U}_\alpha \text{ for each } \alpha \in A\}.$$

If $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{U}_\alpha$ are coverings, then $\bar{\mathfrak{U}}$, $\mathfrak{U} \wedge \mathfrak{V}$ and $\wedge \{\mathfrak{U}_\alpha \mid \alpha \in A\}$ are also coverings.

We need further notations to be used in the theory of coverings. Let p, P and $\mathfrak{U}$ be a point, a set and a collection respectively, then

$$S(p,\mathfrak{U}) = S^1(p,\mathfrak{U}) = \cup\{U \mid p \in U \in \mathfrak{U}\} \cup \{p\},$$

$$S(P,\mathfrak{U}) = S^1(P,\mathfrak{U}) = \cup\{U \mid U \in \mathfrak{U}, U \cap P \neq \emptyset\} \cup P,$$

$$S^n(p,\mathfrak{U}) = S(S^{n-1}(p,\mathfrak{U}), \mathfrak{U}), \quad S^\infty(p,\mathfrak{U}) = \bigcup_{n=1}^{\infty} S^n(p,\mathfrak{U}),$$

$$S^n(P,\mathfrak{U}) = S(S^{n-1}(P,\mathfrak{U}), \mathfrak{U}), \quad S^\infty(P,\mathfrak{U}) = \bigcup_{n=1}^{\infty} S^n(P,\mathfrak{U}),$$

$$\mathfrak{U}^\Delta = \{S(p,\mathfrak{U}) \mid p \in R\}, \quad \mathfrak{U}^{\Delta\Delta} = (\mathfrak{U}^\Delta)^\Delta,$$

$$\mathfrak{U}^* = \{S(U,\mathfrak{U}) \mid U \in \mathfrak{U}\}, \quad \mathfrak{U}^{**} = (\mathfrak{U}^*)^*.$$

If $\mathfrak{U}$ is a covering, then $\mathfrak{U}^\Delta$ and $\mathfrak{U}^*$ are also coverings. It is clear that $\mathfrak{U} < \mathfrak{U}^\Delta < \mathfrak{U}^* < \mathfrak{U}^{\Delta\Delta}$.

C) A T_1-space R is normal if and only if for every finite open covering $\mathfrak{U}$ there exists a finite open covering $\mathfrak{V}$ such that $\mathfrak{V}^* < \mathfrak{U}$.

Definition I. 2. *Let R be a T_1-space. If for every open covering* $\mathfrak{U}$ *of* R *there exists an open covering* $\mathfrak{V}$ *such that* $\mathfrak{V}^* < \mathfrak{U}$ *, then we call* R *a fully normal space.*

Now we can formulate

Theorem I. 1 (A. H. Stone's Theorem) [3]. *A T_2-space R is fully normal if and only if it is paracompact.*

Corollary. *Every metric space is paracompact.*

Practically, the following theorem is proved in A. H. Stone [1].

Theorem I.1'. *Let* $\mathfrak{U}$, $\mathfrak{U}_1$, $\mathfrak{U}_2$, ... *be a sequence of open coverings such that* $\mathfrak{U} > \mathfrak{U}_1^* > \mathfrak{U}_1 > \mathfrak{U}_2^* > \dots$ *; then* $\mathfrak{U}$ *has a locally finite open refinement. If* $\mathfrak{U}$ *is a locally finite open covering of a normal space, then there is a locally finite open covering* $\mathfrak{V}$ *such that* $\mathfrak{V}^* < \mathfrak{U}$.

I. 2. Metrization

Since one of the main purposes of this book is to study the recent development in dimension theory for general metric spaces, in this section we shall give a brief account of the theory of metric spaces and of metrization. In its methods the latter has a close connection with modern dimension theory.

As is well-known a topological space R is called *metrizable* if one can introduce a topology-preserving metric in R . As for necessary and sufficient conditions for a topological space to be metrizable the following classical theorem is still fundamental.

Theorem I. 2 (Alexandroff-Urysohn's Metrization Theorem). *A T_1-space R is metrizable if and only if there exists a sequence* $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \dots$ *of open coverings* $\mathfrak{U}_i$ *such that* $\{ S(p,\mathfrak{U}_i) \mid i = 1,2,\dots \}$ *is a neighbourhood basis for each point* p *of* R .

It is generally agreed that an open collection $\mathfrak{U}$ in a topological space R is called an *open basis* if for every neighbourhood $V(p)$ of every point p of R there exists an element U of $\mathfrak{U}$ such that $p \in U \subset V(p)$.

In metric spaces σ - locally finite open bases play a major role as indicated by the following theorem.

[3] Concerning the proof see A. H. Stone [1], or J. Nagata [8].

Theorem I. 3 (Nagata-Smirnov's Metrization Theorem) [4]. *A regular space* R *is metrizable if and only if there exists a* σ *- locally finite open basis of* R .

Corollary (Urysohn's Metrization Theorem). *A second countable space is metrizable if and only if it is regular, where we mean by a second countable space a topological space which has a countable open basis.*

Theorem I. 4 (Bing's Metrization Theorem) [5]. *A regular space* R *is metrizable if and only if there exists a* σ *-discrete basis of* R .

From now on throughout this section let us denote by R a metric space with the metric $\rho(x,y)$. (Later we may use a symbol like $\langle R,\rho \rangle$ to emphasize that we consider the metric space R together with the fixed metric ρ , especially when more than one metric compatible with the topology of R are involved in the discussion.) An open covering $\mathcal{U}$ of R is called a *uniform covering* if there exists $\varepsilon > 0$ such that

$$\{ S_\varepsilon(p) \mid p \in R \} < \mathcal{U} ,$$

where $S_\varepsilon(p)$ denotes the spherical neighbourhood of p with radius ε , i.e.

$$S_\varepsilon(p) = \{ q \mid \rho(p,q) < \varepsilon \} .$$

A) Every open covering of a compact metric space is a uniform covering.

We denote by $\delta(U)$ the *diameter* of a subset U of R and by mesh $\mathcal{U}$ the number $\sup \{ \delta(U) \mid U \in \mathcal{U} \}$ for a collection $\mathcal{U}$. Let $\mathcal{F}$ be a *filter* of R , i.e. $\mathcal{F}$ is a nonvacuous collection which satisfies

$$(1)\ \emptyset \notin \mathcal{F} , \quad (2)\ F,G \in \mathcal{F} \text{ implies } F \cap G \in \mathcal{F} , \quad (3)\ G \supset F \in \mathcal{F} \text{ implies } G \in \mathcal{F} .$$

If for every $\varepsilon > 0$ there exists an element F of $\mathcal{F}$ with $\delta(F) < \varepsilon$, then we call $\mathcal{F}$ a *Cauchy filter*. Let $\mathcal{F}$ and $\mathcal{G}$ be two Cauchy filters of R . If for every $\varepsilon > 0$ there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\delta(F \cup G) < \varepsilon$, then we call $\mathcal{F}$ and $\mathcal{G}$ *equivalent*.

We can classify all Cauchy filters of R by this equivalence relation. Then we denote by R^* the set of all classes. Let a and b be two points of R^* and let $\mathcal{F} \in a$, $\mathcal{G} \in b$. Then one can easily see that

$$\rho^*(a,b) = \sup \{ \rho(F,G) \mid F \in \mathcal{F} , G \in \mathcal{G} \}$$

is uniquely determined by a and b , where

$$\rho(F,G) = \inf \{ \rho(x,y) \mid x \in F , y \in G \} .$$

One can also easily see that $\rho^*(a,b)$ gives a metric of R^* . If we identify each point p of R with the class containing the Cauchy filter $\mathcal{P} = \{ F \mid p \in F \}$,

[4] Concerning the proof see J. Nagata [1], Yu. Smirnov [1] or J. Nagata [8].
[5] See R. H. Bing [1] or J. Nagata [8].

then R can be considered as a sub-metric space of the metric space R^* .

It is easy to see that R^* satisfies

i) R is dense in R^* ,

ii) R^* is a *complete metric space* i.e. every Cauchy filter of R^* converges,

iii) every uniformly continuous mapping f of R into a complete metric space S can be extended to a uniformly continuous mapping f^* of R^* into S , i.e.

$$f(x) = f^*(x) \text{ for } x \in R .$$

This metric space R^* is called the *completion* of the metric space R .

B) Let A be a subset of a metric space R .
We denote by $\bar{A}^*$ the closure of A in the completion R^* . Then $\bar{A}^*$ consists of all the classes which contain a filter F such that $A \in F$.

C) In connection with B), let $\mathcal{U} = \{ U_\gamma \mid \gamma \in \Gamma \}$ be a uniform covering of R ; then $\mathcal{U}' = \{ R^* - \overline{R - U_\gamma}^* \mid \gamma \in \Gamma \}$ is a uniform covering of R^* .

D) The completion of a totally bounded metric space is compact.

A metric space R is called *totally bounded* if for every $\varepsilon > 0$ the covering $\{ S_\varepsilon(x) \mid x \in R \}$ has a finite subcovering. It is well known that a metric space is compact if and only if it is complete and totally bounded, which implies the above proposition. It is also well known that a metric space R is *separable* (i.e. it has a countable dense set) if and only if it is second countable if and only if one can introduce a totally bounded metric onto R . As for complete metric spaces the following well-known theorem is also applicable in dimension theory.

Theorem I. 5 (Baire's Theorem). *Let* U_n , $n = 1, 2, \ldots$ *be open dense subsets of a complete metric space* R . *Then* $\cap_{n=1}^{\infty} U_n$ *is also dense in* R .

The *Hilbert cube*

$$I^\omega = \{ (x_1, x_2, \ldots) \mid |x_i| \leq 1/i , \; i = 1, 2, \ldots \}$$

is an important example of a compact metric space and accordingly of a separable metric space.

E) A metric space R is separable if and only if R is homeomorphic to a subspace of I^ω .

Let us give another important example of a metric space. We denote by Ω a given set and let $N(\Omega) = \{ (\alpha_1, \alpha_2, \ldots) \mid \alpha_k \in \Omega , \; k = 1, 2, \ldots \}$, $\rho((\alpha_1, \alpha_2, \ldots), (\beta_1, \beta_2, \ldots)) =$

$= 1/\min \{ k \mid \alpha_k \neq \beta_k \}$ for $(\alpha_1, \alpha_2, \dots) \in N(\Omega)$, $(\beta_1, \beta_2, \dots) \in N(\Omega)$. We can easily see that ρ is a metric of $N(\Omega)$. This metric space $N(\Omega)$ is called a *generalized Baire zero-dimensional space.* We shall deal with this space again in Chapter III.

F) A metric space R has a σ - star-finite open basis if and only if R is homeomorphic to a subspace of a topological product $N(\Omega) \times I^{\omega}$ for a suitable Ω , where an open basis is called a σ - *star-finite open basis* if it is the union of countably many star-finite open *coverings* (due to K. Morita).

G) Every locally compact metric space has the star-finite property, and accordingly a σ-star-finite open basis.

I. 3. Mappings

The theory of mappings is also a powerful tool for dimension theory. Let g be a continuous mapping of a topological space R into a topological space S . Let f be a continuous mapping of a subset A of R into S such that $f(x) = g(x)$ for every $x \in A$. Then g is called a *continuous extension of* f *over* R , and f is called the *restriction of* g *to* A , being denoted as $f = g \mid A$. We denote by $f\colon A \to S$ a mapping f of A into S .

Theorem I. 6 (Urysohn's Lemma). *Let* F *and* G *be disjoint closed sets of a normal space* R *; then there exists a real-valued continuous function* f *such that* $f(F) = 0$, $f(G) = 1$, $0 \leq f \leq 1$.

Throughout this book we shall mean by $f(F) = a$ that $f(p) = a$ for every point $p \in F$.

Corollary. *Let* F *and* G *be disjoint closed sets of a metric space* R*; then there exists a real-valued continuous function* f *such that* $f^{-1}(0) = F$, $f^{-1}(1) = G$, $0 \leq f \leq 1$.

Theorem I. 7 (Tietze's Extension Theorem). *Let* f *be a continuous mapping of a closed subset* F *of a normal space* R *into the n-dimensional Euclidean cube* I^n . *Then there exists a continuous extension* g *of* f *over* R .

Corollary. *Let* f *be a continuous mapping of a closed subset* F *of a normal space* R *into the n-sphere* S^n . *Then there exists a continuous extension of* f *over an open set* U *containing* F .

Theorem I. 8 (M. H. Stone-Gelfand-Silov's Approximation Theorem)[6]. *Let* C *be a ring of real-valued continuous functions of a compact* T_2*-space* R . *Let* C *satisfy* *i) every constant function belongs to* C ,

ii) for any distinct points p , q *of* R *there exists* $g \in C$ *such that* $g(p) \neq g(q)$. *Then for any real-valued continuous function* f *of* R *and for any* $\varepsilon > 0$ *there exists* $f_\varepsilon \in C$ *which satisfies* $|f(p) - f_\varepsilon(p)| < \varepsilon$ *for every* $p \in R$.

Theorem I. 9 (A. H. Stone-Morita-Hanai's Theorem)[6]. *Let* f *be a closed continuous mapping of a metric space* R *onto a topological space* S *(namely* f *maps every closed set of* R *to a closed set of* S *). Then* S *is metrizable if and only if* $B(f^{-1}(y))$ *is compact for every* $y \in S$.

I. 4. Dimension [7].

In their book published in 1941 Hurewicz and Wallman had to limit themselves to separable metric spaces because it seemed impossible at that time to establish a theory of dimension for more general spaces. However, such a more general theory has been made possible by the developments in general topology which took place since 1948.

In 1948, A. H. Stone proved Theorem I.1 which was a considerable step forward in the theory of open coverings especially of locally finite coverings in metric spaces. Furthermore Theorem I.3, which was developed from Theorem I.1, determined the importance of locally finite open coverings in general metric spaces. Thus Theorem I.1 made an epoch not only for modern general topology but for modern dimension theory. On the foundation of the developed covering theory for metric spaces, M. Katětov [2] in 1952 and K. Morita [4] in 1954 independently succeeded in extending the principal results of the classical dimension theory to general metric spaces and in proving $\operatorname{Ind} R = \dim R$ for every metric space R [8] . It is an interesting fact that some of the results which have been established in dimension theory for general metric spaces since Katětov-Morita's work are quite new even for separable metric spaces. We are now inclined to think that we have obtained the final answers to the major problems in dimension theory for general metric spaces, though a few questions remain, and dimension theory for non-metrizable spaces has also greatly developed in these years.

The most important dimension functions for general metric spaces are *covering* (or *Lebesgue) dimension* $\dim R$ and *strong inductive* (or *large inductive* or *Čech*) *dimension* $\operatorname{Ind} R$.

Definition I. 3. *Let* $\mathfrak{U}$ *be a collection in a topological space* R *and* p *a point of* R . *Then we mean by the order of* $\mathfrak{U}$ *at* p *the number of members of* $\mathfrak{U}$ *which contain* p , *and we denote it by* $\operatorname{ord}_p \mathfrak{U}$. *If there exist infinitely many such members, then* $\operatorname{ord}_p \mathfrak{U} = +\infty$.

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[6] For a proof see J. Nagata [8].

[7] As for historical review on concepts of dimension as well as the earlier developments of dimension theory see W. Hurewicz and H. Wallman [1], P. Alexandroff [6], and also S. P. Zervos [1]. For modern developments, see J. Nagata [9], [10], [11].

[8] The definitions of $\dim R$ and $\operatorname{Ind} R$ will be found in Definitions I.4 and I.5, respectively.

The order of $\mathcal{U}$ *will be the supremum of* $\text{ord}_p\,\mathcal{U}$ *and be denoted by* $\text{ord}\,\mathcal{U}$,

$$\text{ord}\,\mathcal{U} = \sup\{\text{ord}_p\,\mathcal{U} \mid p \in R\} .$$

Definition I. 4. *If for any finite open covering* $\mathcal{U}$ *of a topological space* R *there exists an open covering* $\mathcal{V}$ *such that* $\mathcal{V} < \mathcal{U}$, $\text{ord}\,\mathcal{V} \leqq n + 1$, *then* R *has covering dimension* $\leqq n$, $\dim R \leqq n$. R *has covering dimension* n , $\dim R = n$ *if it is true that* $\dim R \leqq n$ *and it is false that* $\dim R \leqq n - 1$. *If* $\dim R \leqq n$ *is false for each integer* n , *then* $\dim R = +\infty$. *We define that* $\dim \emptyset = -1$.

Definition I. 4'. *If we replace 'open covering' in Definition I.4 with 'finite functionally open covering', then we obtain the definition of Katětov-Smirnov covering dimension (an open set* U *of* R *is called functionally open or a cozero set if for some real-valued continuous function* f *defined on* R $U = \{x \in R \mid f(x) \neq 0\}$*). The complement of a functionally open set is called functionally closed or a zero set. It is easy to see that Katětov-Smirnov dimension coincides with Lebesgue dimension if the space is normal. In this book,* $\dim R$ *means Lebesgue dimension unless otherwise specified. In a Tychonoff space* X *the Katětov-Smirnov dimension of* X *is* $\dim \beta X$ *where* βX *denotes the Stone-Čech compactification of* X .

Definition I. 5. *i) A topological space* R *has strong inductive dimension* -1 , $\text{Ind}\,R = -1$, *if* $R = \emptyset$.

ii) If for any disjoint closed sets F *and* G *of a topological space* R *there exists an open set* U *such that* $F \subset U \subset R - G$, $\text{Ind}\,B(U) \leqq n - 1$, *then* R *has strong inductive dimension* $\leqq n$, $\text{Ind}\,R \leqq n$.
If it is true that $\text{Ind}\,R \leqq n$ *and it is false that* $\text{Ind}\,R \leqq n - 1$, *then* $\text{Ind}\,R = n$.
If $\text{Ind}\,R \leqq n$ *is false for each* n , *then* $\text{Ind}\,R = +\infty$.

The following notion of *weak inductive* (or *small inductive* or *Urysohn-Menger) dimension* is no longer so important as the preceding two notions, because as proved by P. Roy, it is not equivalent to the preceding dimensions for general metric spaces (though it is for separable metric spaces), and because some basic properties required for dimension are not possessed by this dimension function.

Definition I. 6. *i) A topological space* R *has weak inductive dimension* -1 , $\text{ind}\,R = -1$, *if* $R = \emptyset$.

ii) If for every neighbourhood $U(p)$ *of every point* p *of* R *there exists an open neighbourhood* V *such that* $p \in V \subset U(p)$, $\text{ind}\,B(V) \leqq n - 1$, *then* R *has weak inductive dimension* $\leqq n$, $\text{ind}\,R \leqq n$.
If it is true that $\text{ind}\,R \leqq n$ *and it is false that* $\text{ind}\,R \leqq n - 1$, *then* $\text{ind}\,R = n$.
If $\text{ind}\,R \leqq n$ *is false for each* n , *then* $\text{ind}\,R = +\infty$.