APPENDIX

Foundations

No problem is so big or so complicated that it can't be run away from.

——LINUS†

In Chapter II we have listed some "features" of a foundational system that would be sufficiently broad and flexible to "handle" category theory. In this appendix, a model for such a system will be exhibited. First, however, it seems appropriate to briefly mention other foundational approaches.

Gödel-Bernays-von Neumann Set Theory

The main feature of this theory is that the basic entities involved (called classes) are of two sorts: those that are "small" (called sets) and those that are "large" (called proper classes). Specifically, a class is called a set provided that it is a member of some class, and it is called a proper class otherwise. This system has more flexibility than the usual Zermelo-Fraenkel set theory; for example, with it one can form the class of all sets, the class of all topological spaces, etc., so that the categories Set, Top, etc. can be legitimately constructed. However, within this system, there can be no entities that have proper classes as members. Because of this, functor categories $[\mathscr{A}, \mathscr{B}]$ cannot be formed when the category $\mathscr{A}$ is a proper class (the objects and morphisms of such a category would be proper classes). Also, one cannot properly state the Yoneda Lemma within this system because to do so would require the formation of entities having proper classes as members.

Grothendieck Universes

In this approach, one adds to the usual Zermelo-Fraenkel axioms the axiom that every set is contained in a "universe", where a universe is defined to be a set that contains the natural numbers, that is closed under the usual set-theoretical constructions of pairing, union, power set, and replacement, and that contains each member of each of its members. Using this as a foundation, one cannot, for example, form the category Set of all sets, but rather for each universe $\mathscr{V}$, the category $\mathbf{Set}_{\mathscr{V}}$ of all sets which belong to $\mathscr{V}$. Here there is the advantage that functor categories can be handled without difficulty. A disadvantage is the obvious additional complication introduced by the universes. For example, one has the tedious task of showing that for each universe $\mathscr{V}$, the "essential" properties of $\mathbf{Set}_{\mathscr{V}}$ do not depend upon $\mathscr{V}$.

The Category of All Categories

Here the idea (due to Lawvere) is to throw out set theory as a foundation for mathematics and replace it by category theory. "Category" and "functor" are considered to be primitive notions. The fundamental axioms thus deal with them and are not concerned with the entities "set" and "membership". This approach has the advantage that most serious difficulties encountered in other approaches can be "legislated away" by the introduction of appropriate axioms. The essential disadvantage is that it has less intuitive appeal than set theory. Also, it is not yet fully developed, and for the purposes of this text (i.e., to provide an introduction to category theory), using it itself as a foundation would certainly lead to undue confusion.

One Universe

This approach has been used and developed by Isbell, Mac Lane, and Feferman and is the foundational approach taken for this text. It consists "essentially" of the addition of one extra stage of flexibility to the Gödel-Bernays-von Neumann approach, as opposed to the addition of a plethora of extra stages as with Grothendieck Universes.† Here (cf. Chapter II) there are three (rather than just two) basic sorts of entities: sets, classes, and conglomerates. Each set is a class and each class is a conglomerate. The sets and classes together form a model for Gödel-Bernays-von Neumann set theory. However, even the proper classes occur as elements of conglomerates, so that there is no difficulty in forming or dealing with functor (quasi)categories (15.1), the quasi-category $\mathscr{CAT}$ of all categories (11.5), or in stating the Yoneda Lemma in its proper generality (30.6). A disadvantage, of course, is that there are two sorts of categories—the usual ones and quasicategories (the latter not necessarily having the property that their objects and morphisms are always sets or the property that the conglomerate of all morphisms between any two objects

† A slightly different approach (due to Osius) is also accomplished by the introduction of one more stage of flexibility to Gödel-Bernays-von Neumann set theory. Here atoms (i.e., sets without elements) and certain proper classes called *C*-classes are introduced. The *C*-classes are in one-to-one correspondence with the atoms and this correspondence allows one to regard them as being "essentially" elements of classes. If a category is called a *C*-category whenever its morphisms constitute a *C*-class, then functor categories of *C*-categories are again *C*-categories and one can form the category of all *C*-categories (which is no longer a *C*-category).

forms a set). However, as can be seen in the course of this text, this poses no real difficulty and indeed none is encountered so long as one's primary interest is in categories, and quasicategories are used mainly as a tool for studying categories.

A model for the "one universe" approach can be obtained as follows:

Begin with the usual axioms for Zermelo-Fraenkel or Gödel-Bernays-von Neumann set theory. However, call what are usually called "sets" in the theory, "conglomerates". One now has, for example, the conglomerate N of all natural numbers and one knows for instance that if A and B are conglomerates, then $\{A, B\}$, (A, B), $A \cup B, \bigcup A$, and $\mathscr{P}(A)$ are also conglomerates.

Now add the additional axiom† that there exists a "universe" where by a universe we mean a conglomerate $\mathscr{U}$ with the following properties:

(i) $\mathrm{N} \in \mathscr{U}$
(ii) $A \in \mathscr{U} \Rightarrow \bigcup A \in \mathscr{U}$.
(iii) $A \in \mathscr{U} \Rightarrow \mathscr{P}(A) \in \mathscr{U}$.
(iv) $I \in \mathscr{U}$ and $f\colon I \to \mathscr{U}$ is a function $\Rightarrow f[I] \in \mathscr{U}$.
(v) $a \in A \in \mathscr{U} \Rightarrow a \in \mathscr{U}$.

Calling the members of $\mathscr{U}$ "sets" and the subconglomerates of $\mathscr{U}$ "classes", all of the features of the foundational system required in Chapter II can now be easily verified. For example:

(1) *Every set is a class.*

Proof: If $x \in \mathscr{U}$ and $z \in x$, then by (v) $z \in \mathscr{U}$; hence, $x \subset \mathscr{U}$. □

(2) *Every subconglomerate of a set is a set.*

Proof: If $x \subset y \in \mathscr{U}$, then $x \in \mathscr{P}(y) \in \mathscr{U}$ ((iii)), so that $x \in \mathscr{U}$ ((v)). □

(3) *If x and y are sets, then $\{x, y\}$ is a set.*

Proof: Define $f\colon \mathrm{N} \to \mathscr{U}$ by $f(0) = x$ and $f(a) = y$ if $a \neq 0$. Then $f[\mathrm{N}] = \{x, y\}$ is a set ((iv)). □

(4) *If x and y are sets, then (x, y) is a set.*

Proof: $(x, y) = \{\{x\}, \{x, y\}\}$. Apply (3). □

(5) *If x and y are sets, then $x \times y$ is a set.*

Proof: $x \times y \subset \mathscr{P}\mathscr{P}(\bigcup\{x, y\})$. □

(6) *If I is a set and $f\colon I \to \mathscr{U}$, then $\bigcup_{i \in I} f(i)$ and $\prod_{i \in I} f(i)$ are sets.*

Proof:

$$\bigcup_{i \in I} f(i) = \bigcup f[I] \in \mathscr{U} \quad \text{by (ii) and (iv).}$$

$$\prod_{i \in I} f(i) \subset \mathscr{P}(I \times \bigcup f[I]); \quad \text{((6), (5), (iii), and (2)).} \quad \square$$

† Adding a new axiom, of course, poses the prospect of introducing an inconsistency. However, hypothesizing the existence of a universe is essentially the same as hypothesizing the existence of a strongly inaccessible cardinal. This is generally felt to be free of inconsistencies.