III

Categories

"The time has come," the Walrus said,
"To talk of many things:
Of shoes-and ships-and sealing-wax—
Of cabbages-and kings—"
——Lewis Carroll†

In this chapter, the general notion of a category is developed. Certain properties of categories are introduced, as well as methods of obtaining new categories from given ones.

§2 CONCRETE CATEGORIES

When deciding what a category ought to be, one should keep in mind the three motivating examples from Chapter I. That is, the notion of a category should be sufficiently general to include as special cases:

(1) the class of all sets together with all functions between them,

(2) the class of all groups together with all homomorphisms between them,

(3) the class of all topological spaces together with all continuous functions between them.

Keeping these examples in mind, one's first vague notion of a category is that of a class of structured sets together with a class of functions between these sets which in some way preserve their structure. We now make this notion precise by defining what is known as a "concrete category". Later (§3), the more general notion of "abstract category", will be introduced.

2.1 DEFINITION

A **concrete category** is a triple $\mathscr{C} = (\mathbb{C}, U, hom)$, where

(i) $\mathbb{C}$ is a class whose members are called **$\mathscr{C}$-objects**;

†From *Alice in Wonderland.*

(ii) $U: \mathcal{O} \to \mathcal{U}$ is a set-valued function,† where for each $\mathscr{C}$-object A, $U(A)$ is called the **underlying set of** A;

(iii) *hom*: $\mathcal{O} \times \mathcal{O} \to \mathcal{U}$ is a set-valued function, where for each pair (A, B) of $\mathscr{C}$-objects, $hom(A, B)$ is called the set of all **$\mathscr{C}$-morphisms with domain A and codomain B**;

such that the following conditions are satisfied:

(1) for each pair (A, B) of $\mathscr{C}$-objects, $hom(A, B)$ is a subset of the set $U(B)^{U(A)}$ of all functions from $U(A)$ to $U(B)$;

(2) for each $\mathscr{C}$-object A, the identity function $1_{U(A)}$ on the set $U(A)$ is a member of $hom(A, A)$;

(3) for each triple (A, B, C) of $\mathscr{C}$-objects, $f \in hom(A, B)$ and $g \in hom(B, C)$ implies that $g \circ f \in hom(A, C)$ (where "$\circ$" denotes the composition of functions).

2.2 EXAMPLES OF CONCRETE CATEGORIES

(1) The category **Set** whose class of objects is the class $\mathcal{U}$ of all sets; $U: \mathcal{U} \to \mathcal{U}$ is the identity function, i.e., $U(A) = A$ for all $A \in \mathcal{U}$; and for $A, B \in \mathcal{U}$, $hom(A, B)$ is the set of all functions from A to B. **Set** is commonly called the **category of sets** (1.2).

(2) The category **Grp** whose class of objects is the class of all groups. For any group A, $U(A)$ is the underlying set of A; and for groups A and B, $hom(A, B)$ is the set of all group homomorphisms from A to B. **Grp** is commonly called the **category of groups**.

(3) The category **Top** whose class of objects is the class of all topological spaces; for any topological space A, $U(A)$ is the underlying set of A, and for topological spaces A and B, $hom(A, B)$ is the set of all continuous functions from A to B. **Top** is commonly called the **category of topological spaces**.

(4) For every ring R, the category R-**Mod** whose class of objects is the class of all left R-modules; $U(A)$ is the underlying set of A, and $hom(A, B)$ is the set of all module homomorphisms (i.e., R-linear transformations) from A to B. R-**Mod** is commonly called the **category of left R-modules**.

(5) In a manner similar to that described above, one obtains the following categories:

Mod-R—right R-modules and module homomorphisms;
POS—partially ordered sets and monotone functions;
Lat—lattices and lattice homomorphisms;
BooAlg—boolean algebras and boolean homomorphisms;
Ab—abelian groups and group homomorphisms;
SGrp—semigroups and semigroup homomorphisms;
Mon—monoids (i.e., semigroups with identity) and identity-preserving semigroup homomorphisms;

† Recall that $\mathcal{U}$ is the class of all sets (1.2).

Rng—rings† and ring homomorphisms;
Field—fields†† and field homomorphisms;
R-**Alg**—R-algebras and R-algebra homomorphisms (where R is a commutative ring);
$\mathbf{Top}_2$—Hausdorff spaces and continuous functions;
$\mathbf{CRegT}_2$—completely regular Hausdorff spaces and continuous functions;
$\mathbf{CompT}_2$—compact Hausdorff spaces and continuous functions;
TopGrp—topological groups and continuous homomorphisms;
LinTop—linear topological Hausdorff spaces and continuous linear transformations;
NLinSp—normed linear spaces and bounded (= continuous) linear transformations;
$\mathbf{BanSp}_1$—complex Banach spaces and bounded linear transformations;
$\mathbf{BanSp}_2$—complex Banach spaces and norm-decreasing linear transformations;
CBanAlg—commutative complex Banach algebras (with unit) and norm-decreasing algebra homomorphisms;
C*-Alg—commutative complex Banach algebras with involution $*$ satisfying $\|a*a\| = \|a\|^2$, and norm-decreasing, involution-preserving algebra homomorphisms.

(6) The **category of sets and injective** (resp. **surjective, bijective**) **functions**, whose class of objects is $\mathscr{U}$. $U: \mathscr{U} \to \mathscr{U}$ is the identity function, and $hom(A, B)$ is the set of all injective (resp. surjective, bijective) functions from A to B.

(7) The **category of topological spaces and open functions**, whose objects are topological spaces, where, for spaces A and B, $hom(A, B)$ is the set of all open functions from A to B.

(8) The category **pSet**, the objects of which are all pairs of the form (A, a) where A is a set and $a \in A$; $U((A, a)) = A$ and

$$hom((A, a), (B, b)) = \{f \mid f: A \to B \text{ and } f(a) = b\}.$$

pSet is commonly called the **category of sets with base point** or the **category of pointed sets**. Similarly, one obtains **pTop**, the **category of pointed topological spaces**, and the **category of bi-pointed sets** whose objects are triples (A, a_0, a_1) where $a_0, a_1 \in A$ and morphisms preserve both distinguished elements; i.e., $f(a_0) = b_0$ and $f(a_1) = b_1$.

§3 ABSTRACT CATEGORIES

Just as it is useful to study abstract groups rather than transformation groups or abstract topological spaces rather than metric spaces, so too it is often desirable to study mathematical structures which are somewhat more general than concrete categories. For example, we will later consider certain

† In this book, all rings have identities, and ring homomorphisms preserve the identities.
†† For fields, we always require that $0 \neq 1$.

constructions which are useful tools for the investigation of categories and which, when applied to (concrete) categories, yield entities that are similar to, but more general than, concrete categories. They will be categories, however, if we slightly broaden the definition of concrete category so as to no longer require that morphisms be functions or that the composition law be composition of functions.

First Definition of Category

3.1 DEFINITION

A **category** is a quintuple $\mathscr{C} = (\mathscr{O}, \mathscr{M}, dom, cod, \circ)$ where

(i) $\mathscr{O}$ is a class whose members are called **$\mathscr{C}$-objects,**
(ii) $\mathscr{M}$ is a class whose members are called **$\mathscr{C}$-morphisms,**
(iii) *dom* and *cod* are functions from $\mathscr{M}$ to $\mathscr{O}$ ($dom(f)$ is called the **domain** of f and $cod(f)$ is called the **codomain** of f),
(iv) $\circ$ is a function from

$$D = \{(f, g) \mid f, g \in \mathscr{M} \text{ and } dom(f) = cod(g)\}$$

into $\mathscr{M}$, called the **composition law of** $\mathscr{C}$ ($\circ(f, g)$ is usually written $f \circ g$ and we say that $f \circ g$ is defined if, and only if, $(f, g) \in D$);

such that the following conditions are satisfied:

(1) **Matching Condition:** If $f \circ g$ is defined, then $dom(f \circ g) = dom(g)$ and $cod(f \circ g) = cod(f)$;

(2) **Associativity Condition:** If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$;

(3) **Identity Existence Condition:** For each $\mathscr{C}$-object A there exists a $\mathscr{C}$-morphism e such that $dom(e) = A = cod(e)$ and

(a) $f \circ e = f$ whenever $f \circ e$ is defined, and
(b) $e \circ g = g$ whenever $e \circ g$ is defined;

(4) **Smallness of Morphism Class Condition:** For any pair (A, B) of $\mathscr{C}$-objects, the class

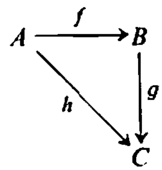
$$hom_{\mathscr{C}}(A, B) = \{f \mid f \in \mathscr{M}, \quad dom(f) = A \quad \text{and} \quad cod(f) = B\}$$

is a set.

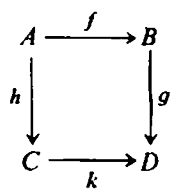
For a given category, $\mathscr{C}$, the class of $\mathscr{C}$-objects will be denoted by $Ob(\mathscr{C})$, whereas, $Mor(\mathscr{C})$ will stand for the class of $\mathscr{C}$-morphisms. Morphisms will usually be denoted by lower-case letters, with upper-case letters being reserved for objects. Also, instead of $hom_{\mathscr{C}}(A, B)$, we often write $hom(A, B)$ when no confusion seems likely.

Although $\mathscr{C}$-morphisms no longer need to be functions, we will continue to use the notations: $f \in hom(A, B)$, $A \xrightarrow{f} B$, and $f\colon A \to B$, interchangeably. Likewise, the composition law will usually be denoted by $\circ$, and we will sometimes use

$A \xrightarrow{f} B \xrightarrow{g} C$ to denote the composition $g \circ f$. Thus, the statement that the triangle



commutes, is equivalent to the statement that $h = g \circ f$ or the statement that $A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{h} C$. When morphisms f and g exist such that the above triangle commutes, we say that h **factors through** B. Similarly, the statement that the square



commutes, means that $g \circ f = k \circ h$. This order of writing the composition of morphisms may sometimes seem to be backwards. However, it comes from the fact that in many of the important examples (e.g., all concrete categories), the composition law is the composition law for functions. Notice that because of the associativity of composition, the notation $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is unambiguous.

3.2 PROPOSITION

Let $\mathscr{C}$ be a category and A be a $\mathscr{C}$-object. Then there exists exactly one $\mathscr{C}$-morphism $e: A \to A$ satisfying the properties 3(*a*) *and* 3(*b*) *of Definition* 3.1*; i.e., such that*

(a) $f \circ e = f$, *whenever* $f \circ e$ *is defined, and*
(b) $e \circ g = g$, *whenever* $e \circ g$ *is defined.*

Proof: Suppose that each of e and $\hat{e}$ is such a morphism. Then by (a), $\hat{e} \circ e = \hat{e}$ and by (*b*), $\hat{e} \circ e = e$; hence, $e = \hat{e}$. □

3.3 DEFINITION

For each object A of a category $\mathscr{C}$, the unique $\mathscr{C}$-morphism $e: A \to A$ satisfying (a) and (b) above is denoted by 1_A and is called the **$\mathscr{C}$-identity** of A.†

3.4 DEFINITION

A category $\mathscr{C}$ is said to be:

(1) **small** provided that $\mathscr{C}$ is a set;

(2) **discrete** provided that all of its morphisms are identities;

† Occasionally (when the domain is well-known or unimportant) an identity is denoted merely by 1.

(3) **connected** provided that for each pair (A, B) of $\mathscr{C}$-objects, $hom_{\mathscr{C}}(A, B) \neq \emptyset$.

Before a list of examples of categories is given, it should be pointed out that a category $\mathscr{C} = (\mathscr{O}, \mathscr{M}, dom, cod, \circ)$ is usually presented in the form $(\mathscr{O}, (hom(A, B))_{(A,B)\in\mathscr{O}\times\mathscr{O}}, \circ)$. Note that if one is given the latter form for $\mathscr{C}$, then the original form can easily be recovered by letting $\mathscr{M}$ be the disjoint union of the morphism sets and by defining $dom: \mathscr{M} \to \mathscr{O}$ and $cod: \mathscr{M} \to \mathscr{O}$ as follows:

$$dom(f) = \text{the unique object } A \text{ such that for some } B, \quad f \in hom(A, B);$$

$$cod(f) = \text{the unique object } B \text{ such that for some } A, \quad f \in hom(A, B).$$

Actually, categories could be defined in general by means of object classes $\mathscr{O}$, the families

$$(hom(A, B))_{(A,B)\in\mathscr{O}\times\mathscr{O}}$$

of morphism sets, and the composition laws. If this is done, however, the morphism sets must be required to be pairwise disjoint, for otherwise a morphism would not necessarily have a unique domain and a unique codomain. (Whether or not a given morphism f would be an identity might then depend upon not f alone, but also upon the chosen domain and codomain of f.) However, if such a triple

$$(\mathscr{O}, (hom(A, B))_{(A,B)\in\mathscr{O}\times\mathscr{O}}, \circ)$$

fails to be a category only because its morphism sets are not pairwise disjoint, then the difficulty can easily be overcome by replacing each set $hom(A, B)$ by a set

$$\overline{hom}(A, B) = \{(A, f, B) \mid f \in hom(A, B)\}.$$

This "disjointifying trick" should be applied in several of the examples below as well as in some constructions later on. Since the trick is "standard", the reader will be expected to apply it himself, whenever its use is appropriate.

3.5 EXAMPLES OF ABSTRACT CATEGORIES

(1) The category naturally associated with any given concrete category $\mathscr{C} = (\mathscr{O}, U, hom)$: whose class of objects is $\mathscr{O}$; whose morphism sets $hom_{\mathscr{C}}(A, B)$ are the sets $hom(A, B)$; and whose composition law is the usual composition of functions.† From now on if no confusion seems likely, we will not distinguish between a given concrete category and the category naturally associated with it.

(2) The **category of sets and relations**: whose class of objects is the class of all sets: whose morphisms sets $hom(A, B)$ are the sets of all relations from A to B; and whose composition law is the usual composition of relations.

(3) The **category of topological bundles, TopBun**: whose class of objects consists

† Notice that for many concrete categories (e.g., **Top** and **Grp**), the morphism sets $hom(A, B)$ are not pairwise disjoint. Thus, the "trick" mentioned above should be applied in these cases.

of all triples (X, p, B) where X and B are topological spaces and $p: X \to B$ is a continuous map. The morphisms from (X, p, B) to (X', p', B') consist of all pairs of the form (r, s) where $r: X \to X'$ and $s: B \to B'$ are continuous maps and $p' \circ r = s \circ p$. For each topological space B there is an associated category $\mathbf{TopBun}_B$ of all **topological bundles with base space** B, whose object class consists of all pairs (X, p) where $p: X \to B$ is continuous. Morphisms in this category from (X, p) to (X', p') are all those continuous maps $r: X \to X'$ for which $p = p' \circ r$.

(4) For a given commutative ring R, the **category of R-matrices**: whose objects are the positive integers; and where each morphism set $hom(m, n)$ is the set of all $n \times m$ matrices with coefficients in R. Composition is the usual multiplication of matrices.

(5) A chain complex of abelian groups is a family $(G_i, d_i)_{i \in \mathbf{Z}}$ indexed by the integers $\mathbf{Z}$ such that for each $i \in \mathbf{Z}$, G_i is an abelian group, $G_i \xrightarrow{d_i} G_{i-1}$ is a morphism in **Ab**, and $d_{i-1} \circ d_i = 0$. The class of all chain complexes of abelian groups is the object class of a category $\mathscr{C}$. A $\mathscr{C}$-morphism f from the chain complex $(G_i, d_i)_{i \in \mathbf{Z}}$ to the chain complex $(G_i', d_i')_{i \in \mathbf{Z}}$ is an indexed family $f = (f_i)_{i \in \mathbf{Z}}$ such that for each $i \in \mathbf{Z}$, $G_i \xrightarrow{f_i} G_i'$ is a morphism in **Ab** and the square

$$\begin{array}{ccc} G_i & \xrightarrow{d_i} & G_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ G_i' & \xrightarrow[d_i']{} & G_{i-1}' \end{array}$$

commutes.

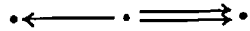
Composition is defined in the obvious way; i.e., $(f_i) \circ (g_i) = (f_i \circ g_i)$. $\mathscr{C}$ is commonly called the **category of chain complexes of abelian groups.**

(6) If $(\mathscr{G}, \leq)$ is a **quasi-ordered class** (i.e., a class $\mathscr{G}$ with a reflexive, transitive relation $\leq$ on $\mathscr{G}$), then $(\mathscr{G}, \leq)$ gives rise to a category $\mathscr{C}$ whose objects are the elements of $\mathscr{G}$ and such that a morphism set $hom_{\mathscr{C}}(A, B)$ contains exactly one element if $A \leq B$, and is empty otherwise. Conversely, any category $\mathscr{C}$ with the property that each morphism set $hom_{\mathscr{C}}(A, B)$ contains at most one member can be obtained in this way. By abuse of the language, we also call these categories **quasi-ordered classes.** Likewise a category $\mathscr{C}$ is called a **partially-ordered class** (resp. **totally-ordered class**) if and only if for each pair (A, B) of $\mathscr{C}$-objects $hom_{\mathscr{C}}(A, B) \cup hom_{\mathscr{C}}(B, A)$ contains at most (resp. exactly) one member.

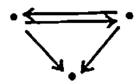
(7) If G is any monoid (i.e., semigroup with identity), then G can be regarded as a category with exactly one object, where the morphisms are precisely the members of G, and the composition law is the semigroup composition operation. Conversely, any category having exactly one object can be regarded as a monoid.

(8) If a category has only a few morphisms, it is sometimes expressed in terms

of a diagram that shows all of the objects as dots and the non-identity morphisms as arrows. Thus



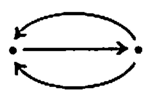
and



can be considered to be categories, but

neither

nor



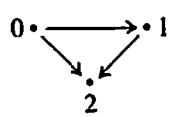
can be categories. (Why?)

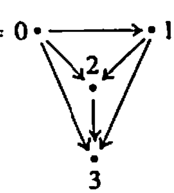
(9) For each natural number n, the set $\{0, 1, 2, \ldots, n - 1\}$ supplied with the usual order can be considered (according to (6) above) to be a category **n**. Thus, we have the special small categories:

0 = The empty category

1 = $\bullet$

2 = $0 \bullet \longrightarrow \bullet\, 1$

3 =


4 =


etc.

It is interesting to observe that for each of the abstract categories $\mathscr{C}$ given above, there is some concrete category $\hat{\mathscr{C}}$ such that the category naturally associated with $\hat{\mathscr{C}}$ (3.5(1)) is "isomorphic" with $\mathscr{C}$. Such categories are called **concretizable.**

(The relationship between concrete categories and concretizable categories is roughly analogous to that between metric spaces and metrizable spaces.) Most of the abstract categories that we will consider will be concretizable. Indeed it is somewhat difficult to exhibit an example of a category that is not concretizable (see Exercise 12L).

A Second Definition of Category

Because of the one-to-one correspondence $A \leftrightarrow 1_A$ between $\mathscr{C}$-objects and $\mathscr{C}$-identity morphisms in any category $\mathscr{C}$, we will be able below to provide an "object-free" definition of category which is equivalent to our earlier definition.

3.6 PROPOSITION (CHARACTERIZATION OF IDENTITIES)

For any morphism e of a category $\mathscr{C}$, the following are equivalent:

(1) *e is a $\mathscr{C}$-identity;*

(2) *$f \circ e = f$ whenever $f \circ e$ is defined;*

(3) *$e \circ g = g$ whenever $e \circ g$ is defined.*

Proof: By the definition of $\mathscr{C}$-identity, (1) implies (2) and (3). Suppose that (2) is true. By the definition of category, we know the existence of an identity h: $cod(e) \to cod(e)$. Hence, by (2) and the fact that h is an identity, we have

$$e = h \circ e = h.$$

Thus, (2) implies (1). Similarly (3) implies (1). □

3.7 DEFINITION

A **partial operation** on a class $\mathscr{M}$ is a function σ from a subset of $\mathscr{M} \times \mathscr{M}$ into $\mathscr{M}$. $\sigma(g, f)$ is usually denoted by $g\sigma f$. $e \in \mathscr{M}$ is called an **identity** with respect to the operation provided that for all $g \in \mathscr{M}$, whenever (g, e) is in the domain of σ (i.e., whenever $g\sigma e$ is **defined**) $g\sigma e = g$ and whenever $e\sigma g$ is defined, $e\sigma g = g$.

Notice that for any category $\mathscr{C} = (\mathcal{C}, \mathscr{M}, dom, cod, \circ)$, the composition law $\circ$ is a partial operation on $\mathscr{M}$ and (by 3.6) the $\mathscr{C}$-identities are precisely the identities of $\mathscr{M}$ with respect to $\circ$.

3.8 SECOND DEFINITION OF CATEGORY

A **category** is a pair $(\mathscr{M}, \circ)$ where $\mathscr{M}$ is a class and $\circ$ is a partial operation on $\mathscr{M}$ satifying the following conditions:

(1) **Matching Condition**: For all $f, g, h \in \mathscr{M}$, if $f \circ g$ and $g \circ h$ are defined, then $f \circ (g \circ h)$ is defined and $(f \circ g) \circ h$ is defined.

(2) **Associativity Condition**: For all $f, g, h \in \mathscr{M}$, $f \circ (g \circ h)$ is defined if and only if $(f \circ g) \circ h$ is defined, and when they are defined, they are equal.

(3) **Identity Existence Condition**: For every $f \in \mathscr{M}$, there exist morphisms e_C and e_D which are identities with respect to $\circ$ such that $e_C \circ f$ and $f \circ e_D$ are defined.†

† Observe that (2) and (3) together imply that the identities of (3) are unique.

(4) Smallness of Morphism Class Condition: For all identities e_C and e_D in $\mathcal{M}$, the class

$$\{f \in \mathcal{M} \mid e_C \circ f \text{ and } f \circ e_D \text{ are defined}\}$$

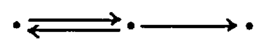
is a set.

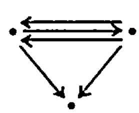
The two definitions of a category are equivalent in the following sense: if $(\mathcal{O}, \mathcal{M}, dom, cod, \circ)$ is a category according to the first definition, then $(\mathcal{M}, \circ)$ is a category according to the second definition, and if $(\mathcal{M}, \circ)$ is a category according to the second definition, then there exists an "essentially unique" category according to the first definition whose morphism class is $\mathcal{M}$ and whose composition law is $\circ$ (see Exercise 3H). Here "essentially unique" means that any two categories satisfying the property are isomorphic in the sense of §14. Because it seems closer to the motivating examples (and thus closer to one's intuition), the first definition of category will be used most often in the sequel. However, since it can sometimes simplify matters, we will reserve the right to use the alternate definition (3.8) when it seems appropriate. Also, from now on, when the symbol $f \circ g$ is written, it will usually mean that the composition makes sense (i.e., $dom(f) = cod(g)$ or "$f \circ g$ is defined") as well as standing for the result of the composition.

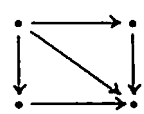
EXERCISES

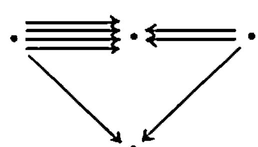
3A. Determine which of the following can be considered as categories and which cannot:

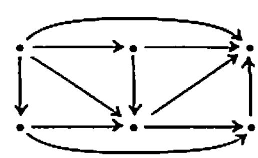
(a)

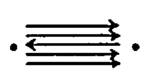
(b) 

(c) 

(d) 

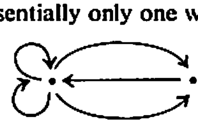
(e) 

(f) 

(g) 

(h) 

3B. Show that there is essentially only one way to define compositions so that



will be a category.

3C. Determine which of the categories of 2.2, 3.5, and 3A are small, which are discrete, and which are connected.

3D. Prove that a category $\mathscr{C}$ is discrete if and only if for all $A, B \in Ob(\mathscr{C})$,

$$hom(A, B) = \begin{cases} \varnothing & \text{if } A \neq B \\ \{1_A\} & \text{if } A = B. \end{cases}$$

3E. Determine all of the categories that are both connected and discrete.

3F. Let $\mathscr{C} = (\mathscr{O}, \mathscr{M}, dom, cod, \circ)$ be a category. Prove that the following statements are equivalent:

(a) $\mathscr{C}$ is small.
(b) $\mathscr{O}$ is a set.
(c) $\mathscr{M}$ is a set.
(d) *dom* is a set.
(e) *cod* is a set.
(f) $\circ$ is a set.

3G. Give an example of a small category whose class of objects is not finite.

3H. Prove that if $(\mathscr{O}, \mathscr{M}, dom, cod, \circ)$ is a category according to Definition 3.1, then $(\mathscr{M}, \circ)$ is a category according to Definition 3.8, and if $(\mathscr{M}, \circ)$ is a category according to 3.8, then there exist functions *dom* and *cod* such that $(\mathscr{I}, \mathscr{M}, dom, cod, \circ)$ is a category according to 3.1 (where $\mathscr{I}$ denotes the class of all identities with respect to $\circ$).

3I. Let $\mathscr{M}$ be a set together with a binary operation $\circ$. Prove that $(\mathscr{M}, \circ)$ is a group if and only if it is a category (according to Definition 3.8) with exactly one identity e such that for every $\mathscr{C}$-morphism f, there exists a $\mathscr{C}$-morphism g such that $g \circ f = e$.

3J. Show that the words "and $(f \circ g) \circ h$ is defined" can be deleted from the Matching Condition (3.8(1)) without changing the definition of category.

3K. Show that if A and B are distinct objects of a category $\mathscr{C}$, then $hom_{\mathscr{C}}(A, B)$ contains no identity morphisms (in the sense of 3.7).

3L. Construct two different categories $\mathscr{C}$ and $\mathscr{D}$ such that the object class, the morphism class, and the domain and codomain functions of $\mathscr{C}$ are the same as those of $\mathscr{D}$.

§4 NEW CATEGORIES FROM OLD

Subcategories

4.1 DEFINITION

A category $\mathscr{B}$ is said to be a **subcategory** of the category $\mathscr{C}$ provided that the following conditions are satisfied:

(1) $Ob(\mathscr{B}) \subset Ob(\mathscr{C})$.

(2) $Mor(\mathscr{B}) \subset Mor(\mathscr{C})$.

(3) The domain, codomain and composition functions of $\mathscr{B}$ are restrictions of the corresponding functions of $\mathscr{C}$.

(4) Every $\mathscr{B}$-identity is a $\mathscr{C}$-identity.

Notice that conditions (2) and (3) imply that for each pair (A, B) of $\mathscr{B}$-objects

$$hom_{\mathscr{B}}(A, B) \subset hom_{\mathscr{C}}(A, B).$$

Also observe that condition (4) does not follow from the other conditions† (see Exercise 4B).

4.2 DEFINITION

A subcategory $\mathscr{B}$ of a category $\mathscr{C}$ is said to be a **full subcategory** of $\mathscr{C}$ provided that for all $A, B \in Ob(\mathscr{B})$, $hom_{\mathscr{B}}(A, B) = hom_{\mathscr{C}}(A, B)$.

4.3 EXAMPLES

(1) Each category is a full subcategory of itself.

(2) The category of finite sets is a full subcategory of Set.

(3) The category of sets and injective (resp. surjective, bijective) functions is a subcategory of **Set** that is not full.

(4) The category of sets and relations is not a subcategory of **Set**.

(5) **BooAlg** is a subcategory of **Lat** and **Lat** is a subcategory of **POS**.†† Neither is full.

(6) **Ab** is a full subcategory of **Grp**, **Grp** is a full subcategory of **Mon**, and **Mon** is a subcategory of **SGrp**, which is not full.

(7) $\mathbf{BanSp_2}$ is a subcategory of $\mathbf{BanSp_1}$, which is not full, $\mathbf{BanSp_1}$ is a full subcategory of **NLinSp**, but **NLinSp** is not a subcategory of **LinTop**.

(8) None of the categories **Grp**, **Top**, **pSet**, **POS**, or **Lat** is a subcategory of **Set**. (Why not?) [However, it is true that each concretizable category is "isomorphic" with some subcategory of **Set** (14.2(10)).]

Quotient Categories

4.4 DEFINITION

An equivalence relation $\sim$ on the class of morphisms of a category $\mathscr{C}$ is called a **congruence** on $\mathscr{C}$ provided that:

(1) every equivalence class under $\sim$ is contained in $hom(A, B)$ for some $A, B \in Ob(\mathscr{C})$, and

(2) whenever $f \sim f'$ and $g \sim g'$ it follows that $g \circ f \sim g' \circ f'$, (whenever the compositions are meaningful).

4.5 PROPOSITION

If $\sim$ is a congruence on a category $\mathscr{C}$, then the class $\mathscr{Q}$ of equivalence classes of morphisms together with the composition law $\tilde{\circ}$ defined by:

$$\tilde{g} \,\tilde{\circ}\, \tilde{f} = \widetilde{g \circ f}$$

(where $\tilde{g}$ denotes the equivalence class of g under $\sim$) is a category (in the sense of Definition 3.8). □

† If we (as some authors do) would identify each object A of a category with the corresponding identity morphism 1_A, then conditions (1) and (4) would obviously be equivalent.

†† A caviling person might say that this is false (only because he has a different definition of boolean algebras or of lattices).

4.6 DEFINITION

If $\sim$ is a congruence on a category $\mathscr{C}$, then the category $(\mathscr{D}, \tilde{\circ})$ described above is called the **quotient category of $\mathscr{C}$ with respect to $\sim$**, and is denoted by $\mathscr{C}/\sim$.

Observe that every quotient category of a category $\mathscr{C}$ has essentially the same objects as $\mathscr{C}$; in particular, a quotient category does *not* result when different objects are identified by an equivalence relation.

4.7 EXAMPLES

(1) If $\mathscr{C}$ is a category and $\sim$ is defined by: $f \sim g$ if and only if $dom(f) = dom(g)$ and $cod(f) = cod(g)$, then $\mathscr{C}/\sim$ is a quasi-ordered class (in the sense of Example 3.5(6)).

(2) Let $\mathscr{C}$ = **Top** and for all $A, B \in Ob(\mathscr{C})$ and $f, g \in hom_{\mathscr{C}}(A, B)$ let $f \sim g$ if and only if f is homotopic to g. Then $\mathscr{C}/\sim$ is called the **homotopy category of topological spaces** and is denoted by **hTop**.

(3) If in the above example $\mathscr{C}$ = **pTop**, and if $f \sim g$ means that there is a base-point-preserving homotopy between f and g, then $\mathscr{C}/\sim$ is called the **homotopy category of topological spaces with base point.**

(4) Let $\mathscr{C}$ = **Grp** and for all $A, B \in Ob(\mathscr{C})$ and $f, g \in hom(A, B)$ let $f \sim g$ if and only if there is some $b \in B$ such that $f(a) = bg(a)b^{-1}$ for all $a \in A$. Then $\mathscr{C}/\sim$ is called the **category of groups and conjugacy classes of homomorphisms.**

Recall that we broadened the concept of categories from concrete ones to abstract ones mainly because some natural constructions applied to concrete (resp. concretizable) categories yield "categories" that are not necessarily concretizable. This is true for quotients. For example, Freyd has shown that even though **Top** is concretizable, the quotient category **hTop** is not. Even more surprising is the fact (due to Kučera) that *every* (abstract) category is actually (isomorphic to) the quotient category of some suitable concretizable category.

Products of Categories

It happens occasionally that one wishes to consider pairs of objects and pairs of morphisms from two given categories as the objects and morphisms of a new category. Below it is seen that such a category can indeed be obtained.

4.8 DEFINITION

If $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ are categories, then the product of the morphism classes

$$Mor\,\mathscr{C}_1 \times Mor\,\mathscr{C}_2 \times \cdots \times Mor\,\mathscr{C}_n$$

together with the composition operation defined by:

$$(f_1, f_2, \ldots, f_n) \circ (g_1, g_2, \ldots, g_n) = (f_1 \circ g_1, f_2 \circ g_2, \ldots, f_n \circ g_n)$$

(i.e., whenever the right side is defined (and only then), the left side is defined and the two sides are equal) is called the **product category of $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$** and is denoted by

$$\mathscr{C}_1 \times \mathscr{C}_2 \times \cdots \times \mathscr{C}_n.$$

4.9 PROPOSITION

Every product category of categories is a category (in the sense of Definition 3.8). □

Sums of Categories

4.10 DEFINITION

If $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ are categories, then the disjoint union† of the morphism classes

$$Mor\,\mathscr{C}_1 \uplus Mor\,\mathscr{C}_2 \uplus \cdots \uplus Mor\,\mathscr{C}_n$$

together with the composition operation defined by:

$$(f, i) \circ (g, j) = (f \circ g, i) \quad \text{if and only if} \quad i = j$$

is called the **sum category of** $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ and is denoted by

$$\mathscr{C}_1 \amalg \mathscr{C}_2 \amalg \cdots \amalg \mathscr{C}_n.$$

4.11 PROPOSITION

Every sum category of categories is a category. □

Opposite Categories

4.12 DEFINITION

For any category $\mathscr{C} = (\mathscr{O}, \mathscr{M}, dom, cod, \circ)$, the **opposite (or dual) category of** $\mathscr{C}$ is the category $\mathscr{C}^{op} = (\mathscr{O}, \mathscr{M}, cod, dom, *)$, where $*$ is defined by $f * g = g \circ f$. (Thus, $\mathscr{C}$ and $\mathscr{C}^{op}$ have the same objects and morphisms, but the domain and codomain functions are switched and the composition laws are the "opposites" of each other.)

4.13 PROPOSITION

The opposite category of any category is a category. □

4.14 PROPOSITION

For any category $\mathscr{C}$, $(\mathscr{C}^{op})^{op} = \mathscr{C}$. □

4.15 THE DUALITY PRINCIPLE

The last proposition allows one to define for any "categorical concept" an opposite or dual concept and for any "categorical statement" a dual statement. We will not bother to explicitly define what categorical concepts and statements are. However, we will give an idea of dual concepts by means of examples. If P is a property concerning morphisms and objects of a category $\mathscr{C}$, the dual property P^{op} is, roughly speaking, the corresponding property of $\mathscr{C}^{op}$ phrased

† The **disjoint union**

$$A_1 \uplus A_2 \uplus \cdots \uplus A_n$$

of a family $(A_1, A_2, \ldots, A_n)$ of classes is the class

$$(A_1 \times \{1\}) \cup (A_2 \times \{2\}) \cup \cdots \cup (A_n \times \{n\}).$$

as a property of $\mathscr{C}$; in other words, the property obtained from P by reversing all arrows.

Take for example the following property $P(X)$ of an object X in $\mathscr{C}$:

For any object Y of $\mathscr{C}$ there exists exactly one $\mathscr{C}$-morphism $f: Y \to X$.

The corresponding property for $\mathscr{C}^{op}$ would be:

For any object Y of $\mathscr{C}^{op}$ there exists exactly one $\mathscr{C}^{op}$-morphism $f: Y \to X$.

Translating this into a property for $\mathscr{C}$, we obtain $P^{op}(X)$:

For any object Y of $\mathscr{C}$ there exists exactly one $\mathscr{C}$-morphism $f: X \to Y$.

In the category **Set**, for example, the above property $P(X)$ holds if and only if X is a singleton set and $P^{op}(X)$ holds if and only if X is the empty set. Quite often, the dual concept P^{op} of a concept P is denoted by "co-P" (cf. constants and coconstants (§8), separators and coseparators (§12), equalizers and coequalizers (§16), and products and coproducts (§18)).

A concept P is called **self-dual** if $P = P^{op}$. What makes duality so interesting and important is the fact that one can "dualize" not only concepts, but also statements. If S is a statement concerning the morphisms and objects of a category, then by definition, the dual statement S^{op} holds in $\mathscr{C}$ if and only if S holds in $\mathscr{C}^{op}$. This, together with the fact that $\mathscr{C} = (\mathscr{C}^{op})^{op}$, immediately implies the so-called "**duality principle for categories**":

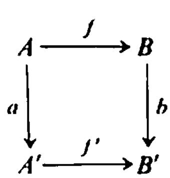
If S is a categorical statement which holds for all categories, then S^{op} also holds for all categories.

We will have numerous occasions to use this principle.

Arrow Categories and Triangle Categories

4.16 DEFINITION

If $\mathscr{C}$ is any category, then the **arrow category for** $\mathscr{C}$ (denoted by $\mathscr{C}^2$) is the category whose class of objects is precisely the class of morphisms of $\mathscr{C}$ and for which a $\mathscr{C}^2$-morphism from $A \xrightarrow{f} B$ to $A' \xrightarrow{f'} B'$ is a pair (a, b) where $A \xrightarrow{a} A'$, $B \xrightarrow{b} B'$ are $\mathscr{C}$-morphisms such that the square



commutes.

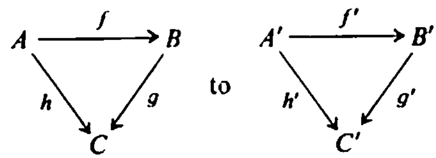
Composition in $\mathscr{C}^2$ is defined by:

$$(\hat{a}, \hat{b}) \circ (a, b) = (\hat{a} \circ a, \hat{b} \circ b);$$

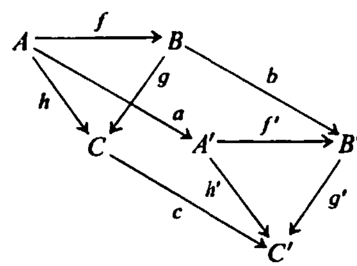
i.e., by pasting the squares together and forgetting the middle arrow.

4.17 DEFINITION

If $\mathscr{C}$ is any category, then the **triangle category for** $\mathscr{C}$ (denoted by $\mathscr{C}^3$) is the category whose class of objects is precisely the class of commutative triangles of $\mathscr{C}$ and for which a $\mathscr{C}^3$-morphism from



is an ordered triple (a, b, c), where $A \xrightarrow{a} A'$, $B \xrightarrow{b} B'$, and $C \xrightarrow{c} C'$ are $\mathscr{C}$-morphisms such that each square in the diagram



commutes.

Composition in $\mathscr{C}^3$ is defined by:

$$(\hat{a}, \hat{b}, \hat{c}) \circ (a, b, c) = (\hat{a} \circ a, \hat{b} \circ b, \hat{c} \circ c);$$

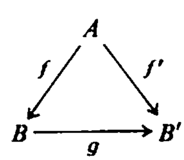
i.e., by pasting prisms together and forgetting the middle face.

Arrow categories and triangle categories are both special cases of the important and useful concept of "functor categories", which will be defined in §15.

Comma Categories

4.18 DEFINITION

If $\mathscr{C}$ is any category and $A \in Ob(\mathscr{C})$, then the **comma category of** A **over** $\mathscr{C}$ is the category $(A, \mathscr{C})$ whose objects are those $\mathscr{C}$-morphisms that have domain A, and whose morphisms from $A \xrightarrow{f} B$ to $A \xrightarrow{f'} B'$ are those $\mathscr{C}$-morphisms $g: B \to B'$ for which the triangle

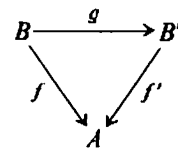


commutes.

Composition in $(A, \mathscr{C})$ is defined according to the composition in $\mathscr{C}$.

4.19 DEFINITION

If $\mathscr{C}$ is any category and $A \in Ob(\mathscr{C})$, then the **comma category of $\mathscr{C}$ over A** is the category $(\mathscr{C}, A)$ whose objects are those $\mathscr{C}$-morphisms which have codomain A, and whose morphisms from $B \xrightarrow{f} A$ to $B' \xrightarrow{f'} A$ are those $\mathscr{C}$-morphisms $g: B \to B'$ for which the triangle



commutes.

Composition in $(\mathscr{C}, A)$ is defined according to the composition in $\mathscr{C}$.

EXERCISES

4A. Prove that if $\mathscr{C}$ is a category with exactly one object (i.e., if $\mathscr{C}$ is a monoid (Example 3.5(7))), then $\mathscr{B}$ is a subcategory of $\mathscr{C}$ if and only if $\mathscr{B}$ is a submonoid of $\mathscr{C}$ or $\mathscr{B}$ is the empty category.

4B. Let $\mathscr{C}$ be a category with exactly one object A and morphism set $hom_{\mathscr{C}}(A, A) = \{a, b\}$, where composition is defined by: $a \circ a = a$; $a \circ b = b \circ a = b \circ b = b$. Let $\mathscr{B}$ also be a category with exactly one object A and morphism set $hom_{\mathscr{B}}(A, A) = \{b\}$, where composition is defined by: $b \circ b = b$. Determine whether or not $\mathscr{B}$ is a subcategory of $\mathscr{C}$.

4C. Show that in the definition of subcategory, the condition that every identity of $\mathscr{B}$ is an identity of $\mathscr{C}$ (4.1(4)) can be replaced by:

"For any $\mathscr{B}$-object B, the identity in $\mathscr{C}$ associated with B is a $\mathscr{B}$-morphism."

4D. Prove that:

(a) if $\mathscr{B}$ is a subcategory of $\mathscr{C}$, and $\mathscr{C}$ is a subcategory of $\mathscr{B}$, then $\mathscr{B} = \mathscr{C}$.

(b) if $\mathscr{B}$ is a (full) subcategory of $\mathscr{C}$, and $\mathscr{C}$ is a (full) subcategory of $\mathscr{D}$, then $\mathscr{B}$ is a (full) subcategory of $\mathscr{D}$.

4E. Show that if $\mathscr{A}$ is a quotient category of $\mathscr{B}$ and $\mathscr{B}$ is a quotient category of $\mathscr{C}$, then $\mathscr{A}$ can be regarded as a quotient category of $\mathscr{C}$.

4F. Show that if $\sim$ is a congruence on $\mathscr{B}$ and $\mathscr{A}$ is a full subcategory of $\mathscr{B}$, then $\sim$ induces a congruence $\sim_{\mathscr{A}}$ on $\mathscr{A}$ such that $\mathscr{A}/\sim_{\mathscr{A}}$ is a subcategory of $\mathscr{B}/\sim$.

4G. Prove that every category has a quotient category which is a quasi-ordered class (in the sense of 3.5(6)).

4H. Let $(\mathscr{C}_i)_{i \in I}$ be a set-indexed family of small categories. Prove that the product of the morphism classes $\Pi(Mor\, \mathscr{C}_i)_{i \in I}$ together with the composition operation defined by:

$$\pi_i(F \circ G) = (\pi_i F) \circ (\pi_i G)$$

is a category in the sense of Definition 3.8.

4I. For any categories $\mathscr{C}_1$ and $\mathscr{C}_2$, show that

$$Ob(\mathscr{C}_1 \times \mathscr{C}_2) = Ob(\mathscr{C}_1) \times Ob(\mathscr{C}_2)$$

and that if $A_1, B_1 \in Ob(\mathscr{C}_1)$ and $A_2, B_2 \in Ob(\mathscr{C}_2)$, then

$$hom_{\mathscr{C}_1 \times \mathscr{C}_2}[(A_1, A_2), (B_1, B_2)] = hom_{\mathscr{C}_1}(A_1, B_1) \times hom_{\mathscr{C}_2}(A_2, B_2).$$

4J. Show that if $\mathscr{A}_i$ is a (full) subcategory of $\mathscr{C}_i$ for each $i = 1, 2, \dots, n$, then

$$\mathscr{A}_1 \times \mathscr{A}_2 \times \cdots \times \mathscr{A}_n$$

is a (full) subcategory of $\mathscr{C}_1 \times \mathscr{C}_2 \times \cdots \times \mathscr{C}_n$.

4K. Prove that if $\mathscr{A}, \mathscr{B}, \mathscr{A}'$, and $\mathscr{B}'$ are non-empty categories, then the product categories $\mathscr{A} \times \mathscr{B}$ and $\mathscr{A}' \times \mathscr{B}'$ are equal if and only if $\mathscr{A} = \mathscr{A}'$ and $\mathscr{B} = \mathscr{B}'$.

4L. Let $(\mathscr{C}_i)_{i \in I}$ be a family of categories. Prove that the disjoint union of the morphism classes

$$\coprod_{i \in I} (Mor\, \mathscr{C}_i),$$

together with the same composition operaiion as that given in 4.10, is a category in the sense of Definition 3.8.

4M. Show that if $\mathscr{A}_i$ is a (full) subcategory of $\mathscr{C}_i$ for each $i = 1, 2, \dots, n$, then

$$\mathscr{A}_1 \amalg \mathscr{A}_2 \amalg \cdots \amalg \mathscr{A}_n$$

is a (full) subcategory of

$$\mathscr{C}_1 \amalg \mathscr{C}_2 \amalg \cdots \amalg \mathscr{C}_n.$$

4N. Show that for each category $\mathscr{C}$, $hom_{\mathscr{C}}(A, B) = hom_{\mathscr{C}^{op}}(B, A)$.

4O. Form the duals of the following statements and determine which of them are self-dual.

(a) $\mathscr{C}$ is connected.
(b) $\mathscr{C}$ is a quasi-ordered class (in the sense of 3.5(6)).
(c) $\mathscr{C}$ is a partially-ordered class.
(d) $\mathscr{C}$ is a totally-ordered class.
(e) $\mathscr{C}$ is a monoid (in the sense of 3.5(7)).
(f) $\mathscr{C}$ is a group.
(g) $f \in Mor\, \mathscr{C}$ and there is some $g \in Mor\, \mathscr{C}$ such that $g \circ f$ is an identity.
(h) $f \in Mor\, \mathscr{C}$ and for all $g, h \in Mor\, \mathscr{C}$ such that $f \circ g$ and $f \circ h$ are defined, $f \circ g = f \circ h$.

4P. Show that each condition of the second definition of category (3.8) is self-dual.

4Q. Establish the following consequence of the duality principle:

If S is a categorical statement, then S holds for all categories satisfying property P if and only if S^{op} holds for all categories satisfying P^{op}.

4R. Show that every category $\mathscr{C}$ can be considered to be a full subcategory of $\mathscr{C}^2$, $\mathscr{C}^2$ can be considered to be a full subcategory of $\mathscr{C}^3$, and for each $A \in Ob(\mathscr{C})$, $(A, \mathscr{C})$ and $(\mathscr{C}, A)$ can be considered to be subcategories of $\mathscr{C}^2$.

4S. Show that the following pairs of categories are "essentially the same". [This sort of "essential sameness" will be defined more precisely later (cf. 14.1)].

(a) **TopBun** and the arrow category $\mathbf{Top}^2$;

(b) $\mathbf{TopBun}_B$ and the comma category (**Top**, B) (for any topological space B);

(c) **pTop** and (P, **Top**) (for any singleton space P);

(d) **pSet** and (P, **Set**) (for any singleton set P);

(e) The category of bi-pointed sets and (A, **Set**) (for any set A consisting of exactly two elements).