XI

Pointed Categories

Virtually all algebraic notions in category theory are parodies of their parents in the most "classical" of categories . . . the category of left *A*-modules.

——H. BASS†

The categories R-**Mod** (for various rings R) are without doubt the most thoroughly investigated categories. Because of their nice properties, they provide a useful tool for the study of other categories as well. For example, algebraic topology is essentially the study of topology by means of functors (homology, cohomology, higher homotopy) from topological categories into categories of R-modules. The reason that the categories R-**Mod** are so nice is surprisingly simple. Besides the fact that they are (considered as concrete categories) finitary algebraic (as are many other more complicated categories such as **SGrp** and **Mon**), they are distinguished by the fact that they have finite "biproducts"; i.e., finite products and finite coproducts whose corresponding object parts coincide, and that they are (normal epi, normal mono) categories. (This latter condition allows one to define the extremely useful concept of "exact sequences".) In addition, for the categories R-**Mod**, each morphism set $hom(A, B)$ can be uniquely supplied with the structure of an abelian group in such a way that morphism composition acts distributively on the left and on the right with respect to the group addition. As we will see, the categorical properties mentioned above are not independent. For example, the existence of a group structure on the morphism sets is closely related to the existence of finite biproducts. The relationship among the above properties will be studied in this chapter. Categories of R-modules will be characterized and those categories that behave "locally" like categories of R-modules (called abelian categories) will be introduced.

† From *The Morita Theorems*.

As the title of this chapter indicates we will assume throughout it that each category is non-empty and pointed. The zero morphisms will usually be denoted by 0 (as will the zero objects, when they occur).

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§39 NORMAL AND EXACT CATEGORIES

In this section we will investigate the question of when a pointed category has especially nice factorization properties. This will be seen to occur precisely when the category is "exact" (Theorem 39.17).

Exact Categories

We begin by restating some results which have essentially already been established (see 16B, 16L, and 27R).

39.1 PROPOSITION

If $f: A \to B$ is a monomorphism and $k: K \to A$ is any morphism, then the following are equivalent:

(1) $(K, k) \approx Ker(f)$.

(2) *K is a zero object*, 0. □

39.2 COROLLARY

Each category that has kernels or cokernels also has a zero object, 0. □

39.3 PROPOSITION

Let $f: A \to B$ be an $\mathscr{A}$-morphism and $m: B \to C$ be a monomorphism in $\mathscr{A}$. Then:

(1) $(K, k) \approx Ker(f)$ *if and only if* $(K, k) \approx Ker(m \circ f)$.

(2) $Ker(Ker(f)) = 0$. □

39.4 PROPOSITION

Let A be an object of the category $\mathscr{A}$ that has kernels and cokernels. Let $\mathscr{S}$ (resp. $\mathscr{Q}$) be the quasi-ordered class of all subobjects (resp. quotient objects) of A. Let $G: \mathscr{S} \to \mathscr{Q}$ be the map that sends each subobject (S, m) of A to $Cok(m)$, and let $F: \mathscr{Q} \to \mathscr{S}$ be the map which sends each quotient object (q, Q) of A to $Ker(q)$. Then $(\mathscr{S}, \mathscr{Q}, G, F)$ is a Galois correspondence (see 27Q). □

39.5 COROLLARY

If $\mathscr{A}$ has kernels and cokernels, then

(1) *For each $\mathscr{A}$-morphism f, $Ker(f) \approx Ker(Cok(Ker(f)))$ and $Cok(f) \approx Cok(Ker(Cok(f)))$.*

(2) *An $\mathscr{A}$-morphism f is a normal monomorphism if and only if $f \approx Ker(Cok(f))$.*

(3) *An $\mathscr{A}$-morphism f is a normal epimorphism if and only if $f \approx Cok(Ker(f))$.*

(4) *For each $\mathscr{A}$-object A, the quasi-ordered class of all normal subobjects of A is anti-isomorphic with the quasi-ordered class of all normal quotient objects of A.* □

39.6 DEFINITION

A category is called:

(1) **normal** provided that it has kernels and cokernels, is (epi, mono)-factorizable, and each of its monomorphisms is a normal monomorphism.

(2) **conormal** provided that it has kernels and cokernels, is (epi, mono)-factorizable, and each of its epimorphisms is a normal epimorphism.

(3) **exact** provided that it is both normal and conormal.

Note that since for categories normality and conormality are notions which are duals of each other, exactness is a self-dual concept.

It should be pointed out that a category with kernels and cokernels that has the property that each monomorphism is a normal monomorphism and each epimorphism is a normal epimorphism need not be exact (see Exercise 39B). However, in case that the category also has equalizers or coequalizers, it must be exact (see Proposition 39.19).

39.7 EXAMPLES

In the following table "+" means that the category has the property in question and "−" means that it does not have it.

Category	Normal	Conormal
(1) R-**Mod**	+	+
(2) **Grp**	−	+
(3) **pSet**	+	−
(4) **Mon**	−	−

(5) The full subcategory of **pTop** consisting of all pointed compact Hausdorff spaces is normal but not conormal.

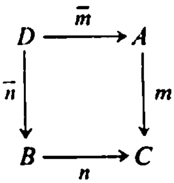
(6) The full subcategory of **Ab** consisting of all abelian groups whose underlying set has at most 27 elements is exact.

Notice that each of the categories (1), (2), (3), and (5) above has the property that in it each monomorphism is a regular monomorphism and each epimorphism is a regular epimorphism.

39.8 PROPOSITION

If $\mathscr{A}$ has kernels, then each pair of normal subobjects of any $\mathscr{A}$-object has an intersection.

Proof: If (A, m) and (B, n) are normal subobjects of C, then $(A, m) \approx Ker(f)$ for some morphism f. Let $(D, \bar{n}) \approx Ker(f \circ n)$. Then there exists a unique morphism $\bar{m}: D \to A$ such that the square



commutes. The above square is a pullback square since if r and s are morphisms such that $m \circ r = n \circ s$, then $f \circ n \circ s = f \circ m \circ r = 0$, so that there is a unique morphism k such that $s = \bar{n} \circ k$. Thus $(D, n \circ \bar{n})$ is an intersection of (A, m) and (B, n). □

39.9 PROPOSITION

Every normal category has a zero object and has finite intersections.

Proof: Immediate from Corollary 39.2 and the above proposition (39.8). □

39.10 PROPOSITION

If $\mathscr{A}$ is exact, then for each $\mathscr{A}$-object A the quasi-ordered classes of all subobjects and of all quotient objects of A are (up to equivalence) anti-isomorphic lattices (possibly on a class) with smallest and largest members.

Proof: By Proposition 39.8, the class $\mathscr{S}$ of all subobjects of A has finite infima (= intersections), and by its dual, the class of all quotient objects of A, has finite infima (= cointersections). Thus since $\mathscr{S}$ and $\mathscr{Q}$ are anti-isomorphic as quasi-ordered classes (39.5(4)), $\mathscr{Q}$ and $\mathscr{S}$ each must have finite suprema. □

39.11 COROLLARY

(1) *An exact category is well-powered if and only if it is co-(well-powered).*

(2) *Every concretizable exact category is well-powered and co-(well-powered).*

Proof: Each concretizable category is regular co-(well-powered) (16N). □

Next we wish to demonstrate that exact categories are distinguished by especially nice factorization properties. This will later enable us to define exact sequences. For this purpose, consider first the category **Grp** that is conormal but not normal. Here each morphism has a unique (normal epi, mono)-factorization, $f = m \circ g$. Moreover g can be chosen as $Cok(Ker(f))$. Also the congruence relation of f is completely determined by $Ker(f)$, and the question of whether or not f is a monomorphism can be decided by knowing only $Ker(f)$. All of these facts are consequences of the fact that **Grp** is conormal. Indeed, it can be shown that under certain conditions on a category $\mathscr{A}$, these are all equivalent to each other and to the fact that $\mathscr{A}$ is conormal (see Theorem 39.13

and Exercise 39C). In fact, the exact categories will be shown to be precisely the (normal epi, normal mono) categories (see Theorem 39.17).

39.12 LEMMA

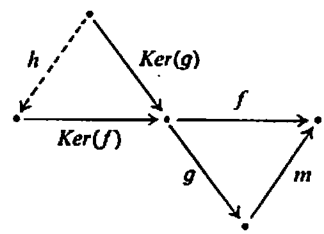
If $\mathcal{A}$ has kernels and cokernels, f, g, and m are $\mathcal{A}$-morphisms such that $f = m \circ g$, and g is a normal epimorphism, then there exists a unique $\mathcal{A}$-morphism $\bar{g}$ such that $Cok(Ker(f)) \approx \bar{g} \circ g$.

Proof: Since

$$f \circ Ker(g) = m \circ g \circ Ker(g) = 0,$$

there exists a unique $\mathcal{A}$-morphism h such that

$$Ker(g) = Ker(f) \circ h.$$



Hence

$$Cok(Ker(f)) \circ Ker(g) = Cok(Ker(f)) \circ Ker(f) \circ h = 0.$$

Since $g \approx Cok(Ker(g))$ (39.5(3)), this implies the existence of a unique morphism $\bar{g}$ with $Cok(Ker(f)) = \bar{g} \circ g$. □

39.13 THEOREM (CHARACTERIZATION OF CONORMAL CATEGORIES)

If $\mathcal{A}$ has kernels and cokernels and each $\mathcal{A}$-epimorphism is a normal epimorphism, then the following are equivalent:

(1) *$\mathcal{A}$ is conormal.*

(2) *If f is an $\mathcal{A}$-morphism such that $Ker(f) = 0$, then f is a monomorphism.*

(3) *If f is an $\mathcal{A}$-morphism such that $f = m \circ Cok(Ker(f))$, then m is a monomorphism.*

(4) *$\mathcal{A}$ is a (normal epi, mono) category.*

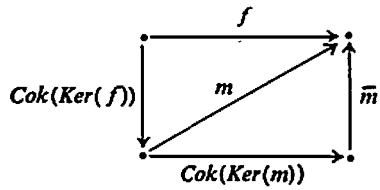
(5) *For each $\mathcal{A}$-morphism f, if either a congruence relation of f or a congruence relation of $Cok(Ker(f))$ exists, then they both exist and coincide.*

Proof: We will show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1) and (3) ⇒ (5) ⇒ (2).
(1) ⇒ *(2).* Suppose that $Ker(f) = 0$. Let $f = m \circ e$ be an (epi, mono)-factorization of f. Then according to the preceding lemma (39.12), there exists a morphism $\bar{g}$ such that

$$\bar{g} \circ e \approx Cok(Ker(f)) = Cok(0) = 1.$$

Hence e is a section, so that $f = m \circ e$ is a monomorphism.

(2) ⇒ *(3)*. Suppose that $f = m \circ Cok(Ker(f))$. Then let $\bar{m}$ be the unique morphism such that $m = \bar{m} \circ Cok(Ker(m))$.



Now $g = Cok(Ker(m)) \circ Cok(Ker(f))$ is an epimorphism and so (by hypothesis) a normal epimorphism. Thus according to the lemma (39.12), there exists a unique morphism $\bar{g}$ such that

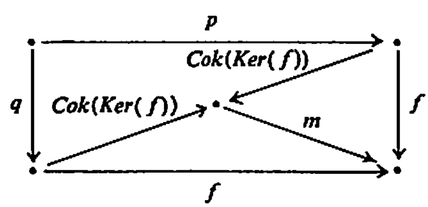
$$Cok(Ker(f)) = \bar{g} \circ g = \bar{g} \circ Cok(Ker(m)) \circ Cok(Ker(f)).$$

Since $Cok(Ker(f))$ is an epimorphism, this implies that $1 = \bar{g} \circ Cok(Ker(m))$. Consequently, $Cok(Ker(m))$ is a section and, hence, an isomorphism. Thus $Ker(m) = 0$ which by (2) implies that m is a monomorphism.

(3) ⇒ *(4)*. Clearly by the definition of cokernel each morphism f has a factorization $f = m \circ Cok(Ker(f))$. By (3) m must be a monomorphism. Hence $\mathscr{A}$ is (regular epi, mono)-factorizable, which implies it is a (regular epi, mono) category (33.4).

(4) ⇒ *(1)*. Immediate from the definition of a conormal category.

(3) ⇒ *(5)*. Let $f = m \circ Cok(Ker(f))$ be the factorization of f induced by the definition of cokernels. If (p, q) is a congruence relation of f, then the diagram



commutes, so that the "inner square" is a pullback square (21.10(1)) and hence (p, q) is a congruence relation for $Cok(Ker(f))$. Conversely if (p, q) is a congruence relation for $Cok(Ker(f))$, then the above square commutes. By (3) m is a monomorphism. Thus the "outer square" is a pullback square (21.10(2)), so that (p, q) is a congruence relation of f.

(5) ⇒ *(2)*. If $Ker(f) = 0$, then $Cok(Ker(f)) = 1$. Hence $(1, 1)$ is a congruence relation of $Cok(Ker(f))$, so that by (5) it is a congruence relation of f. Consequently f is a monomorphism (21.17). □

39.14 COROLLARY

If $\mathscr{A}$ has kernels and cokernels, then the following conditions are equivalent:

(1) *$\mathscr{A}$ is conormal.*

(2) *$\mathscr{A}$ is a balanced (normal epi, mono) category.* □

39.15 THEOREM

If $\mathcal{A}$ is exact, then

(1) *$\mathcal{A}$ is an (epi, mono) category, and for each $\mathcal{A}$-morphism f,*

$$f = Ker(Cok(f)) \circ Cok(Ker(f))$$

is the unique (epi, mono)-factorization of f. Consequently,

$$Im(f) \approx Ker(Cok(f))$$

and

$$Coim(f) \approx Cok(Ker(f)).$$

(2) *For each $\mathcal{A}$-morphism f,*

$$f \text{ is a monomorphism} \Leftrightarrow Ker(f) = 0 \Leftrightarrow f \approx Im(f) \Leftrightarrow Coim(f) = 1.$$

(3) *For each $\mathcal{A}$-morphism f,*

$$f \text{ is an epimorphism} \Leftrightarrow Cok(f) = 0 \Leftrightarrow f \approx Coim(f) \Leftrightarrow Im(f) = 1.$$

(4) *For each $\mathcal{A}$-morphism f,*

$$f \text{ is an isomorphism} \Leftrightarrow Ker(f) = Cok(f) = 0.$$

Proof: Immediate from the above characterization theorem (39.13), its dual, and the definition of exactness. □

39.16 LEMMA

If m is a normal monomorphism and $g \approx Cok(m)$, then $m \approx Ker(g)$.

Proof: Since m is normal, $m \approx Ker(f)$ for some morphism f. Since $f \circ m = 0$ and $g \approx Cok(m)$, there exists some morphism $\bar{f}$ such that $f = \bar{f} \circ g$. Now if r is a morphism such that $g \circ r = 0$, then

$$f \circ r = \bar{f} \circ g \circ r = 0$$

so that there exists a unique morphism h such that $r = m \circ h$. Thus

$$m \approx Ker(g).$$

□

39.17 THEOREM (CHARACTERIZATION OF EXACT CATEGORIES)

For any category, $\mathcal{A}$, the following are equivalent:

(1) *$\mathcal{A}$ is exact.*

(2) *$\mathcal{A}$ is (normal epi, normal mono)-factorizable.*

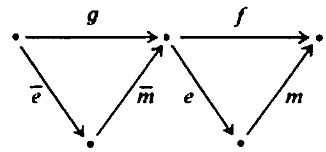
(3) *$\mathcal{A}$ is a (normal epi, normal mono) category.*

Proof:

(1) ⇒ *(2)*. Immediate from the above theorem (39.15).

(2) ⇒ *(3)*. Immediate from the fact that every (regular epi, mono)-factorizable category is a (regular epi, mono) category (33.4).

(3) ⇒ (1). Since (3) is self-dual, we need only show that $\mathscr{A}$ has kernels and each $\mathscr{A}$-monomorphism is normal. Let f be an $\mathscr{A}$-morphism and let $f = m \circ e$ be its (normal epi, normal mono)-factorization. Then $e \approx Cok(g)$ for some morphism g. Let $g = \bar{m} \circ \bar{e}$ be the (normal epi, normal mono)-factorization of g. We wish to show that $\bar{m} \approx Ker(f)$.



Since $\bar{e}$ is an epimorphism, it follows that

$$e \approx Cok(g) \approx Cok(\bar{m} \circ \bar{e}) \approx Cok(\bar{m}) \quad (39.3(1) \text{ dual}).$$

Since $\bar{m}$ is a normal monomorphism, by the lemma

$$\bar{m} \approx Ker(Cok(\bar{m})) = Ker(e) = Ker(m \circ e) = Ker(f) \quad (39.3(1)).$$

Consequently $\mathscr{A}$ has kernels.

If h is any $\mathscr{A}$-monomorphism and $h = \hat{m} \circ \hat{e}$ is its (normal epi, normal mono)-factorization, then $\hat{e}$ is a monomorphism and a normal epimorphism; hence an isomorphism. Consequently, h is a normal monomorphism. □

39.18 PROPOSITION

If $\mathscr{A}$ has kernels and coequalizers and if each $\mathscr{A}$-epimorphism is a normal epimorphism, then $\mathscr{A}$ is a conormal category.

Proof: We will establish condition (2) of the above characterization for conormality (39.13). Suppose that $Ker(f) = 0$ and let (r, s) be a pair of morphisms for which $f \circ r = f \circ s$. Now let $(g, C) \approx Coeq(r, s)$. Thus there exists a morphism h such that $f = h \circ g$. Consequently, Lemma 39.12 implies that there exists a unique morphism $\bar{g}$ such that $\bar{g} \circ g \approx Cok(Ker(f)) = Cok(0) = 1$. Hence g is a section and an epimorphism; thus an isomorphism, so that $r = s$. Therefore f is a monomorphism. □

39.19 PROPOSITION

Suppose that $\mathscr{A}$ has equalizers and coequalizers, each $\mathscr{A}$-monomorphism is a normal monomorphism, and each $\mathscr{A}$-epimorphism is a normal epimorphism. Then $\mathscr{A}$ is exact.

Proof: Immediate from the above proposition (39.18) and its dual. □

Exact Sequences

39.20 DEFINITION

Let $\mathscr{A}$ be exact. A sequence $(f_n)_{n \in I}$ of $\mathscr{A}$-morphisms indexed by a (finite or infinite) interval of integers is said to be an **exact sequence** provided that for each n, $n+1 \in I$

(1) $cod(f_n) = dom(f_{n+1})$, *and*

(2) $Im(f_n) \approx Ker(f_{n+1})$.

39.21 PROPOSITION

If $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ *are morphisms in an exact category, then the following are equivalent:*

(1) (f, g) *is an exact sequence; i.e.,* $Im(f) \approx Ker(g)$.

(2) $Cok(f) \approx Coim(g)$.

(3) $g \circ f = 0$ *and* $Cok(f) \circ Ker(g) = 0$.

Proof:

(*1*) ⇒ (*2*). By (1) and Theorem 39.15(1),

$$Ker(Cok(f)) \approx Im(f) \approx Ker(g).$$

Thus

$$Cok(f) \approx Cok(Ker(Cok(f))) \approx Cok(Ker(g)) \approx Coim(g) \quad (39.5(1)).$$

(*2*) ⇒ (*3*).

(i) $g \circ f = (Im(g) \circ Coim(g)) \circ f = Im(g) \circ (Cok(f) \circ f) = Im(g) \circ 0 = 0.$

(ii) $Cok(f) \circ Ker(g) = Coim(g) \circ Ker(g) = Cok(Ker(g)) \circ Ker(g) = 0.$

(*3*) ⇒ (*1*). Since $g \circ f = 0$, we have

$$g \circ (Im(f) \circ Coim(f)) = 0 = 0 \circ Coim(f).$$

Hence $g \circ Im(f) = 0$, so that by the definition of kernels there exists a morphism h such that $Im(f) = Ker(g) \circ h$. Hence $Im(f) \leq Ker(g)$. Similarly since $Cok(f) \circ Ker(g) = 0$, there exists a morphism k such that

$$Ker(g) = Ker(Cok(f)) \circ k = Im(f) \circ k \quad (39.15(1)).$$

Thus $Ker(g) \leq Im(f)$, so that $Ker(g) \approx Im(f)$. □

39.22 PROPOSITION

If f and g are morphisms in an exact category, then

(1) $0 \to \bullet \xrightarrow{f} \bullet$ *is exact if and only if f is a monomorphism.*

(2) $\bullet \xrightarrow{g} \bullet \to 0$ *is exact if and only if g is an epimorphism.*

(3) $0 \to \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ *is exact if and only if* $f \approx Ker(g)$.

(4) $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \to 0$ *is exact if and only if* $g \approx Cok(f)$.

(5) $0 \to \bullet \xrightarrow{f} \bullet \to 0$ *is exact if and only if f is an isomorphism.*

(6) *The following are equivalent:*

   (*i*) $0 \to \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \to 0$ *is exact.*

   (*ii*) *f is a monomorphism and* $g \approx Cok(f)$.

   (*iii*) *g is an epimorphism and* $f \approx Ker(g)$.

(7) *The following are equivalent:*

   (*i*) $\bullet \xrightarrow{f} \bullet \xrightarrow{1} \bullet$ *is exact.*

   (*ii*) $\bullet \xrightarrow{1} \bullet \xrightarrow{f} \bullet$ *is exact.*

   (*iii*) $f = 0$.

(8) $A \xrightarrow{1} A \xrightarrow{1} A$ *is exact if and only if* $A = 0$. □

Exact Functors

39.23 DEFINITION

A functor $F: \mathscr{A} \to \mathscr{B}$ between exact categories is called an **exact functor** provided that it preserves exact sequences; i.e., whenever $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ is exact, then $\bullet \xrightarrow{F(f)} \bullet \xrightarrow{F(g)} \bullet$ is exact.

39.24 PROPOSITION

Each exact functor preserves zero objects, zero morphisms, kernels, cokernels, epimorphisms, monomorphisms, images, coimages, and (epi, mono)-factorizations.

Proof: Immediate from Proposition 39.22. □

39.25 PROPOSITION

If F is a functor between exact categories, then the following are equivalent:

(1) *F is exact.*

(2) *F preserves exact sequences of the form* $0 \to \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \to 0$.

(3) *F preserves kernels and epimorphisms.*

(4) *F preserves cokernels and monomorphisms.*

(5) *F preserves kernels and images.*

Proof: Clearly (1) implies (2), and (5) implies (1). By Proposition 39.22(6), (2), (3), and (4) are equivalent. Thus we need only show that (2) implies (5).

If (2) holds, then by the equivalence of (2), (3), and (4), F preserves kernels and cokernels. Thus since for each morphism f, $Im(f) \approx Ker(Cok(f))$ (39.15(1)), F must preserve images also. □

EXERCISES

39A. Prove that each normal category is balanced.

39B. Let $\mathscr{A}$ be the full subcategory of the category **pTop** of pointed topological spaces, whose only objects are:

(1) a pointed one-element space, and

(2) a pointed three-element space in which the following sets are open: $\varnothing$, the entire set, the set consisting of the distinguished point, and the set consisting of the two non-distinguished points.

Prove that $\mathscr{A}$ has kernels and cokernels, that each monomorphism in $\mathscr{A}$ is normal, and that each epimorphism in $\mathscr{A}$ is normal, but that $\mathscr{A}$ is not exact.

39C. Suppose that $\mathscr{A}$ is a category that has kernels and cokernels and that in $\mathscr{A}$ the class of normal epimorphisms is closed under composition. Prove that if $\mathscr{A}$ either has equalizers or is (epi, mono)-factorizable, then the following are equivalent:

(a) Each regular epimorphism in $\mathscr{A}$ is normal.

(b) For each $\mathscr{A}$-morphism f, f is a monomorphism if and only if $Ker(f) = 0$.

(c) For each $\mathscr{A}$-morphism f, the unique morphism m with $f = m \circ Cok(Ker(f))$, is a monomorphism.
(d) $\mathscr{A}$ is (normal epi, mono)-factorizable.
(e) $\mathscr{A}$ is a (normal epi, mono) category.
(f) For each $\mathscr{A}$-morphism f, if either a congruence relation of f or a congruence relation of $Cok(Ker(f))$ exists, then they both exist and coincide.

39D. Prove that if

$$\begin{array}{ccc} C & \xrightarrow{q} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & D \end{array}$$

is a pullback square in a normal category, then f is a monomorphism if and only if p is a monomorphism.

39E. Prove that for any morphism f in an exact category, the sequence

$$0 \longrightarrow \bullet \xrightarrow{Ker(f)} \bullet \xrightarrow{f} \bullet \xrightarrow{Cok(f)} \bullet \longrightarrow 0 \text{ is exact.}$$

39F. Suppose that in an exact category

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is a diagram with exact rows. Prove that the following are equivalent:
(a) There is a morphism $A \to A'$ that makes the above diagram commute.
(b) There is a morphism $C \to C'$ that makes the above diagram commute.

39G. *Nine Lemma*
Suppose that in an exact category

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A' & & A & & A'' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

is a diagram that commutes and has exact rows and columns. Prove that there is an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

which, inserted as the top row of the above diagram, makes the diagram commute.

39H. *First Noether Isomorphism Theorem*
Suppose that

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow D \longrightarrow E \longrightarrow 0$$

are exact sequences in an exact category. Prove that there is an object Q and exact sequences

$$0 \longrightarrow A \longrightarrow D \longrightarrow Q \longrightarrow 0$$

and

$$0 \longrightarrow C \longrightarrow Q \longrightarrow E \longrightarrow 0$$

such that the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & Q & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

commutes.

39I. Prove that every faithful functor between exact categories reflects exact sequences.

39J. Show that if F is an exact functor between exact categories, then F is faithful if and only if it reflects exact sequences.

39K. Let $\mathscr{A}$ be a full subcategory of an exact category $\mathscr{B}$ with embedding functor $E: \mathscr{A} \hookrightarrow \mathscr{B}$. Prove that the following are equivalent:

(1) $\mathscr{A}$ and E are exact.
(2) $\mathscr{A}$ contains 0 and is closed under the formation of kernels and cokernels.
(3) $\mathscr{A}$ contains 0 and is closed under the formation of kernels and images.

39L. *Half-Exact Functors*
A functor F between exact categories is said to be:

(α) a left-exact functor provided that it preserves exact sequences of the form

$$0 \to \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot.$$

(β) a right-exact functor provided that it preserves exact sequences of the form

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \to 0.$$

(γ) a half-exact functor provided that whenever

$$0 \to \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \to 0$$

is exact, then

$$\cdot \xrightarrow{F(f)} \cdot \xrightarrow{F(g)} \cdot$$

is exact.

(a) Prove that every half-exact functor preserves zero objects and zero morphisms.
(b) Prove that a functor between exact categories is
 (i) left-exact if and only if it preserves kernels.
 (ii) right-exact if and only if it preserves cokernels.
 (iii) exact if and only if it is both left-exact and right-exact.
(c) Let A be an R-module. Prove that:
 (i) $Hom(A, \_)$: R-**Mod** $\to$ **Ab** is left-exact.
 (ii) $Hom(A, \_)$: R-**Mod** $\to$ **Ab** is exact if and only if A is projective.
 (iii) $A \otimes \_$: R-**Mod** $\to$ **Ab** is right-exact.

39M. Let $\mathscr{A}$ and $\mathscr{B}$ be exact categories. Determine whether or not
 (i) $[\mathscr{A}, \mathscr{B}]$ is exact.
 (ii) the full subcategory of $[\mathscr{A}, \mathscr{B}]$ consisting of all 0-preserving functors is exact.
 (iii) the full subcategory of $[\mathscr{A}, \mathscr{B}]$ consisting of all exact functors is exact.

§40 ADDITIVE CATEGORIES

As has been mentioned before, in any category of (left or right) R-modules, the morphism sets $hom(A, B)$ can be supplied with the structure of an abelian group in such a way that morphism composition acts distributively from the left and from the right. In addition, in these categories finite products (= direct products) coincide with finite coproducts (= direct sums). In this section we will see how these two seemingly unrelated properties are linked. In particular it will be shown that a category $\mathscr{A}$ having finite products has biproducts if and only if there is a (unique) semiadditive structure on $\mathscr{A}$.

Recall that throughout this chapter, all categories are assumed to be pointed.

Biproducts and Semiadditive Structures

40.1 DEFINITION

(1) An **additive structure** [resp. **semiadditive structure**] on a category $\mathscr{A}$ is a function $+$ that associates with each pair (f, g) of $\mathscr{A}$-morphisms with common domain A and common codomain B, an $\mathscr{A}$-morphism $f + g$ with domain A and codomain B such that the following conditions (A1), (A2), and (A3) [resp. (A1′), (A2), and (A3)] are satisfied:

(A1) For each pair (A, B) of $\mathscr{A}$-objects, $+$ induces on $hom(A, B)$ the structure of an abelian group.

(A1′) For each pair (A, B) of $\mathscr{A}$-objects, $+$ induces on $hom(A, B)$ the structure of a commutative monoid.

(A2) Composition is left and right distributive over $+$; i.e., whenever

$$A \xrightarrow{k} B \underset{h}{\overset{g}{\rightrightarrows}} C \xrightarrow{f} D$$

are $\mathscr{A}$-morphisms, it follows that

$$f \circ (g + h) = (f \circ g) + (f \circ h)$$

and

$$(g + h) \circ k = (g \circ k) + (h \circ k).$$

(A3) The zero morphisms of $\mathscr{A}$ act as monoid identities with respect to $+$; i.e., for each $\mathscr{A}$-morphism f,

$$0 + f = f + 0 = f.$$

(2) If $+$ is an additive structure (resp. semiadditive structure) on a category $\mathscr{A}$, then we call $(\mathscr{A}, +)$ [and by an abuse of notation also $\mathscr{A}$] an **additive category** (resp. **semiadditive category**).

Concerning the above definitions, it should be noted that if $\mathscr{A}$ has finite products or finite coproducts and $(\mathscr{A}, +)$ is semiadditive, then $+$ is completely determined by $\mathscr{A}$ (40.13). This tends to justify the above notational abuse.

It is also worth mentioning that (A3) above follows from (A1) and (A2) but does not follow from (A1′) and (A2) (see 40A).

40.2 EXAMPLES

(1) R-**Mod** is additive (for every ring R).

(2) **Grp** is not additive.

(3) If R is any ring,† then R can be regarded as an additive category with exactly one object. Conversely, each additive category having only one object can be regarded as a ring (cf. 3.5(7)).

(4) All full subcategories, all quotient categories, and all product categories of (semi)additive categories are (semi)additive.

(5) If $\mathscr{A}$ is (semi)additive, then so are $\mathscr{A}^{op}$ and $\mathscr{A}^{\mathscr{C}}$, for any category $\mathscr{C}$.

40.3 NOTATIONAL REMARK

If $(A_i)_I$ is a family of $\mathscr{A}$-objects, then for each $j, k \in I$ we let

$$\delta_{jk}: A_j \to A_k \quad \text{be} \quad \begin{cases} 1_{A_j} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

40.4 DEFINITION

(1) Let $(A_i)_I$ be a family of $\mathscr{A}$-objects. Then the family $(\mu_i, B, \pi_i)_I$ is called a **biproduct of** $(A_i)_I$ provided that the following conditions hold:

(i) $(B, \pi_i)_I$ is a product of $(A_i)_I$.
(ii) $(\mu_i, B)_I$ is a coproduct of $(A_i)_I$.
(iii) $\pi_k \circ \mu_j = \delta_{jk}$ for each $j, k \in I$.

† Recall our convention that all rings have identities and ring homomorphisms preserve identities.

(2) A category $\mathscr{A}$ **has (finite) biproducts** provided that each (finite) set-indexed family of $\mathscr{A}$-objects has a biproduct.

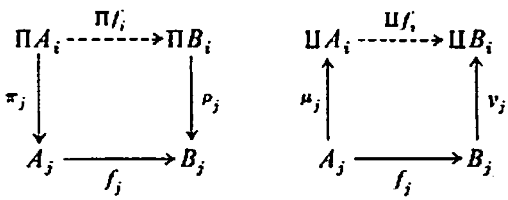
The (object-part of a) biproduct is usually denoted by $\oplus A_i$. In particular the biproduct of a pair of objects, (A, B), is often denoted by

$$(\mu_A, \mu_B, A \oplus B, \pi_A, \pi_B).$$

40.5 EXAMPLE

The empty family of $\mathscr{A}$-objects has a biproduct if and only if $\mathscr{A}$ has a zero object.

Recall that if $(A_i \xrightarrow{f_i} B_i)_I$ is a family of morphisms and ΠA_i, $\amalg A_i$, ΠB_i, and $\amalg B_i$ are the products and coproducts of $(A_i)_I$ and $(B_i)_I$, then we have defined Πf_i and $\amalg f_i$ to be the unique morphisms which for each $j \in I$ make the squares



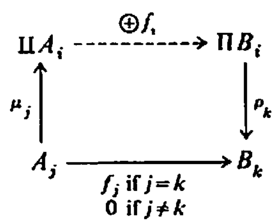
commute; i.e., $\Pi f_i = \langle f_i \circ \pi_i \rangle$ and $\amalg f_i = [\nu_i \circ f_i]$ (18.5 and 18.15). Since we are now assuming that all categories are pointed, we also have the following naturally occurring morphism:

40.6 DEFINITION

If $(A_i \xrightarrow{f_i} B_i)_I$ is a family of morphisms and $(\mu_i, \amalg A_i)_I$ and $(\Pi B_i, \rho_i)$ are the coproduct and product of $(A_i)_I$ and $(B_i)_I$, respectively, then

$$\oplus f_i \colon \amalg A_i \to \Pi B_i$$

is the unique morphism from $\amalg A_i$ to ΠB_i such that for each $j, k \in I$, the square



commutes.

40.7 PROPOSITION

If $(A_i \xrightarrow{f_i} B_i)_I$ is a family of morphisms and $(\mu_i, \oplus A_i, \pi_i)$ and $(\nu_i, \oplus B_i, \rho_i)$ are biproducts of $(A_i)_I$ and $(B_i)_I$, respectively, then Πf_i, $\amalg f_i$, and $\oplus f_i$ are all the same; i.e.,

$$\Pi f_i = \amalg f_i = \oplus f_i.$$

Proof:

$$\rho_k \circ \Pi f_i \circ \mu_j = f_k \circ \pi_k \circ \mu_j = f_k \circ \delta_{jk} = \begin{cases} f_j & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$$\rho_k \circ \amalg f_i \circ \mu_j = \rho_k \circ \nu_j \circ f_j = \delta_{jk} \circ f_j = \begin{cases} f_j & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$$\rho_k \circ \oplus f_i \circ \mu_j = \begin{cases} f_j & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

Thus since products are mono-sources and coproducts are epi-sinks,

$$\Pi f_i = \amalg f_i = \oplus f_i.$$

□

40.8 PROPOSITION

Let $(\mathscr{A}, +)$ *be a semiadditive category, let* $(A_i)_I$ *be a finite family of* $\mathscr{A}$*-objects and let*

$$A_i \xrightarrow{\mu_i} B \xrightarrow{\pi_i} A_i$$

be $\mathscr{A}$*-morphisms. Then the following conditions are equivalent:*

(1) $(\mu_i, B, \pi_i)_I$ *is a biproduct of* $(A_i)_I$.
(2) $(B, \pi_i)_I$ *is a product of* $(A_i)_I$ *and for all* $j, k \in I$, $\pi_k \circ \mu_j = \delta_{jk}$.
(3) $(\mu_i, B)_I$ *is a coproduct of* $(A_i)_I$ *and for* $j, k \in I$, $\pi_k \circ \mu_j = \delta_{jk}$.
(4) $\sum_{i \in I} (\mu_i \circ \pi_i) = 1_B$ *and for all* $j, k \in I$, $\pi_k \circ \mu_j = \delta_{jk}$.

Proof: Clearly each of (1) and (4) is self-dual, (2) and (3) are dual to each other, and together are equivalent to (1). Thus we need only show that (2) and (4) are equivalent.

(2) ⇒ *(4)*. By distributivity of composition over addition (A2) it follows that for each $k \in I$

$$\pi_k \circ \sum_i (\mu_i \circ \pi_i) = \sum_i (\pi_k \circ \mu_i \circ \pi_i) = \sum_i (\delta_{ik} \circ \pi_i) = \pi_k = \pi_k \circ 1_B.$$

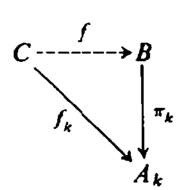
Thus since products are mono-sources,

$$\sum_i (\mu_i \circ \pi_i) = 1_B.$$

(4) ⇒ *(2)*. Let $(C \xrightarrow{f_i} A_i)_I$ be a family of morphisms. Define $f = \sum_i (\mu_i \circ f_i)$. Then for each $k \in I$ we have, by distributivity

$$\pi_k \circ f = \pi_k \circ \sum_i (\mu_i \circ f_i) = \sum_i (\pi_k \circ \mu_i \circ f_i) = \sum_i (\delta_{ik} \circ f_i) = f_k$$

i.e., for each $k \in I$ the triangle



commutes. Also f is unique with respect to this property since if for each k, $\pi_k \circ g = f_k$, then

$$g = 1 \circ g = (\sum_i \mu_i \circ \pi_i) \circ g = \sum_i (\mu_i \circ \pi_i \circ g) = \sum_i (\mu_i \circ f_i) = f.$$

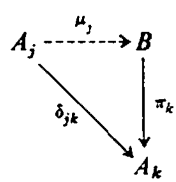
Thus $(B, \pi_i)_I$ is a product of $(A_i)_I$. □

Notice that the above proposition remains valid in the case that $I = \varnothing$, if we interpret $\sum_{i \in I} (\mu_i \circ \pi_i)$ to be the zero morphism $0: B \to B$.

40.9 PROPOSITION

If $(\mathscr{A}, +)$ is a semiadditive category and $(B, \pi_i)_I$ is a product in $\mathscr{A}$ of the finite family $(A_i)_I$, then (B, π_i) can be completed in a unique way to a biproduct $(\mu_i, B, \pi_i)_I$ of $(A_i)_I$.

Proof: By the definition of product, for each $j \in I$ there exists a unique morphism $\mu_j: A_j \to B$ such that for each $k \in I$ the triangle



commutes. But by the above proposition (40.8(2)) $(\mu_i)_I$ is the unique family for which (μ_i, B, π_i) is a biproduct of $(A_i)_I$. □

40.10 COROLLARY

If $(\mathscr{A}, +)$ is a semiadditive category, then the following are equivalent:

(1) *$\mathscr{A}$ has finite products.*

(2) *$\mathscr{A}$ has finite coproducts.*

(3) *$\mathscr{A}$ has finite biproducts.* □

40.11 LEMMA

If $(\mu_1, \mu_2, A_1 \oplus A_2, \pi_1, \pi_2)$ and $(\nu_1, \nu_2, B_1 \oplus B_2, \rho_1, \rho_2)$ are biproducts and $f: A_1 \to B_1$, $h: A_1 \to B_2$, $g: A_2 \to B_1$ and $k: A_2 \to B_2$ are $\mathscr{A}$-morphisms, then the morphisms

$$x = [\langle f, h\rangle, \langle g, k\rangle]: A_1 \oplus A_2 \to B_1 \oplus B_2,$$

and

$$y = \langle [f, g], [h, k]\rangle: A_1 \oplus A_2 \to B_1 \oplus B_2$$

are the same.

Proof:

$$\rho_1 \circ x \circ \mu_1 = \rho_1 \circ \langle f, h\rangle = f = [f, g] \circ \mu_1 = \rho_1 \circ y \circ \mu_1$$
$$\rho_2 \circ x \circ \mu_1 = \rho_2 \circ \langle f, h\rangle = h = [h, k] \circ \mu_1 = \rho_2 \circ y \circ \mu_1$$
$$\rho_2 \circ x \circ \mu_2 = \rho_2 \circ \langle g, k\rangle = k = [h, k] \circ \mu_2 = \rho_2 \circ y \circ \mu_2$$
$$\rho_1 \circ x \circ \mu_2 = \rho_1 \circ \langle g, k\rangle = g = [f, g] \circ \mu_2 = \rho_1 \circ y \circ \mu_2$$

Thus since products are mono-sources and coproducts are epi-sinks, it follows that $x = y$. $\square$

40.12 PROPOSITION

If $\mathscr{A}$ has finite biproducts, then there exists a unique semiadditive structure $+$ on $\mathscr{A}$, given by each of the following:

If $A \overset{f}{\underset{g}{\rightrightarrows}} B$, *then*

$$A \xrightarrow{f+g} B = \begin{cases} A \xrightarrow{\Delta} A \oplus A \xrightarrow{[f,g]} B = \\ A \xrightarrow{\langle f,g\rangle} B \oplus B \xrightarrow{\nabla} B = \\ A \xrightarrow{\Delta} A \oplus A \xrightarrow{f\oplus g} B \oplus B \xrightarrow{\nabla} B. \end{cases}$$

Proof: Let the biproducts be

$$(\mu_1, \mu_2, A \oplus A, \pi_1, \pi_2) \quad \text{and} \quad (\nu_1, \nu_2, B \oplus B, \rho_1, \rho_2).$$

First we show uniqueness; i.e., if $+$ is a semiadditive structure on $\mathscr{A}$, then the above equalities hold. From the distributivity of composition over sums and the above characterization of biproducts (40.8(4)), we have:

$$\begin{aligned} [f, g] \circ \Delta &= [f, g] \circ 1 \circ \Delta = [f, g] \circ (\textstyle\sum \mu_i \circ \pi_i) \circ \Delta \\ &= \textstyle\sum ([f, g] \circ \mu_i \circ \pi_i \circ \Delta) = \sum ([f, g] \circ \mu_i) = f + g. \\ \nabla \circ \langle f, g\rangle &= \nabla \circ 1 \circ \langle f, g\rangle = \nabla \circ (\textstyle\sum \nu_i \circ \rho_i) \circ \langle f, g\rangle \\ &= \textstyle\sum (\nabla \circ \nu_i \circ \rho_i \circ \langle f, g\rangle) = \sum (\rho_i \circ \langle f, g\rangle) = f + g. \\ \nabla \circ (f \oplus g) \circ \Delta &= \nabla \circ (\textstyle\sum \nu_i \circ \rho_i) \circ (f \oplus g) \circ (\sum \mu_i \circ \pi_i) \circ \Delta \\ &= \sum_{\substack{j=1,2 \\ k=1,2}} (\nabla \circ \nu_j \circ \rho_j \circ (f \oplus g) \circ \mu_k \circ \pi_k \circ \Delta) \\ &= \sum_{\substack{j=1,2 \\ k=1,2}} \rho_j \circ (f \oplus g) \circ \mu_k = f + 0 + g + 0 = f + g. \end{aligned}$$

Secondly we show that each of the above equalities actually does define a semiadditive structure on $\mathscr{A}$. Denote the first two operations given above as $+$ and $*$, respectively; i.e.,

$$f + g = [f, g] \circ \Delta$$

and

$$f * g = \nabla \circ \langle f, g\rangle.$$

For any morphism $f: A \to B$,

$$[f, 0] \circ \mu_1 = f = f \circ \pi_1 \circ \mu_1,$$

and

$$[f, 0] \circ \mu_2 = 0 = f \circ \pi_1 \circ \mu_2.$$

Thus since coproducts are epi-sinks, $[f, 0] = f \circ \pi_1$. Likewise it can be shown that $[0, f] = f \circ \pi_2$. Thus

$$f + 0 = [f, 0] \circ \Delta = f \circ \pi_1 \circ \Delta = f = f \circ \pi_2 \circ \Delta = [0, f] \circ \Delta = 0 + f.$$

Similarly, it can be shown that $\langle f, 0 \rangle = \nu_1 \circ f$ and $\langle 0, f \rangle = \nu_2 \circ f$, so that

$$f * 0 = \nabla \circ \langle f, 0 \rangle = \nabla \circ \nu_1 \circ f = f = \nabla \circ \nu_2 \circ f = \nabla \circ \langle 0, f \rangle = 0 * f.$$

Hence 0 acts as a left and right identity with respect to $+$ and with respect to $*$, so that (A3) is established for $+$ and $*$. To show distributivity (A2), suppose that $p: C \to A$ and $q: B \to D$. Then

$$(f * g) \circ p = \nabla \circ \langle f, g \rangle \circ p = \nabla \circ \langle f \circ p, g \circ p \rangle = f \circ p + g \circ p$$

and

$$q \circ (f + g) = q \circ [f, g] \circ \Delta = [q \circ f, q \circ g] \circ \Delta = q \circ f + q \circ g.$$

Now suppose that $f, g, h, k: A \to B$. Then

$$(f + g) * (h + k) = \nabla \circ \langle f + g, h + k \rangle = \nabla \circ \langle [f, g] \circ \Delta, [h, k] \circ \Delta \rangle$$
$$= \nabla \circ \langle [f, g], [h, k] \rangle \circ \Delta.$$

But by the lemma (40.11), this is

$$\nabla \circ [\langle f, h \rangle, \langle g, k \rangle] \circ \Delta = [\nabla \circ \langle f, h \rangle, \nabla \circ \langle g, k \rangle] \circ \Delta$$
$$= [f * h, g * k] \circ \Delta = (f * h) + (g * k).$$

Hence

$$(f + g) * (h + k) = (f * h) + (g * k).$$

Letting $g = 0 = h$, and using (A3), we obtain

$$f * k = f + k.$$

Thus $* = +$, so that distributivity (A2) is established. Letting $g = 0$ in the above equation, we obtain

$$f + (h + k) = (f + h) + k$$

so that $+$ is associative. Letting $f = 0$ and $k = 0$, we obtain

$$g + h = h + g$$

so that $+$ is commutative. Thus $+$ induces the structure of a commutative monoid on $hom(A, B)$, so that (A1′) holds. ☐

40.13 THEOREM

If $\mathscr{A}$ has finite products or finite coproducts, then the following conditions are equivalent:

(1) *There exists a semiadditive structure on $\mathscr{A}$.*

(2) *There exists a unique semiadditive structure on $\mathscr{A}$.*

(3) *$\mathscr{A}$ has finite biproducts.* ☐

Additive Functors

40.14 DEFINITION

If $\mathscr{A}$ and $\mathscr{B}$ are semiadditive categories, then $F: \mathscr{A} \to \mathscr{B}$ is called an **additive functor** provided that for each pair (A, B) of $\mathscr{A}$-objects, F induces a monoid homomorphism from $hom(A, B)$ to $hom(F(A), F(B))$.

40.15 EXAMPLES

(1) A function between two rings (considered as additive categories) is an additive functor if and only if it is a ring homomorphism.

(2) For any ring R, the forgetful functor F: R-**Mod** $\to$ **Z-Mod** is additive.

40.16 THEOREM

If $F: \mathscr{A} \to \mathscr{B}$ is a functor between semiadditive categories and $\mathscr{A}$ has finite products, then the following are equivalent:

(1) *F is additive.*

(2) *F preserves finite products.*

(3) *F preserves finite coproducts.*

(4) *F preserves finite biproducts.*

Proof: The equivalence of (2), (3), and (4) follows immediately from Propositions 40.8 and 40.9. That (1) implies (4) follows immediately from the characterization of biproducts (40.8(4)) and the fact that each additive functor must preserve zero morphisms. To see that (4) implies (1), note that since F preserves empty biproducts, it preserves zero morphisms. Also by the uniqueness of the semiadditive structure defined in terms of biproducts (40.12), F must preserve addition. □

Module-Valued Functors

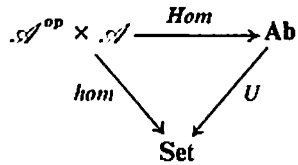
For any pair $(\mathscr{A}, \mathscr{B})$ of additive categories one can define $Add[\mathscr{A}, \mathscr{B}]$ to be the full subcategory of $[\mathscr{A}, \mathscr{B}]$ whose objects are the additive functors from $\mathscr{A}$ to $\mathscr{B}$. Likewise one can define the quasicategory of all additive categories and additive functors, and the category of all small additive categories and additive functors. Many of the results of general categories translate into analogous results in the realm of additive categories. We leave the task of such translations to the reader, and restrict ourselves to pointing out the important fact that the role **Set**-valued functors play in the study of arbitrary categories is played by **Ab**-valued (or more generally **Mod**-R-valued) functors in the study of additive categories.

40.17 PROPOSITION

If $\mathscr{A}$ is an additive category and (**Ab**, U) *is the concrete category of abelian groups, then $\mathscr{A}^{op} \times \mathscr{A}$ is additive and there exists an additive functor*

$$Hom: \mathscr{A}^{op} \times \mathscr{A} \to \mathbf{Ab}$$

such that the triangle



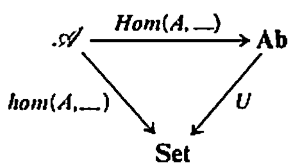
commutes.

In fact, Hom preserves limits.

Proof: The verification that $\mathscr{A} \times \mathscr{A}^{op}$ is additive is straightforward (40B). (A1) enables us to supply each set *hom*(A, B) with the structure of an abelian group, and (A2) guarantees that the morphisms *hom*(f, g) can be considered as group homomorphisms. Since *hom* preserves limits and U reflects them (32.12), *Hom* must preserve them (24E). □

40.18 COROLLARY

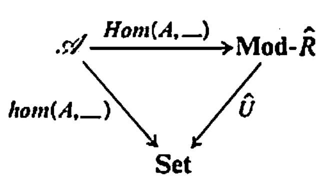
If $\mathscr{A}$ is additive and A is an $\mathscr{A}$-object, then there exists an additive functor $Hom(A, \_\_)\colon \mathscr{A} \to$ **Ab** *such that the triangle*



commutes. □

40.19 PROPOSITION

If $\mathscr{A}$ is additive and A is an $\mathscr{A}$-object, then the full subcategory of $\mathscr{A}$ whose sole object is A can be considered as a ring $\hat{R}$, and there exists an additive functor $Hom(A, \_\_)\colon \mathscr{A} \to$ **Mod-**$\hat{R}$ *such that the triangle*



commutes (where $\hat{U}$ denotes the forgetful functor).

Proof: That for each $\mathscr{A}$-object B, *hom*(A, B) can be considered as a right $\hat{R}$-module follows from the fact that it has the structure of an abelian group (A1) and from the associativity of composition and the distributivity of composition over addition (A2). The last two conditions also guarantee that the morphisms *hom*(A, f) can be considered as linear transformations. *Hom*$(A, \_\_)$ is additive since *hom*$(A, \_\_)$ preserves limits and $\hat{U}$ reflects them. □

Next we would like to be able to characterize those $\mathscr{A}$-objects A for which the above constructed functor $Hom(A, \_\_)$ preserves coproducts. To accomplish this, we first need the following lemma.

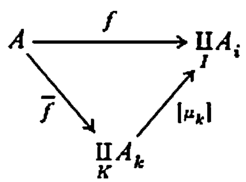
40.20 LEMMA

Let $\mathscr{A}$ be an additive category that has products and coproducts, let $(A_i)_I$ be a set-indexed family of $\mathscr{A}$-objects, and let $(\mu_i, \coprod_I A_i)$ and $(\prod_I A_i, \pi_i)$ be the coproduct and product, respectively, of $(A_i)_I$. Then for each morphism

$$f: A \to \coprod_I A_i,$$

the following are equivalent:

(1) f can be factored through a finite coproduct; i.e., there is a finite set $K \subset I$, a coproduct $(\nu_k, \coprod_K A_k)$ of $(A_k)_K$, and a morphism $\bar{f}: A \to \coprod_K A_k$ such that the triangle



commutes.

(2) $\sum_I (\mu_i \circ \pi_i \circ \oplus 1_{A_i} \circ f) = f$; i.e., for all but finitely many $i \in I$,

$$\mu_i \circ \pi_i \circ \oplus 1_{A_i} \circ f = 0,$$

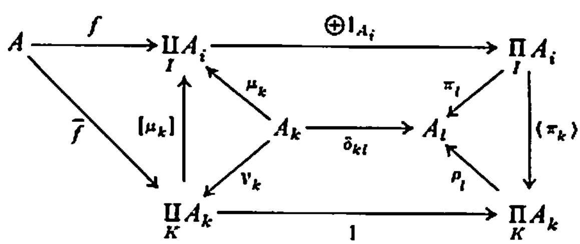
and the remainder have a sum that is f.

Proof:

(1) ⇒ *(2)*. Complete the finite coproduct $(\nu_k, \coprod_K A_k)$ to a biproduct

$$(\nu_k, \coprod_K A_k = \prod_K A_k, \rho_k)$$

(40.9 dual). Then for each $k, l \in K$, the diagram



commutes.

Let $j \in I - K$. Then for each $k \in K$, we have

$$\mu_j \circ \pi_j \circ \oplus 1_{A_i} \circ [\mu_k] \circ \nu_k = \mu_j \circ \pi_j \circ \oplus 1_{A_i} \circ \mu_k = \mu_j \circ \delta_{kj}$$
$$= \mu_j \circ 0 = 0 \circ \nu_k.$$

Since coproducts are epi-sinks, this implies that for each $j \in I-K$,

$$\mu_j \circ \pi_j \circ \oplus 1_{A_i} \circ [\mu_k] = 0,$$

so that

$$\mu_j \circ \pi_j \circ \oplus 1_{A_i} \circ f = \mu_j \circ \pi_j \circ \oplus 1_{A_i} \circ [\mu_k] \circ \bar{f} = 0 \circ \bar{f} = 0.$$

Thus

$$\sum_I (\mu_i \circ \pi_i \circ \oplus 1_{A_i} \circ f) = \sum_K (\mu_k \circ \pi_k \circ \oplus 1_{A_i} \circ f)$$
$$= \sum_K (\mu_k \circ \pi_k \circ \oplus 1_{A_i} \circ [\mu_k] \circ \bar{f}).$$

Using the commutativity of the above diagram and 40.8, this becomes:

$$\sum_K (\mu_k \circ \rho_k \circ \bar{f}) = \sum_K ([\mu_k] \circ \nu_k \circ \rho_k \circ \bar{f}) = [\mu_k] \circ (\sum_K \nu_k \circ \rho_k) \circ \bar{f}$$
$$= [\mu_k] \circ 1 \circ \bar{f} = f.$$

(2) $\Rightarrow$ *(1)*. Assuming (2), the set $K = \{i \in I \mid \mu_i \circ \pi_i \circ \oplus 1_{A_i} \circ f \neq 0\}$ is finite. Using the notation above for this K, let $\bar{f} = \sum_K (\nu_k \circ \pi_k \circ \oplus 1_{A_i} \circ f)$. Then

$$[\mu_k] \circ \bar{f} = [\mu_k] \circ \sum_K (\nu_k \circ \pi_k \circ \oplus 1_{A_i} \circ f) = \sum_K ([\mu_k] \circ \nu_k \circ \pi_k \circ \oplus 1_{A_i} \circ f)$$
$$= \sum_K (\mu_k \circ \pi_k \circ \oplus 1_{A_i} \circ f) = \sum_I (\mu_i \circ \pi_i \circ \oplus 1_{A_i} \circ f) = f. \quad \square$$

40.21 THEOREM

Let $(\mathscr{A}, hom(A, \_))$ be an algebraic category. If $\mathscr{A}$ is additive, $\hat{R}$ is the ring $Hom(A, A)$, and $Hom(A, \_)\colon \mathscr{A} \to$ **Mod-$\hat{R}$** *is the functor constructed in* 40.19, *then the following conditions are equivalent:*

(1) *$F = Hom(A, \_)$ preserves coproducts.*

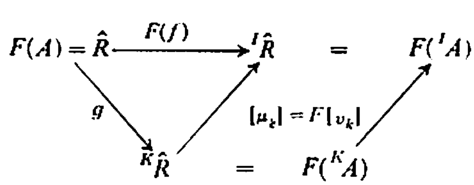
(2) *$hom(A, \_)$ is a finitary functor* (see 22E and 32G).

Proof:

(1) $\Rightarrow$ *(2)*. Suppose that $f\colon A \to {}^I A$. Then since F preserves coproducts we have

$$\hat{R} = F(A) \xrightarrow{F(f)} F({}^I A) = {}^I F(A) = {}^I \hat{R},$$

so that since **Mod-$\hat{R}$** is finitary, there exists a finite set $K \subset I$ and a morphism $g\colon \hat{R} \to {}^K \hat{R}$ such that the triangle



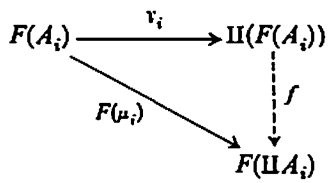
commutes.

Define $\bar{g} = g(1_A): A \to {}^K A$. Then

$$[v_k] \circ \bar{g} = [v_k] \circ g(1_A) = F([v_k])(g(1_A)) = [\mu_k](g(1_A))$$
$$= ([\mu_k] \circ g)(1_A) = F(f)(1_A) = f \circ 1_A = f.$$

Thus *hom* $(A, —)$ must be finitary (32G(iv)).

(2) ⇒ (*1*). Let $(\mu_i, \amalg A_i)$ be the coproduct of $(A_i)_I$ in $\mathscr{A}$ and let $(v_i, \amalg F(A_i))$ be the coproduct of $(F(A_i))_I$ in Mod-$\hat{R}$. Then there exists a unique linear transformation f such that for each i, the triangle



commutes.

If $g_i \in F(A_i) = Hom(A, A_i)$, then

$$f(v_i(g_i)) = F(\mu_i)(g_i) = \mu_i \circ g_i.$$

Since f is a linear transformation, for each $\sum v_i(g_i) \in \amalg F(A_i)$ we have

$$f(\sum v_i(g_i)) = \sum f(v_i(g_i)) = \sum (\mu_i \circ g_i).$$

Thus whenever $f(\sum v_i(g_i)) = 0$, we have

$$\sum (\mu_i \circ g_i) = 0$$

so that for each i, $g_i = 0$ (40J). Thus f is injective. If $k \in F(\amalg A_i)$, then by Exercise (32G) and the lemma (40.20) $k = \sum (\mu_i \circ \pi_i \circ \oplus 1_{A_i} \circ k)$. Define $\hat{k} = \sum (v_i(\pi_i \circ \oplus 1_{A_i} \circ k))$. Then $f(\hat{k}) = k$, so that f is surjective. Thus f is bijective and so is an isomorphism. Consequently F preserves coproducts. □

EXERCISES

40A. Let $\mathscr{A}$ be a category and let $+$ be a function which associates with each pair (f, g) of $\mathscr{A}$-morphisms with common domain A and common codomain B, an $\mathscr{A}$-morphism $f + g$ with domain A and codomain B.

(a) Prove that for such a function $+$, conditions (A1) and (A2) of Definition 40.1 imply condition (A3).

(b) Prove that conditions (A1′) and (A2) do not imply (A3). [*Hint:* Consider the following category that has precisely one object X, with morphism set $hom(X, X) = \{1_X, 0, b\}$ and composition defined by

$\circ$	1_X	0	b
1_X	1_X	0	b
0	0	0	0
b	b	0	b

Define a function $+$ by:

$+$	1_X	0	b
1_X	b	0	1_X
0	0	0	0
b	1_X	0	b

Show that $(\mathscr{A}, +)$ satisfies (A1′) and (A2) but not (A3).]

40B. Show that for any categories $\mathscr{A}$ and $\mathscr{B}$

(a) $\mathscr{A}$ is additive if and only if $\mathscr{A}^{op}$ is additive.

(b) If $\mathscr{A}$ is additive, then so is $\mathscr{A}^{\mathscr{B}}$.

(c) If $\mathscr{A}$ and $\mathscr{B}$ are additive, then so is $\mathscr{A} \times \mathscr{B}$.

40C. Prove that if $f, g: A \to B$ are morphisms in an additive category, then $Ker(f - g) \approx Equ(f, g)$; i.e., if either exists, then they both do, and are the same.

40D. Prove that if $A \xrightarrow{f} B$, $A \xrightarrow{g} C$, $B \xrightarrow{h} D$, and $C \xrightarrow{k} D$ are morphisms in a semiadditive category, then

$$A \xrightarrow{\langle f,g \rangle} B \oplus C \xrightarrow{[h,k]} D = h \circ f + k \circ g.$$

40E. Using Lemma 40.11 and Exercise 40D, represent morphisms between biproducts of pairs of objects in any category $\mathscr{A}$ as 2×2 matrices and show that if $\mathscr{A}$ is semiadditive, their composition is usual matrix multiplication; i.e.,

$$A_1 \oplus A_2 \xrightarrow{\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}} B_1 \oplus B_2 \xrightarrow{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}} C_1 \oplus C_2$$

$$= A_1 \oplus A_2 \xrightarrow{\begin{pmatrix} g_{11} \circ f_{11} + g_{12} \circ f_{21} & g_{11} \circ f_{12} + g_{12} \circ f_{22} \\ g_{21} \circ f_{11} + g_{22} \circ f_{21} & g_{21} \circ f_{12} + g_{21} \circ f_{22} \end{pmatrix}} C_1 \oplus C_2$$

40F. Generalize Exercise 40E by representing all morphisms between finite biproducts of objects in any category $\mathscr{A}$ as finite matrices and showing that if $\mathscr{A}$ is semiadditive, their composition is usual matrix multiplication.

40G. Let $F \dashv G: (\mathscr{A}, \mathscr{B})$ be an adjoint situation, where $\mathscr{A}$ and $\mathscr{B}$ are additive categories.

(a) Show that F is additive if and only if G is additive.

(b) Show that if $\mathscr{A}$ has finite products, then F and G are both additive.

40H. Let $\mathscr{A}$ be additive, A an $\mathscr{A}$-object, $G: \mathscr{A} \to$ **Ab** an additive functor, U: **Ab** $\to$ **Set** the forgetful functor, and $\eta: hom(A, \_) \to U \circ G$ a natural transformation. Show that there exists a natural transformation $\bar{\eta}: Hom(A, \_) \to G$ such that $U * \bar{\eta} = \eta$.

40I. Let $\mathscr{A}$ be additive, A an $\mathscr{A}$-object, and $G: \mathscr{A} \to$ **Ab** an additive functor.

(a) Prove that the Yoneda functions (30.6) induce an isomorphism between the abelian groups $G(A)$ and $[Hom(A, \_), G]$.

(b) Show that the Yoneda embedding (30.8) induces a full additive embedding $E: \mathscr{A}^{op} \hookrightarrow Add[\mathscr{A}, \mathbf{Ab}]$.

40J. Let $\mathscr{A}$ be a semiadditive category that has products and coproducts, let $(\mu_i, \amalg A_i)$ be a coproduct of $(A_i)_I$, and let $(A \xrightarrow{g_i} A_i)_I$ be a family of $\mathscr{A}$-morphisms that are 0 except for finitely many $i \in I$. Show that if $\sum_I (\mu_i \circ g_i) = 0$, then $g_i = 0$ for all $i \in I$.

§41 ABELIAN CATEGORIES

In this section we combine the notions of exactness and additivity of the previous two sections to obtain the important concept of abelian category.

Recall that throughout this chapter all categories are assumed to be pointed.

Definition and General Properties

41.1 DEFINITION

A category is called an **abelian category** provided that it is exact and has finite biproducts.

41.2 EXAMPLES

(1) R-**Mod** is abelian (for every ring R).

(2) The full subcategory of **Ab** consisting of finite abelian groups is abelian.

(3) The full subcategory of **TopGrp** consisting of compact abelian groups is abelian.

(4) The full subcategory of **Ab** consisting of abelian groups whose underlying set has at most 27 elements is not abelian (even though it is exact and additive).

(5) The opposite of any abelian category is abelian.

41.3 THEOREM

If $\mathscr{A}$ is exact and has finite products or finite coproducts, then the following conditions are equivalent:

(1) *There exists an additive structure on $\mathscr{A}$.*

(2) *There exists a unique additive structure on $\mathscr{A}$.*

(3) *$\mathscr{A}$ is an abelian category.*

Proof: In view of Theorem 40.13, it is sufficient to show that if $\mathscr{A}$ has finite biproducts, then the unique semiadditive structure on $\mathscr{A}$ (40.12) actually induces the structure of an abelian group on each set $hom(A, B)$. That is, we wish to show that each $\mathscr{A}$-morphism $f: A \to B$ has an additive inverse. Given $f: A \to B$, consider the morphism

$$A \oplus B \xrightarrow{h=\langle[1_A,0],[f,1_B]\rangle} A \oplus B.$$

Let $(K, k) \approx Ker(h)$. Then

$$\begin{aligned}\langle 0, 0\rangle = 0 = h \circ k &= \langle [1_A, 0] \circ k, [f, 1_B] \circ k\rangle \\ &= \langle [1_A, 0] \circ \langle \pi_A \circ k, \pi_B \circ k\rangle, [f, 1_B] \circ \langle \pi_A \circ k, \pi_B \circ k\rangle\rangle.\end{aligned}$$

But since $\mathscr{A}$ is semiadditive (40.12), this is

$$\langle \pi_A \circ k + 0,\ f \circ \pi_A \circ k + \pi_B \circ k\rangle \quad \text{(40D)}.$$

Thus $\pi_A \circ k = 0$ and $\pi_B \circ k = 0$, so that $k = 0$. Hence $Ker(h) = 0$. Similarly if $(c, C) \approx Cok(h)$, then

$$\begin{aligned}[0, 0] = 0 = c \circ h &= [c \circ \langle 1_A, f\rangle, c \circ \langle 0, 1_B\rangle] \\ &= [[\mu_1 \circ c, \mu_2 \circ c] \circ \langle 1_A, f\rangle, [\mu_1 \circ c, \mu_2 \circ c] \circ \langle 0, 1_B\rangle] \\ &= [\mu_1 \circ c + \mu_2 \circ c \circ f,\ 0 + \mu_2 \circ c].\end{aligned}$$

Thus $Cok(h) = 0$. Consequently, h must be an isomorphism (39.15). Now

$$1_A = \pi_A \circ \mu_A = \pi_A \circ h \circ h^{-1} \circ \mu_A = [1_A, 0] \circ h^{-1} \circ \mu_A.$$

But since $1_{A \oplus B} = \mu_A \circ \pi_A + \mu_B \circ \pi_B$ (40.8), this is

$$[1_A, 0] \circ (\mu_A \circ \pi_A + \mu_B \circ \pi_B) \circ h^{-1} \circ \mu_A = \pi_A \circ h^{-1} \circ \mu_A + 0 = \pi_A \circ h^{-1} \circ \mu_A.$$

Let $g = \pi_B \circ h^{-1} \circ \mu_A$. Then

$$\begin{aligned}0 = \pi_B \circ \mu_A &= \pi_B \circ h \circ h^{-1} \circ \mu_A = [f, 1_B] \circ h^{-1} \circ \mu_A \\ &= [f, 1_B] \circ 1 \circ h^{-1} \circ \mu_A = [f, 1_B] \circ (\mu_A \circ \pi_A + \mu_B \circ \pi_B) \circ h^{-1} \circ \mu_A \\ &= f \circ \pi_A \circ h^{-1} \circ \mu_A + \pi_B \circ h^{-1} \circ \mu_A.\end{aligned}$$

But by the above equality, this is

$$f \circ 1_A + g = f + g.$$

Thus $f + g = 0$, so that f has an additive inverse. □

41.4 COROLLARY

A category $\mathscr{A}$ is abelian if and only if the following conditions are satisfied:

(1) *$\mathscr{A}$ is exact.*

(2) *$\mathscr{A}$ is (semi)additive.*

(3) *$\mathscr{A}$ has finite products (or finite coproducts).*

Proof: That every abelian category satisfies these conditions follows from the definition (41.1) and the above theorem (41.3). Conversely, every semi-additive category with finite products (or finite coproducts) must have finite biproducts (40.10). □

41.5 PROPOSITION

Every abelian category $\mathscr{A}$ is finitely complete and finitely cocomplete.

Proof: By definition, $\mathscr{A}$ has finite products and finite coproducts, and since $\mathscr{A}$ is exact it must have finite intersections and finite cointersections (39.9 and its dual). Thus $\mathscr{A}$ must have all finite limits and finite colimits (23.7(5) and its dual). □

41.6 DEFINITION

A square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

is called a **pulation square** (or, for minors, **Doolittle diagram**) provided that it is both a pullback square and a pushout square.

41.7 LEMMA

Let $\mathscr{A}$ be a category in which each section is a normal monomorphism, and let $(\mu_1, \mu_2, A_1 \amalg A_2)$ be a coproduct of the $\mathscr{A}$-objects A_1 and A_2. Then

(1) $A_1 \amalg A_2 \xrightarrow{[0,1]} A_2 = Cok(A_1 \xrightarrow{\mu_1} A_1 \amalg A_2)$.

(2) $A_1 \xrightarrow{\mu_1} A_1 \amalg A_2 = Ker(A_1 \amalg A_2 \xrightarrow{[0,1]} A_2)$.

(3) *For each $\mathscr{A}$-object B, the square*

$$\begin{array}{ccc} B & \xrightarrow{0} & A_2 \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ A_1 & \xrightarrow[\mu_1]{} & A_1 \amalg A_2 \end{array}$$

is a pulation square.

Proof:

(*1*). Clearly $[0, 1] \circ \mu_1 = 0$. Suppose that $f \circ \mu_1 = 0$. Since coproducts are epi-sinks, the morphism $f \circ \mu_2$ is such that

$$(f \circ \mu_2) \circ [0, 1] = f.$$

Also it is unique with respect to this property since if $g \circ [0, 1] = f$, then

$$g = g \circ [0, 1] \circ \mu_2 = f \circ \mu_2.$$

(*2*). Since $[1, 0] \circ \mu_1 = 1$, μ_1 is a section; and so by hypothesis it is a normal monomorphism. Hence since $[0, 1] \approx Cok(\mu_1)$, we must have $\mu_1 \approx Ker([0, 1])$ (39.5(2)).

(*3*). From the definition of coproduct, it is clear that the square is a pushout square.

Now suppose that $f_1 : C \to A_1$ and $f_2 : C \to A_2$ are morphisms such that $\mu_1 \circ f_1 = \mu_2 \circ f_2$. Then

$$\begin{aligned} f_1 &= [1, 0] \circ \mu_1 \circ f_1 = [1, 0] \circ \mu_2 \circ f_2 = 0 \circ f_2 = 0 = 0 \circ f_1 \\ &= [0, 1] \circ \mu_1 \circ f_1 = [0, 1] \circ \mu_2 \circ f_2 = f_2. \end{aligned}$$

Hence the zero morphism from C to B is the unique morphism h for which $0 \circ h = f_1$ and $0 \circ h = f_2$. Thus the square is a pullback square. □

41.8 COROLLARY

If $\mathscr{A}$ *is abelian and* $(\mu_1, \mu_2, A_1 \oplus A_2, \pi_1, \pi_2)$ *is a biproduct of* A_1 *and* A_2, *then*

$$0 \to A_1 \xrightarrow{\mu_1} A_1 \oplus A_2 \xrightarrow{[0,1]=\pi_2} A_2 \to 0$$

and

$$0 \to A_2 \xrightarrow{\mu_2} A_1 \oplus A_2 \xrightarrow{\pi_1} A_1 \to 0$$

are exact sequences, and

$$\begin{array}{ccc} 0 & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle \mu_2} \\ A_1 & \xrightarrow[\mu_1]{} & A_1 \oplus A_2 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ A_1 & \longrightarrow & 0 \end{array}$$

are pulation squares. □

41.9 THEOREM

For any category $\mathscr{A}$, *the following are equivalent:*

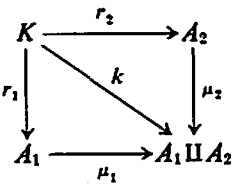
(1) $\mathscr{A}$ *is abelian.*

(2) $\mathscr{A}$ *has pullbacks, pushouts, and a zero object, and in* $\mathscr{A}$ *each monomorphism is a normal monomorphism and each epimorphism is a normal epimorphism.*

Proof: Clearly since abelian categories are finitely complete and finitely cocomplete, (1) implies (2). Conversely, the conditions of (2) clearly imply that $\mathscr{A}$ is finitely complete and finitely cocomplete (23.7 and dual); and hence exact (39.19). Let A_1 and A_2 be $\mathscr{A}$-objects. Consider the morphism

$$A_1 \amalg A_2 \xrightarrow{h = 1_{A_1} \oplus 1_{A_2}} A_1 \times A_2.$$

By part (2) of the lemma, $(A_1, \mu_1) \approx Ker(\pi_2 \circ h)$. Similarly $(A_2, \mu_2) \approx Ker(\pi_1 \circ h)$. Let $(K, k) \approx Ker(h)$. Then since $\pi_2 \circ h \circ k = 0 = \pi_1 \circ h \circ k$, there exist morphisms r_1 and r_2 such that the diagram



commutes.

Hence by part (3) of the lemma, $k = 0$.

Dually, it can be shown that $Cok(h) = 0$, so that h is an isomorphism (39.15). Hence $\mathscr{A}$ has finite biproducts. □

41.10 PROPOSITION

Let $A_1 \underset{\mu_2}{\overset{\mu_1}{\rightrightarrows}} B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A_2$ *be morphisms in an abelian category. Then* $(\mu_1, \mu_2, B, \pi_1, \pi_2)$ *is a biproduct if and only if the following conditions are satisfied:*

(1) $\pi_1 \circ \mu_1 = 1_{A_1}$ *and* $\pi_2 \circ \mu_2 = 1_{A_2}$.

(2) *The sequences*

$$A_1 \xrightarrow{\mu_1} B \xrightarrow{\pi_2} A_2 \quad \text{and} \quad A_2 \xrightarrow{\mu_2} B \xrightarrow{\pi_1} A_1$$

are exact.

Proof: That (1) and (2) are satisfied if we have a biproduct is immediate from the definition of biproduct and Corollary 41.8. Conversely, suppose that (1) and (2) are satisfied. Then clearly, for $k, j = 1, 2$,

$$\pi_k \circ \mu_j = \delta_{jk}.$$

Let

$$f = 1_B - (\mu_1 \circ \pi_1) - (\mu_2 \circ \pi_2).$$

Then

$$\pi_1 \circ f = (\pi_1 \circ 1_B) - (\pi_1 \circ \mu_1 \circ \pi_1) - (\pi_1 \circ \mu_2 \circ \pi_2) = \pi_1 - \pi_1 - 0 = 0,$$

and

$$\pi_2 \circ f = \pi_2 \circ 1_B = (\pi_2 \circ \mu_1 \circ \pi_1) - (\pi_2 \circ \mu_2 \circ \pi_2) = \pi_2 - 0 - \pi_2 = 0.$$

By (1) μ_1 is a section, so that $\mu_1 \approx Im(\mu_1)$. Thus by exactness,

$$\mu_1 \approx Im(\mu_1) \approx Ker(\pi_2).$$

Hence since $\pi_2 \circ f = 0$, there exists a unique morphism h such that $\mu_1 \circ h = f$. Now

$$h = 1_{A_1} \circ h = \pi_1 \circ \mu_1 \circ h = \pi_1 \circ f = 0.$$

Thus $0 = \mu_1 \circ h = f = 1_B - (\mu_1 \circ \pi_1) - (\mu_2 \circ \pi_2)$.

Consequently, $1_B = \mu_1 \circ \pi_1 + \mu_2 \circ \pi_2$, so that $(\mu_1, \mu_2, B, \pi_1, \pi_2)$ is a biproduct (40.8). □

Exact Functors

41.11 THEOREM

Every (half) exact functor $F: \mathscr{A} \to \mathscr{B}$ between abelian categories is additive.

Proof: Let (P, π_1, π_2) be a product in $\mathscr{A}$ of the $\mathscr{A}$-objects A_1 and A_2. Complete this to a biproduct $(\mu_1, \mu_2, P, \pi_1, \pi_2)$ (40.9). Then $\pi_1 \circ \mu_1 = 1_{A_1}$, $\pi_2 \circ \mu_2 = 1_{A_2}$ and the sequences

$$0 \to A_1 \xrightarrow{\mu_1} P \xrightarrow{\pi_2} A_2 \to 0$$

and

$$0 \to A_2 \xrightarrow{\mu_2} P \xrightarrow{\pi_1} A_1 \to 0$$

are exact.

Therefore, since F is a functor

$$F(\pi_1) \circ F(\mu_1) = F(1_{A_1}) = 1_{F(A_1)}$$

$$F(\pi_2) \circ F(\mu_2) = F(1_{A_2}) = 1_{F(A_2)}$$

and since F is (half) exact, the sequences

$$F(A_1) \xrightarrow{F(\mu_1)} F(P) \xrightarrow{F(\pi_2)} F(A_2)$$

and

$$F(A_2) \xrightarrow{F(\mu_2)} F(P) \xrightarrow{F(\pi_1)} F(A_1)$$

are exact.

Thus by the above proposition (41.10)

$$(F(\mu_1), F(\mu_2), F(P), F(\pi_1), F(\pi_2))$$

is a biproduct.

Hence F preserves finite products so that it is additive (40.16). □

41.12 THEOREM

Let $F: \mathscr{A} \to \mathscr{B}$ be a zero-preserving functor between abelian categories. Then the following are equivalent:

(1) *F is exact.*

(2) *F preserves finite limits and finite colimits.*

(3) *F preserves pullbacks and epimorphisms.*

(4) *F preserves kernels and epimorphisms.*

Proof:

(1) ⇒ *(2)*. By the above theorem (41.11) each exact functor is additive, and so preserves finite products and finite coproducts (40.16). However, exact functors also preserve kernels and cokernels (39.24); and hence, in this case, also equalizers and coequalizers (40C and its dual). Consequently F preserves finite limits and finite colimits (23.7 and its dual).

(2) ⇒ *(3)*. Immediate since pullbacks are particular finite limits and in an exact category f is an epimorphism if and only if $f \approx Cok(Ker\, f)$ (39.15).

(3) $\Rightarrow$ *(4)*. Since F preserves pullbacks and the zero object, it must preserve finite limits (23.7) and, hence, kernels.
(4) $\Rightarrow$ *(1)*. Immediate from Proposition 39.25. □

Module-Valued Functors

For the next three propositions assume that $\mathscr{A}$ is an abelian category with object A. $Hom(A, A)$ is considered to be the ring $\hat{R}$ and $Hom(A, \_\_): \mathscr{A} \to$ **Mod-**$\hat{R}$ to be the functor introduced in Proposition 40.19.

41.13 PROPOSITION
$Hom(A, \_\_)$ is left-exact (39L).

Proof: Suppose that $0 \to B \xrightarrow{f} C \xrightarrow{g} D$ is an exact sequence in $\mathscr{A}$; i.e., $f \approx Ker(g)$. We wish to show that

$$Hom(A, f) \approx Ker(Hom(A, g));$$

i.e., that

$$0 \to Hom(A, B) \xrightarrow{Hom(A,f)} Hom(A, C) \xrightarrow{Hom(A,g)} Hom(A, D)$$

is exact.

Let $h \in Hom(A, B)$. Then $Hom(A, g) \circ Hom(A, f)(h) = g \circ f \circ h = 0$, since $g \circ f = 0$. Thus $Hom(A, g) \circ Hom(A, f) = 0$. Also $Hom(A, f)$ is a monomorphism since f is. Now suppose that $k: M \to Hom(A, C)$ such that $Hom(A, g) \circ k = 0$; i.e., for each $x \in M$, $g \circ (k(x)) = 0$. Then since $f \approx Ker(g)$, for each $x \in M$ there exists a unique morphism $y_x: A \to B$ such that $f \circ y_x = k(x)$. Since k is a linear transformation and f is a monomorphism, it is easy to see that $y: M \to Hom(A, B)$ defined by $y(x) = y_x$ is a linear transformation such that $Hom(A, f) \circ y = k$, and is unique with respect to this property. □

41.14 PROPOSITION
$Hom(A, \_\_)$ is faithful if and only if A is a separator for $\mathscr{A}$. □

41.15 PROPOSITION
The following are equivalent:

(1) *$Hom(A, \_\_)$ is exact.*
(2) *$Hom(A, \_\_)$ preserves epimorphisms.*
(3) *A is a projective object in $\mathscr{A}$.*

Proof:

(1) $\Leftrightarrow$ *(2)*. Since $Hom(A, \_\_)$ preserves kernels (41.13), it is exact if and only if it preserves epimorphisms (39.25).
(2) $\Leftrightarrow$ *(3)*. Since the forgetful functor $\hat{U}$: **Mod-**$\hat{R} \to$ **Set** obviously preserves and reflects epimorphisms, $Hom(A, \_\_)$ preserves epimorphisms if and only if $hom(A, \_\_) = \hat{U} \circ Hom(A, \_\_)$ does; i.e., if and only if A is projective (12.14). □

It should be remarked that the small abelian categories can be characterized as precisely those categories $\mathscr{A}$ for which there exists an exact embedding $E: \mathscr{A} \hookrightarrow$ **Ab**; or, equivalently, precisely those categories $\mathscr{A}$ for which there is some ring R and a full, exact embedding $E: \mathscr{A} \hookrightarrow R$-**Mod**. The proofs of these assertions are not within the scope of this presentation. However, we are now able to give a categorical characterization for the categories of R-modules.

41.16 THEOREM

Let $(\mathscr{A}, U)$ be a concrete category. Then the following are equivalent:

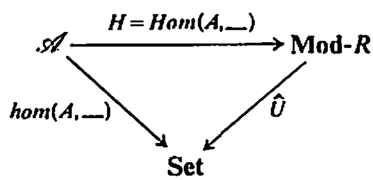
(1) *$(\mathscr{A}, U)$ is equivalent*† *to* (**Mod**-$\hat{R}$, $\hat{U}$) *for some ring* $\hat{R}$.

(2) *$(\mathscr{A}, U)$ is a finitary algebraic abelian category.*

Proof:

(1) ⇒ *(2)*. Clear.

(2) ⇒ *(1)*. Since $(\mathscr{A}, U)$ is algebraic, $U \approx hom(A, \_\!\_)$ for some $\mathscr{A}$-object A (30.20). Let $\hat{R}$ be the ring of morphisms $Hom(A, A)$ (40.19). Then the triangle



commutes (where $\hat{U}$ is the usual forgetful functor for **Mod**-$\hat{R}$). It remains to be shown that $H = Hom(A, \_\!\_)$ is an equivalence. Since $(\mathscr{A}, hom(A, \_\!\_))$ and (**Mod**-$\hat{R}$, $\hat{U}$) are algebraic, H must be algebraic (32.20); in particular, it is faithful (32.17) and preserves (regular) epimorphisms; i.e., is exact (41.15). Also, since $(\mathscr{A}, U)$ is finitary, H must preserve coproducts (40.21).

To show that H is full, let X be any $\mathscr{A}$-object and let $\mathscr{B}_X$ be the class of all $\mathscr{A}$-objects Y for which the restriction of H to

$$hom_{\mathscr{A}}(Y, X) \to hom_{\text{Mod-}\hat{R}}(H(Y), H(X))$$

is surjective.

First of all, $A \in \mathscr{B}_X$. To see this, let $f: H(A) \to H(X)$ be a linear transformation. Consider $f(1_A): A \to X$. Then for each $h: A \to A$,

$$H(f(1_A))(h) = f(1_A) \circ h = f(1_A \circ h) = f(h).$$

Thus $H(f(1_A)) = f$. Therefore $A \in \mathscr{B}_X$. Next let I be an index set and $(\mu_i, {}^I A)$ be the Ith copower of A. Then ${}^I A$ belongs to $\mathscr{B}_X$. To see this, let

$$g: H({}^I A) \to H(X)$$

be a linear transformation. Since H preserves coproducts, $(H(\mu_i), H({}^I A))$ is the Ith copower of $H(A)$. Now since $A \in \mathscr{B}_X$ it follows that for each $i \in I$

† I.e., there exists an equivalence $H: \mathscr{A} \to$ **Mod**-$\hat{R}$ such that U and $\hat{U} \circ H$ are naturally isomorphic (14J).

there exists a morphism $g_i\colon A \to X$ such that $H(g_i) = g \circ H(\mu_i)$. Since $(\mu_i, {}^I A)$ is a coproduct, there exists a unique morphism $\bar{g}\colon {}^I A \to X$ such that $\bar{g} \circ \mu_i = g_i$ for each $i \in I$. Hence, for each $i \in I$,

$$H(\bar{g}) \circ H(\mu_i) = H(\bar{g} \circ \mu_i) = H(g_i) = g \circ H(\mu_i)$$

so that since coproducts are epi-sinks, $H(\bar{g}) = g$. Therefore ${}^I A \in \mathscr{B}_X$. Finally if (q, Q) is a regular quotient of some element Y of $\mathscr{B}_X$, then Q belongs to $\mathscr{B}_X$. To see this, suppose that $g\colon H(Q) \to H(X)$ is a linear transformation. Since $Y \in \mathscr{B}_X$, there is some morphism $\hat{g}\colon Y \to X$ such that $H(\hat{g}) = g \circ H(q)$. Thus, since H is faithful, there exists a morphism $\bar{g}\colon Q \to X$ such that $H(\hat{g}) = H(\bar{g}) \circ H(q)$. Consequently, since H preserves epimorphisms, $H(\bar{g}) = g$.

Now since A represents U, A is free over a singleton set (31.4), so that the objects of the form ${}^I A$ are exactly the free objects of $\mathscr{A}$ (31C). Thus the members of $\mathscr{B}_X$ are precisely the $\mathscr{A}$-objects (31.9). Hence $Hom(A, \_)$ is full.

It remains to be shown that H is dense; i.e., that each right $\hat{R}$-module M is isomorphic to $H(X)$ for some $\mathscr{A}$-object X. To see this, observe that there exist linear transformations

$${}^I\hat{R} \overset{f}{\underset{g}{\rightrightarrows}} {}^K\hat{R} \xrightarrow{q} M$$

where $(q, M) \approx Coeq(f, g)$. Furthermore, since H is full and preserves coproducts, there are $\mathscr{A}$-morphisms

$${}^I A \overset{\bar{f}}{\underset{\bar{g}}{\rightrightarrows}} {}^K A$$

such that $H(\bar{f}) = f$ and $H(\bar{g}) = g$. Let $(\bar{q}, Q)$ be the coequalizer of $\bar{f}$ and $\bar{g}$ in $\mathscr{A}$. Since H is exact, it preserves coequalizers (41.12) so that $H(Q)$ must be isomorphic to M. □

EXERCISES

41A. Prove that a category $\mathscr{A}$ is abelian if and only if it is exact and has finite products.

41B. In any abelian category $\mathscr{A}$, consider the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow[r]{} & D \end{array}$$

and the sequence

$$A \xrightarrow{\langle f,g\rangle} B \oplus C \xrightarrow{[-s,r]} D.$$

Prove the following:

(a) $[-s, r] \circ \langle f, g \rangle = 0$ if and only if the square commutes.

(b) $0 \to A \xrightarrow{\langle f,g \rangle} B \oplus C \xrightarrow{[-s,r]} D$ is exact if and only if the square is a pullback square.

(c) $A \xrightarrow{\langle f,g \rangle} B \oplus C \xrightarrow{[-s,r]} D \to 0$ is exact if and only if the square is a pushout square.

(d) $0 \to A \xrightarrow{\langle f,g \rangle} B \oplus C \xrightarrow{[-s,r]} D \to 0$ is exact if and only if the square is a pulation square.

41C. Prove that in abelian categories the pullback of an epimorphism is an epimorphism (cf. 21.13 and 21N). [*Hint:* Use 41B and 39D dual.]

41D. Prove that if $(\mathscr{A}, U)$ is an algebraic abelian category and $\mathscr{B}$ is a small full, exact subcategory of $\mathscr{A}$, then there is some ring R and a full, exact embedding

$$E: \mathscr{B} \hookrightarrow \textbf{Mod-}R.$$

[*Hint:* There is some $\mathscr{A}$-object A such that $U \approx hom(A, \_)$, and there is some set I such that each $\mathscr{B}$-object is a quotient of ${}^IA$. Let $R = Hom({}^IA, {}^IA)$, and let E be the restriction of $Hom({}^IA, \_)$ to $\mathscr{B}$.]

41E. Prove that if $\mathscr{A}$ is semiadditive, then $Add[\mathscr{A}, \textbf{Ab}]$ is abelian.

41F. Prove that for any ring R, considered as a one object semiadditive category, $Add[R, \textbf{Ab}] \approx R\textbf{-Mod}$.

41G. Show that the (concrete) category of compact abelian groups is algebraic and abelian but not finitary.