X

Reflective Subcategories

Every theorem in category theory is either a pushover or its dual—a put-on.
——W. E. ANON

In Chapter VII we claimed that adjoint situations are abundant throughout mathematics and that adjoint functor theorems belong to the most useful results of category theory. In Chapter VIII we concentrated our study on the special case of set-valued functors and their adjoints. The results of Chapter IX concerning $(\mathscr{E}, \mathscr{M})$ categories, (extremal) subobjects and factorizations now enable us to study in more detail the special case of embedding functors and their left and right adjoints.

§36 GENERAL REFLECTIVE SUBCATEGORIES

Definitions and General Properties

36.1 DEFINITION

Let $\mathscr{A}$ be a subcategory of $\mathscr{B}$ with embedding functor $E: \mathscr{A} \hookrightarrow \mathscr{B}$:

(1) An E-universal map (r_B, A_B) for a $\mathscr{B}$-object B is called an **$\mathscr{A}$-reflection of B.**

(2) $\mathscr{A}$ is called **reflective in $\mathscr{B}$** or a **reflective subcategory of $\mathscr{B}$** if and only if there exists an $\mathscr{A}$-reflection for each $\mathscr{B}$-object; i.e., if and only if E has a left adjoint, $R: \mathscr{B} \to \mathscr{A}$. In this case, R is called a **reflector for $\mathscr{A}$.**

(3) If $\mathscr{E}$ is a class of $\mathscr{B}$-morphisms, then $\mathscr{A}$ is called **$\mathscr{E}$-reflective in $\mathscr{B}$** provided that for each $\mathscr{B}$-object B there exists an $\mathscr{A}$-reflection (r_B, A_B) such that each $r_B \in \mathscr{E}$. For the case that $\mathscr{E}$ is the class of all epimorphisms [resp. monomorphisms; extremal epimorphisms] of $\mathscr{B}$ we say that $\mathscr{A}$ is **epireflective** [resp. **monoreflective**; **(extremal epi)-reflective**] **in $\mathscr{B}$.**

DUAL NOTIONS: (i.e., with respect to E co-universal maps and right adjoints to E) **$\mathscr{A}$-coreflection of B**; **coreflective in $\mathscr{B}$** (or a **coreflective subcategory of $\mathscr{B}$**); **coreflector for $\mathscr{A}$**; **$\mathscr{E}$-coreflective in $\mathscr{B}$**, especially **monocoreflective** [resp. **epi-coreflective, (extremal mono)-coreflective**] **in $\mathscr{B}$**.

36.2 EXAMPLES

Each of the examples of reflections given in 26.2(2) is in fact an epireflection. Also each of the examples of co-universal maps given in 26.2(4) actually describes a mono-coreflective situation. In addition:

(1) If $\mathscr{B}$ is the category of ordered fields and order-preserving field homomorphisms, and $\mathscr{A}$ is the full subcategory of $\mathscr{B}$ consisting of all real-closed fields, then $\mathscr{A}$ is reflective in $\mathscr{B}$.

(2) If $\mathscr{B}$ is the category **Field** of all fields and all field homomorphisms and $\mathscr{A}$ is the full subcategory of $\mathscr{B}$ consisting of all algebraically closed fields, then $\mathscr{A}$ is *not* a reflective in $\mathscr{B}$. (Why not?)

(3) If $\mathscr{B}$ is the category **TopGrp** of topological groups and continuous homomorphisms and $\mathscr{A}$ is the full subcategory of $\mathscr{B}$ consisting of all compact Hausdorff groups, then $\mathscr{A}$ is reflective in $\mathscr{B}$. (The reflector is called the **Bohr compactification** functor.)

36.3 PROPOSITION

Each full monoreflective subcategory of a category is also epireflective.

Proof: Let $\mathscr{A}$ be monoreflective in $\mathscr{B}$, let $B \xrightarrow{r_B} A_B$ be the $\mathscr{A}$-reflection of B and let $A_B \underset{t}{\overset{s}{\rightrightarrows}} C$ be $\mathscr{B}$-morphisms with the property $s \circ r_B = t \circ r_B$.

If $C \xrightarrow{r_C} A_C$ is the $\mathscr{A}$-reflection of C, then

$$r_C \circ s \circ r_B = r_C \circ t \circ r_B.$$

Since A_C belongs to $\mathscr{A}$, this equation together with the uniqueness property in the definition of universal maps implies that $r_C \circ s = r_C \circ t$. Since r_C is a monomorphism, it follows that $s = t$. Hence r_B is an epimorphism. □

36.4 PROPOSITION

If $\mathscr{A}$ is a subcategory of $\mathscr{B}$, then the following are equivalent:

(1) *$\mathscr{A}$ is a full subcategory of $\mathscr{B}$.*

(2) *For each $\mathscr{A}$-object A, the pair $(1_A, A)$ is an $\mathscr{A}$-reflection of A.* □

36.5 PROPOSITION

Let $\mathscr{A}$ be a full reflective subcategory of $\mathscr{B}$ with embedding $E\colon \mathscr{A} \hookrightarrow \mathscr{B}$ and reflector $R\colon \mathscr{B} \to \mathscr{A}$. Then

(1) *$R \circ E \cong 1_{\mathscr{A}}$, and*

(2) *$(E \circ R) \circ (E \circ R) \cong E \circ R$.* □

The last two propositions show that if $\mathscr{A}$ is a full reflective subcategory of $\mathscr{B}$ with embedding E and reflector R, then $E \circ R$ is "quasi idempotent" and "projects" $\mathcal{B}$ onto $\mathcal{A}$. Also if (r, A) is an $\mathcal{A}$-reflection of B, then A is an $\mathcal{A}$-object that in a certain sense "best approximates" B in $\mathcal{A}$ (via the morphism r). All of this is false for reflective subcategories that are not full. Because of these and many other results that are true for full reflective subcategories but not for reflective subcategories in general, and because of the fact that most reflective subcategories "occurring in nature" are full and isomorphism-closed, we adopt the following:

36.6 CONVENTION

Throughout the remainder of this chapter, all subcategories will be assumed to be both **full** and **isomorphism-closed**. In particular, $\mathcal{A}$ will be assumed to be a full, isomorphism-closed subcategory of $\mathcal{B}$, with embedding $E: \mathcal{A} \hookrightarrow \mathcal{B}$.

As will become evident in that which follows, relatively little is known about general reflective subcategories. However, a satisfactory theory of epireflective subcategories has emerged.

Relationship to Subobjects

36.7 PROPOSITION

Let B be a $\mathcal{B}$-object, and let (r, A) be an $\mathcal{A}$-reflection for B. Then the following are equivalent:

(1) *r is a monomorphism (resp. extremal monomorphism; section).*

(2) *B is a subobject (resp. extremal subobject; sect) of some $\mathcal{A}$-object.*

Proof: Clearly (1) implies (2). That (2) implies (1) follows from the fact that whenever $f = g \circ h$ is a monomorphism (resp. extremal monomorphism; section), then h is also a monomorphism (resp. extremal monomorphism; section) (6.5, 17C, 5.5). ☐

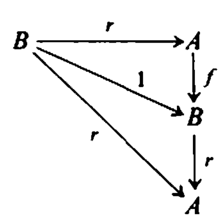
36.8 PROPOSITION

If (r, A) is an $\mathcal{A}$-reflection for B, then the following are equivalent:

(1) *r is an isomorphism.*

(2) *r is a section.*

Proof: If $f \circ r = 1$, then the diagram



commutes, so that by the uniqueness property in the definition of universal map, $r \circ f = 1$. Thus r is an isomorphism. ☐

36.9 COROLLARY

If $\mathscr{A}$ is reflective in $\mathscr{B}$, then $\mathscr{A}$ is closed under the formation of sects in $\mathscr{B}$.

Proof: Immediate from the Propositions 36.7, 36.8, and the fact that $\mathscr{A}$ is isomorphism-closed in $\mathscr{B}$. □

36.10 PROPOSITION

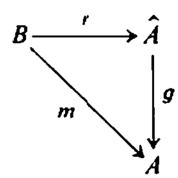
If $\mathscr{A}$ is epireflective in $\mathscr{B}$, then $\mathscr{A}$ is closed under the formation of extremal subobjects in $\mathscr{B}$.

Proof: Let $f: B \to A$ be an extremal monomorphism in $\mathscr{B}$, where A is an $\mathscr{A}$-object. If $(r, \hat{A})$ is the $\mathscr{A}$-reflection of B, then there exists some morphism g such that $f = g \circ r$. Since r is an epimorphism and f is extremal, r must be an isomorphism. Thus, since $\mathscr{A}$ is isomorphism-closed, B must be an $\mathscr{A}$-object. □

36.11 PROPOSITION

If $\mathscr{B}$ is an $(\mathscr{E}, \mathscr{M})$ category and $\mathscr{A}$ is $\mathscr{E}$-reflective in $\mathscr{B}$, then $\mathscr{A}$ is closed under the formation of $\mathscr{M}$-subobjects in $\mathscr{B}$.

Proof: Let $m: B \to A$ be a morphism in $\mathscr{M}$, where A is an $\mathscr{A}$-object. If $(r, \hat{A})$ is the $\mathscr{A}$-reflection of B, then there exists some morphism g such that the triangle



commutes.

Thus, since $m \in \mathscr{M}$ and $r \in \mathscr{E}$, r must be an isomorphism (33.6). □

Relationship to Limits

If $\mathscr{A}$ is reflective in $\mathscr{B}$, the embedding functor $E: \mathscr{A} \hookrightarrow \mathscr{B}$ has a left adjoint, so (among other things) it preserves limits. I.e., if $D: I \to \mathscr{A}$ has a limit $(L, (l_i))$, then $(L, (l_i))$ is also the limit of $E \circ D: I \to \mathscr{B}$. Next we will see that E also "detects" limits.

36.12 Recall that if $E: \mathscr{A} \hookrightarrow \mathscr{B}$ is a full embedding functor and I is a category, then $\mathscr{A}$ is said to be **closed under the formation of I-limits in $\mathscr{B}$** provided that for each functor $D: I \to \mathscr{A}$ and each limit $(L, (l_i))$ of $E \circ D$, the following equivalent conditions are fulfilled:

(1) L is an $\mathscr{A}$-object.

(2) $(L, (l_i))$ is a limit of D.

DUAL NOTION: **closed under the formation of I^{op}-colimits in $\mathscr{B}$** (23.5).

36.13 THEOREM

If $\mathscr{A}$ is reflective in $\mathscr{B}$, then for each category I, $\mathscr{A}$ is closed under the formation of I-limits in $\mathscr{B}$.

Proof: Let $D: I \to \mathscr{A}$ be a functor, $(L, (l_i))$ be the limit of $E \circ D$, and $r: L \to A_L$ be the $\mathscr{A}$-reflection of L. By the definition of universal map, for each l_i there exists a unique $\bar{l}_i: A_L \to D(i)$ such that $l_i = \bar{l}_i \circ r$. From the uniqueness of the morphisms $\bar{l}_i$ it follows that $(A_L, (\bar{l}_i))$ is a natural source for $E \circ D$. Thus there exists some $h: A_L \to L$ such that for each $i, \bar{l}_i = l_i \circ h$. Hence for each i,

$$l_i \circ h \circ r = \bar{l}_i \circ r = l_i = l_i \circ 1,$$

so that since $(L, (l_i))$ is a mono-source (20.4), $h \circ r = 1$. Consequently, r is a section, hence an isomorphism (36.8), so that since $\mathscr{A}$ is isomorphism-closed, L is an $\mathscr{A}$-object. □

36.14 COROLLARY

Each reflective subcategory of an I-complete category is I-complete. □

For epireflective subcategories we know even more. Even if a functor is only "initially in $\mathscr{A}$" then its limit object in $\mathscr{B}$ must belong to $\mathscr{A}$ (36.15). As we will see later, this property actually characterizes the epireflective subcategories of "nice" categories (37.2).

36.15 DEFINITION

A functor $D: I \to \mathscr{B}$ is said to be **initially in $\mathscr{A}$** provided that for each I-object i there is an I-object j such that $D(j) \in Ob(\mathscr{A})$ and $hom(j, i) \neq \varnothing$. $\mathscr{A}$ is said to be **strongly closed under the formation of I-limits in $\mathscr{B}$** provided that for each functor $D: I \to \mathscr{B}$ that is initially in $\mathscr{A}$ and each limit $(L, (l_i))$ of D, the object L is an $\mathscr{A}$-object.

DUAL NOTIONS: **finally in $\mathscr{A}$; strongly closed under the formation of I^{op}-colimits in $\mathscr{B}$.**

36.16 THEOREM

If $\mathscr{A}$ is epireflective in $\mathscr{B}$, then for each category I, $\mathscr{A}$ is strongly closed under the formation of I-limits in $\mathscr{B}$.

Proof: Suppose that $D: I \to \mathscr{B}$ is initially in $\mathscr{A}$, $(L, (l_i))$ is a limit of D and $r: L \to A_L$ is the $\mathscr{A}$-reflection of L. Let J be the class of all I-objects j for which $D(j)$ is an $\mathscr{A}$-object. Then for each $j \in J$ there exists a unique morphism $\bar{l}_j: A_L \to D(j)$ such that $l_j = \bar{l}_j \circ r$. Since D is initially in $\mathscr{A}$, for each I-object i there is some $j_i \in J$ and some $f_i: j_i \to i$. Now for each I-object i, let

$$\bar{l}_i = D(f_i) \circ \bar{l}_{j_i}.$$

Since r is an epimorphism it can be shown without difficulty that $\bar{l}_i$ is independent of the choice of f_i and j_i and that $(A_L, (\bar{l}_i))$ is a natural source for D.

Hence there is a morphism $h: A_L \to L$ such that for each i, $l_i \circ h = \bar{l}_i$. Consequently

$$l_i \circ h \circ r = \bar{l}_i \circ r = D(f_i) \circ \bar{l}_{j_i} \circ r = D(f_i) \circ l_{j_i} = l_i = l_i \circ 1,$$

so that since $(L, (l_i))$ is a mono-source (20.4), $h \circ r = 1$. It follows that r is an isomorphism (36.8), so that since $\mathscr{A}$ is isomorphism-closed, L is an $\mathscr{A}$-object. □

Relationship to Colimits

36.17 PROPOSITION

If $\mathscr{A}$ is reflective in $\mathscr{B}$ (with embedding $E: \mathscr{A} \hookrightarrow \mathscr{B}$ and reflector $R: \mathscr{B} \to \mathscr{A}$), and if $D: I \to \mathscr{A}$ is a functor, $((k_i), K)$ is the colimit of $E \circ D$, and $r: K \to A_K$ is the $\mathscr{A}$-reflection of K, then $((r \circ k_i), A_K)$ is the colimit of D.

Proof: This is immediate since the sink

$$((r \circ k_i), A_K)$$

is the sink

$$((R(k_i)), R(K)),$$

and R, having a right adjoint, must preserve colimits (27.7). □

36.18 COROLLARY

Each reflective subcategory of an I-cocomplete category is I-cocomplete. □

We have seen that a reflective subcategory $\mathscr{A}$ of a complete and cocomplete category $\mathscr{B}$ is both complete and cocomplete (36.14, 36.18) and that the limits in $\mathscr{A}$ are formed in the same way as the limits in $\mathscr{B}$ (36.13). However, $\mathscr{A}$ is not necessarily closed under the formation of *colimits* in $\mathscr{B}$; i.e., if $((k_i), K)$ is a colimit of $E \circ D$ for some $D: I \to \mathscr{A}$, $((k_i), K)$ is not necessarily the colimit of D. To see this, let $(A_i)_I$ be an infinite family of compact Hausdorff spaces. The coproduct of this family in $\mathbf{Top}_2$ is the disjoint topological sum, which is not compact, even though $\mathbf{CompT}_2$ is epireflective in $\mathbf{Top}_2$. The coproduct of the family $(A_i)_I$ in $\mathbf{CompT}_2$ is actually the Stone-Čech compactification of the disjoint topological sum (as is evident from Proposition 36.17).

EXERCISES

36A. *Non-Full Subcategories*

Consider the following two (non-full) subcategories $\mathscr{A}$ and $\mathscr{B}$ of **POS**. Objects of $\mathscr{A}$ (resp. $\mathscr{B}$) are those partially ordered sets which have the property that each subset (resp. non-empty subset) has a supremum. A monotone function between two $\mathscr{A}$-objects (resp. $\mathscr{B}$-objects) belongs to $\mathscr{A}$ (resp. $\mathscr{B}$) if and only if it preserves suprema of all subsets (resp. non-empty subsets). Prove the following:

(a) $\mathscr{A}$ and $\mathscr{B}$ are each (extremal mono)-reflective in **POS**, but neither is epireflective in **POS**. (Does this contradict Proposition 36.3?)

(b) For no $\mathscr{A}$-object A is the pair $(1_A, A)$ an $\mathscr{A}$-reflection of A. (What is the $\mathscr{A}$-reflection of a complete lattice?)

(c) $(1_B, B)$ is a $\mathscr{B}$-reflection of the $\mathscr{B}$-object B if and only if B is inversely well-ordered.

36B. Prove that for a complete category $\mathscr{B}$ the following are equivalent:

(a) For each small category I, $\mathscr{A}$ is strongly closed under the formation of I-limits in $\mathscr{B}$.
(b) $\mathscr{A}$ is closed under the formation of products in $\mathscr{B}$, and $\mathscr{A}$ is strongly closed under the formation of equalizers in $\mathscr{B}$ (cf. Theorem 23.8).

36C. In the proof of Theorem 36.16 show that $(A_L, (\bar{l}_i))$ is a natural source for D.

36D. Let $\mathscr{C}$ be a category which has a subcategory $\mathscr{A}$ that is cocomplete as a category in its own right. Let $\mathscr{B}$ be the full subcategory of $\mathscr{C}$ consisting of all those $\mathscr{C}$-objects for which there exists an $\mathscr{A}$-reflection. Show that $\mathscr{B}$ is a cocomplete subcategory of $\mathscr{C}$ (23.5).

§37 CHARACTERIZATION AND GENERATION OF $\mathscr{E}$-REFLECTIVE SUBCATEGORIES

Even though we are not able to give a satisfactory characterization of reflective subcategories of even such nice categories as **Top**, we will in this section characterize the $\mathscr{E}$-reflective (in particular the epireflective) subcategories of "nice" categories. This will naturally lead to the concept of the smallest $\mathscr{E}$-reflective subcategory of a category that contains a given class of objects of the category (called the $\mathscr{E}$-reflective hull of the class).

Recall that throughout this chapter $\mathscr{A}$ is considered to be a (full, isomorphism-closed) subcategory of the category $\mathscr{B}$. Also in this section $\mathscr{E}$ (resp. $\mathscr{M}$) will denote a class of epimorphisms (resp. monomorphisms) that is closed under composition with isomorphisms.

37.1 CHARACTERIZATION THEOREM I

If $\mathscr{B}$ is an $\mathscr{E}$-co-(well-powered) $(\mathscr{E}, \mathscr{M})$ category that has products, then the following are equivalent:

(1) *$\mathscr{A}$ is $\mathscr{E}$-reflective in $\mathscr{B}$.*

(2) *$\mathscr{A}$ is closed under the formation of products and $\mathscr{M}$-subobjects in $\mathscr{B}$.*

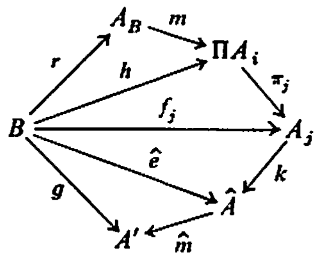
Proof: That (1) implies (2) is immediate from results of the last section (36.13 and 36.11). To show the converse, suppose that (2) is satisfied and B is a $\mathscr{B}$-object. We need only find an $\mathscr{A}$-reflection for B. Since $\mathscr{B}$ is $\mathscr{E}$-co-(well-powered), there is a representative set $(f_i, A_i)_I$ of quotient objects of $\mathscr{B}$ of the form (f, A) where $f \in \mathscr{E}$ and $A \in Ob(\mathscr{A})$. Let $(\Pi A_i, \pi_i)$ be the product in $\mathscr{B}$ of the family $(A_i)_I$. By hypothesis ΠA_i is an $\mathscr{A}$-object, and by the definition of product there is a $\mathscr{B}$-morphism $h\colon B \to \Pi A_i$ such that for each i, $\pi_i \circ h = f_i$. Now let

$$B \xrightarrow{h} \Pi A_i = B \xrightarrow{r} A_B \xrightarrow{m} \Pi A_i$$

be the $(\mathscr{E}, \mathscr{M})$-factorization of h. We claim that (r, A_B) is the $\mathscr{A}$-reflection of B. To see this, let $g\colon B \to A'$ be a $\mathscr{B}$-morphism, where A' is an $\mathscr{A}$-object. If

$$B \xrightarrow{g} A' = B \xrightarrow{\hat{e}} \hat{A} \xrightarrow{\hat{m}} A'$$

is the $(\mathscr{E}, \mathscr{M})$-factorization of g, then there is some $j \in I$ and some isomorphism $k: A_j \to \hat{A}$ such that $\hat{e} = k \circ f_j$. Hence the diagram



commutes, and since r is an epimorphism, $\hat{m} \circ k \circ \pi_j \circ m$ is the unique morphism x from A_B to A' for which $x \circ r = g$. Thus (r, A_B) is an $\mathscr{A}$-reflection for B. □

37.2 CHARACTERIZATION THEOREM II

If $\mathscr{B}$ is complete, well-powered, and co-(well-powered), then the following are equivalent:

(1) *$\mathscr{A}$ is epireflective in $\mathscr{B}$.*

(2) *$\mathscr{A}$ is strongly closed under the formation of I-limits in $\mathscr{B}$ for each small category I.*

(3) *$\mathscr{A}$ is strongly closed under the formation of products and pullbacks in $\mathscr{B}$.*

(4) *$\mathscr{A}$ is strongly closed under the formation of products and inverse images in $\mathscr{B}$.*

(5) *$\mathscr{A}$ is strongly closed under the formation of products and finite intersections in $\mathscr{B}$.*

(6) *$\mathscr{A}$ is strongly closed under the formation of products and intersections in $\mathscr{B}$.*

(7) *$\mathscr{A}$ is strongly closed under the formation of products and inverse images of extremal monomorphisms in $\mathscr{B}$.*

(8) *$\mathscr{A}$ is strongly closed under the formation of products and finite intersections of extremal subobjects in $\mathscr{B}$.*

(9) *$\mathscr{A}$ is strongly closed under the formation of products and intersections of extremal subobjects in $\mathscr{B}$.*

(10) *$\mathscr{A}$ is strongly closed under the formation of products and equalizers in $\mathscr{B}$.*

(11) *$\mathscr{A}$ is closed under the formation of (extremal mono)-sources; i.e., if $(B, (f_i))$ is an (extremal mono)-source such that the codomain of each f_i is an $\mathscr{A}$-object, then B is an $\mathscr{A}$-object.*

(12) *$\mathscr{A}$ is closed under the formation of products and extremal subobjects in $\mathscr{B}$.*

Proof: The proof of the equivalence of conditions (2) through (10) is left as an exercise which can be accomplished in a manner analogous to the proof for the characterizations of completeness (23.8). It remains to be shown that (2) ⇒ (1) ⇒ (11) ⇒ (12) ⇒ (10).

(2) ⇒ (1). Let $f: B \to A$ be a $\mathscr{B}$-morphism with $A \in Ob(\mathscr{A})$, and let $\mathscr{M}$ be the class of all $\mathscr{A}$-morphisms that are monomorphisms in $\mathscr{B}$. Since $\mathscr{A}$ is strongly

closed under the formation of equalizers and intersections, $\mathscr{M}$ satisfies the conditions for $\mathscr{M}$ in the Factorization Lemma (34.3). Thus f has an (epi, $\mathscr{M}$)-factorization. Now let $(B \xrightarrow{e_i} A_i)_I$ be a set-indexed representative family of all epimorphisms with domain B and codomain some $\mathscr{A}$-object. By the factorization property proved above, $(e_i, A_i)_I$ is a solution set for B. Also (by (2)) the inclusion functor preserves limits. Hence by the First Adjoint Functor Theorem (28.3), the inclusion has a left adjoint; i.e., $\mathscr{A}$ is reflective in $\mathscr{B}$. To show that it is epireflective, suppose that $r: B \to A_B$ is the $\mathscr{A}$-reflection of B and

$$B \xrightarrow{r} A_B = B \xrightarrow{e} \bar{A} \xrightarrow{m} A_B$$

is its (epi, $\mathscr{M}$)-factorization. By the universality there exists some $h: A_B \to \bar{A}$ such that $h \circ r = e$. Hence

$$m \circ h \circ r = m \circ e = r = 1 \circ r$$

so that by the uniqueness condition for universal maps, $m \circ h = 1$. Thus m is a retract and a monomorphism, hence an isomorphism. Consequently, r is an epimorphism.

(1) $\Rightarrow$ *(11)*. Let $(B, B \xrightarrow{f_i} A_i)$ be an (extremal mono)-source where each A_i is an $\mathscr{A}$-object. If $B \xrightarrow{r} A_B$ is an $\mathscr{A}$-epireflection for B, then for each i there is a morphism $\bar{f}_i: A_B \to A_i$ such that $f_i = \bar{f}_i \circ r$. This then provides a factorization for the source, so that since r is an epimorphism, it must be an isomorphism. Hence B is an $\mathscr{A}$-object.

(11) $\Rightarrow$ *(12)*. This is trivial since products and extremal subobjects are special cases of (extremal mono)-sources.

(12) $\Rightarrow$ *(10)*. This is immediate since every regular monomorphism is extremal. □

The above characterization theorems guarantee the existence of most of the reflections given in the examples 26.2(2) (independently of any special constructions). In particular, a subcategory of **Grp** (resp. **Top**, $\mathbf{Top}_2$, etc.) is epireflective if and only if it is closed under the formation of products and subgroups (resp. products and subspaces, products and closed subspaces, etc.).

In many cases (e.g., in **Top**) all coreflective subcategories are automatically monocoreflective. The reason for this is shown in the following theorem.

37.3 CHARACTERIZATION THEOREM III

If $\mathscr{B}$ is cocomplete, well-powered, and co-(well-powered) and if $\mathscr{A}$ contains a separator for $\mathscr{B}$, then the following are equivalent:

(1) *$\mathscr{A}$ is coreflective in $\mathscr{B}$.*

(2) *$\mathscr{A}$ is both monocoreflective and epicoreflective in $\mathscr{B}$.*

(3) *$\mathscr{A}$ is strongly closed under the formation of I-colimits in $\mathscr{B}$ for each small category I.*

(4) *$\mathscr{A}$ is closed under the formation of colimits in $\mathscr{B}$.*

(5) *$\mathscr{A}$ is closed under the formation of coproducts and coequalizers in $\mathscr{B}$.*

(6) *$\mathscr{A}$ is strongly closed under the formation of coproducts and coequalizers in $\mathscr{B}$.*

(7) *$\mathscr{A}$ is closed under the formation of coproducts and extremal quotient objects in $\mathscr{B}$.*

Proof: We will show (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (1). That (3) ⇒ (4) and (4) ⇒ (5) is immediate. Furthermore the implications (2) ⇒ (3), (6) ⇒ (7), and (7) ⇒ (1) follow immediately from the dual statement of the Characterization Theorem II (37.2). Thus we need only show that (1) ⇒ (2) and (5) ⇒ (6).

(1) ⇒ (2). If S is a separator for $\mathscr{B}$ that belongs to $\mathscr{A}$, then each $\mathscr{B}$-object is a quotient of some copower ${}^IS$ of S (19.6 dual) that again belongs to $\mathscr{A}$ (37.2 dual). Hence $\mathscr{A}$ must be epicoreflective in $\mathscr{B}$ (36.7 dual). Thus $\mathscr{A}$ must also be monocoreflective in $\mathscr{B}$ (36.3 dual).

(5) ⇒ (6). Let (c, C) be a coequalizer of $B \underset{g}{\overset{f}{\rightrightarrows}} A$, where A is an $\mathscr{A}$-object. It is sufficient to show that C is an $\mathscr{A}$-object. If S is an $\mathscr{A}$-object that is a separator for $\mathscr{B}$, then there exists some copower ${}^IS$ of S and an epimorphism $e\colon {}^IS \to B$ (19.6 dual). Since e is an epimorphism, (c, C) must be the coequalizer of ${}^IS \underset{g\circ e}{\overset{f\circ e}{\rightrightarrows}} A$ (16B dual). Since ${}^IS$ and A both belong to $\mathscr{A}$ (37.2 dual), the hypothesis (5) implies that C belongs to $\mathscr{A}$. □

We now turn our attention to the notion of generation of $\mathscr{E}$-reflective subcategories of "nice" categories.

37.4 THEOREM

If $\mathscr{B}$ is an $(\mathscr{E}, \mathscr{M})$ category that has products and is $\mathscr{E}$-co-(well-powered), then

(1) *The intersection of any class of $\mathscr{E}$-reflective subcategories of $\mathscr{B}$ is also $\mathscr{E}$-reflective in $\mathscr{B}$.*

(2) *Each subcategory $\mathscr{A}$ of $\mathscr{B}$ can be embedded in a smallest $\mathscr{E}$-reflective subcategory $\mathscr{E}(\mathscr{A})$ of $\mathscr{B}$, the objects of which are precisely the $\mathscr{M}$-subobjects of products of $\mathscr{A}$-objects in $\mathscr{B}$.*

Proof: The first assertion, the existence of $\mathscr{E}(\mathscr{A})$, and the fact that $\mathscr{E}(\mathscr{A})$ contains all $\mathscr{M}$-subobjects of products of $\mathscr{A}$-objects in $\mathscr{B}$ follow immediately from the Characterization Theorem I (37.1). The reverse containment follows from the fact that under these circumstances the class $\mathscr{M}$ is closed under the formation of products and compositions (33F and 33.1). □

37.5 DEFINITION

If $\mathscr{A}$ is a subcategory of $\mathscr{B}$, $\mathscr{E}$ is a class of epimorphisms of $\mathscr{B}$, and $\mathscr{A}$ is contained in a smallest $\mathscr{E}$-reflective subcategory $\mathscr{E}(\mathscr{A})$ of $\mathscr{B}$, then $\mathscr{E}(\mathscr{A})$ is called the **$\mathscr{E}$-reflective hull of $\mathscr{A}$ in $\mathscr{B}$**. For the case that $\mathscr{E}$ is the class of *all* epimorphisms of $\mathscr{B}$, $\mathscr{E}(\mathscr{A})$ is called the **epireflective hull of $\mathscr{A}$ in $\mathscr{B}$**.

DUAL NOTIONS: **$\mathscr{M}$-coreflective hull of $\mathscr{A}$ in $\mathscr{B}$**; **monocoreflective hull of $\mathscr{A}$ in $\mathscr{B}$**.

37.6 COROLLARY

If $\mathscr{B}$ is complete, well-powered and co-(well-powered), then

(1) *The intersection of any class of epireflective subcategories of $\mathscr{B}$ is epireflective in $\mathscr{B}$.*

(2) *Each subcategory $\mathscr{A}$ of $\mathscr{B}$ has an epireflective hull in $\mathscr{B}$ whose objects are precisely all extremal subobjects of products of $\mathscr{A}$-objects.*

Proof: Immediate from the fact that under the given hypotheses, $\mathscr{B}$ is an (epi, extremal mono) category (34.5). □

37.7 EXAMPLES

(1) $\mathbf{CompT_2}$ is the epireflective hull of the closed unit interval [0, 1] in $\mathbf{Top_2}$.

(2) $\mathbf{CRegT_2}$ is the epireflective hull of [0, 1] (or of $\mathbf{R}$) in **Top**.

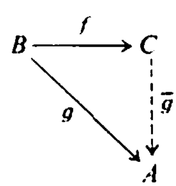
(3) If X is the topological space with two points and three open sets, then the epireflective hull of X in **Top** is the category of all T_0 spaces.

(4) The monocoreflective hull of $\mathbf{Z}_n$ in **Ab** is the subcategory of all abelian groups G with the property that $nG = \{0\}$.

(5) The monocoreflective hull of the category of all finite abelian groups in **Ab** is the category of all abelian torsion groups.

37.8 DEFINITION

A $\mathscr{B}$-morphism $f: B \to C$ is called **$\mathscr{A}$-extendable** provided that for each $\mathscr{B}$-morphism $g: B \to A$, where A is an $\mathscr{A}$-object, there exists some $\bar{g}: C \to A$ such that the triangle



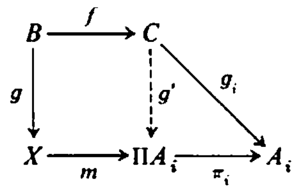
commutes.

37.9 PROPOSITION

Let $\mathscr{B}$ be an $(\mathscr{E}, \mathscr{M})$ category that has products and is $\mathscr{E}$-co-(well-powered), and let $\mathscr{E}(\mathscr{A})$ be the $\mathscr{E}$-reflective hull of $\mathscr{A}$. Then any morphism f in $\mathscr{E}$ is $\mathscr{A}$-extendable if and only if it is $\mathscr{E}(\mathscr{A})$-extendable.

Proof: Clearly since $\mathscr{A}$ is contained in $\mathscr{E}(\mathscr{A})$, each $\mathscr{E}(\mathscr{A})$-extendable morphism is also $\mathscr{A}$-extendable. To show the converse, suppose that $f: B \to C$ is a morphism in $\mathscr{E}$ that is $\mathscr{A}$-extendable, X is an object of $\mathscr{E}(\mathscr{A})$ and $g: B \to X$ is a $\mathscr{B}$-morphism. According to Theorem 37.4, there is a set-indexed family $(A_i)_I$ of $\mathscr{A}$-objects and a morphism $m: X \to \Pi A_i$ that belongs to $\mathscr{M}$. Since f is $\mathscr{A}$-extendable, for each $i \in I$ there is a morphism $g_i: C \to A_i$ such that $g_i \circ f = \pi_i \circ m \circ g$. Now by the definition of product there is a morphism $g': C \to \Pi A_i$

such that $g_i = \pi_i \circ g'$ for each i. Since $(\Pi A_i, \pi_i)$ is a mono-source, the square in the diagram



commutes. Hence by the $(\mathscr{E}, \mathscr{M})$-diagonalization property, there exists a morphism $\bar{g}: C \to X$ such that $\bar{g} \circ f = g$. Consequently f is $\mathscr{E}(\mathscr{A})$-extendable. ☐

37.10 COROLLARY

If $\mathscr{B}$ is complete, well-powered and co-(well-powered) and if $\mathscr{C}$ is the epireflective hull of $\mathscr{A}$ in $\mathscr{B}$, then a $\mathscr{B}$-epimorphism is $\mathscr{A}$-extendable if and only if it is $\mathscr{C}$-extendable. ☐

EXERCISES

37A. Prove the equivalence of items (2) through (10) in the Characterization Theorem II (37.2).

37B. Prove that a full subcategory of $\mathbf{CompT_2}$ is reflective in $\mathbf{CompT_2}$ if and only if it is cocomplete as a category in its own right.

37C. *Fitting Properties*

Let $\mathscr{A} \hookrightarrow \mathscr{B} \hookrightarrow \mathscr{C}$ be full embeddings and let $\mathscr{A}$ be monoreflective in $\mathscr{C}$.

(a) Prove that the following are equivalent:

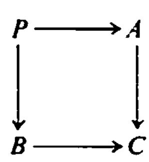
(i) $\mathscr{B}$ is reflective in $\mathscr{C}$.

(ii) $\mathscr{B}$ is monoreflective in $\mathscr{C}$.

(iii) $\mathscr{B}$ is epireflective in $\mathscr{C}$.

(b) Show that if $\mathscr{C}$ is finitely complete, then (1) and (2) below are equivalent:

(1) If



is a pullback square, A is an $\mathscr{A}$-object, and B is a $\mathscr{B}$-object, then P must be a $\mathscr{B}$-object.

(2) (α) If $X \xrightarrow{m} B$ is a regular monomorphism and B is a $\mathscr{B}$-object, then X must be a $\mathscr{B}$-object, and

(β) If A is an $\mathscr{A}$-object and B is a $\mathscr{B}$-object, then $A \times B$ must be a $\mathscr{B}$-object.

(If $\mathscr{B}$ has these properties, it is called $\mathscr{A}$-**fitting.**)

(c) Now let $\mathscr{M}$ be some class of monomorphisms in $\mathscr{C}$ that is left-cancellable (i.e., whenever $f \circ g \in \mathscr{M}$, then $g \in \mathscr{M}$) and assume that $\mathscr{A}$ is $\mathscr{M}$-reflective in $\mathscr{C}$. Prove that (i), (ii), and (iii) above are equivalent to the following:

(iv) $\mathscr{B}$ is $\mathscr{M}$-reflective in $\mathscr{C}$.

If, in addition, $\mathscr{C}$ is complete, well-powered, and co-(well-powered), and $\mathscr{M}$ is closed under the formation of intersections and pullbacks and has the property that whenever $f \circ g$ is an epimorphism and $f \in \mathscr{M}$, then g is an epimorphism; show that (i), (ii), (iii), and (iv) above are equivalent to the following:

(v) $\mathscr{B}$ is closed under the formation of products and extremal subobjects.

(vi) $\mathscr{B}$ is $\mathscr{A}$-fitting and strongly closed under the formation of intersections.

(vii) $\mathscr{B}$ is $\mathscr{A}$-fitting and closed under the formation of intersections.

(viii) $\mathscr{B}$ is closed under the formation of finite products and arbitrary intersections.

(d) Assume that $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{M}$ have the above properties, $\mathscr{B}$ is $\mathscr{A}$-fitting, and $\bar{\mathscr{B}}$ is the epireflective hull of $\mathscr{B}$ in $\mathscr{C}$. Prove that the $\bar{\mathscr{B}}$-reflection of a $\mathscr{C}$-object C can be obtained as the intersection of all those $\mathscr{M}$-subobjects of the $\mathscr{A}$-reflection of C that belong to $\mathscr{B}$ and contain C.

(e) Apply the above results to the case where $\mathscr{A}$ = **CompT$_2$**, $\mathscr{C}$ = **CRegT$_2$**, and $\mathscr{M}$ = the topological embeddings.

37D. (a) Prove that a full, isomorphism-closed subcategory of **Top** that is simultaneously reflective and coreflective in **Top** coincides with **Top**.

(b) Characterize all full, isomorphism-closed subcategories of **Ab** that are simultaneously epireflective and monocoreflective in **Ab** [Cf. Exercise 23C(c)].

37E. *Reflective Hulls*

Let $\mathscr{C}$ be complete, well-powered and co-(well-powered), $\mathscr{A}$ a full isomorphism-closed subcategory of $\mathscr{C}$, and $\mathscr{B}$ the epireflective hull of $\mathscr{A}$ in $\mathscr{C}$.

Prove that:

(a) $\mathscr{B}$ is complete and well-powered.

(b) If $\mathscr{B}$ is co-(well-powered), then $\mathscr{A}$ has a "reflective hull" in $\mathscr{C}$ that coincides with the epireflective hull of $\mathscr{A}$ in $\mathscr{B}$.

37F. Let $\mathscr{A}$ be a reflective subcategory of a complete, well-powered, co-(well-powered) category $\mathscr{B}$, let $R: \mathscr{B} \to \mathscr{A}$ be the reflector, and let $r_B: B \to R(B)$ be the $\mathscr{A}$-reflection of B. Show that for any $\mathscr{B}$-epimorphism $f: B \to C$, the following are equivalent:

(a) f is $\mathscr{A}$-extendable.

(b) $R(f)$ is an isomorphism.

(c) There exists a $\mathscr{B}$-morphism $g: C \to R(B)$ such that $g \circ f = r_B$.

(d) There exists a $\mathscr{B}$-morphism $g: C \to R(B)$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ {\scriptstyle r_B}\downarrow & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle r_C} \\ R(B) & \xrightarrow[R(f)]{} & R(C) \end{array}$$

commutes.

37G. Let $\mathscr{B}$ be an $(\mathscr{E}, \mathscr{M})$ category that has products and is $\mathscr{E}$-co-(well-powered), let $\mathscr{E}(\mathscr{A})$ be the $\mathscr{E}$-reflective hull of $\mathscr{A}$, and let B be a $\mathscr{B}$-object. Prove that the following are equivalent:

(a) B is an object of $\mathscr{E}(\mathscr{A})$.
(b) B is an $\mathscr{M}$-subobject of a product of $\mathscr{A}$-objects.
(c) Each $\mathscr{A}$-extendable $\mathscr{E}$-morphism in $\mathscr{B}$ is $\{B\}$-extendable.
(d) Each $\mathscr{A}$-extendable $\mathscr{E}$-morphism $f: B \to C$ is an isomorphism.
(e) Each $\mathscr{A}$-extendable morphism $f: B \to C$ is an $\mathscr{M}$-morphism.

37H. If $\mathscr{B}$ is complete, well-powered, and co-(well-powered), and if $\mathscr{C}$ is the epireflective hull of $\mathscr{A}$ in $\mathscr{B}$, then show that for each $\mathscr{B}$-object B, the following are equivalent:

(a) B is a $\mathscr{C}$-object.
(b) B is an extremal subobject of a product of $\mathscr{A}$-objects.
(c) Each $\mathscr{A}$-extendable epimorphism in $\mathscr{B}$ is $\{B\}$-extendable.
(d) Each $\mathscr{A}$-extendable epimorphism $f: B \to C$ is an isomorphism.
(e) Each $\mathscr{A}$-extendable morphism $f: B \to C$ is an extremal monomorphism.

§38 ALGEBRAIC SUBCATEGORIES

In this section we concern ourselves with the question of when a subcategory of an algebraic (or varietal (38.3)) category is itself algebraic (or varietal). (Recall that in this chapter all subcategories are assumed to be both full and isomorphism-closed.)

38.1 THEOREM

If $(\mathscr{B}, U)$ is an algebraic category and $\mathscr{A}$ is a subcategory of $\mathscr{B}$ with embedding $E: \mathscr{A} \hookrightarrow \mathscr{B}$, then the following are equivalent:

(1) *$(\mathscr{A}, U \circ E)$ is algebraic.*

(2) *$\mathscr{A}$ is reflective in $\mathscr{B}$ and contains with each morphism its (regular epi, mono)-factorization in $\mathscr{B}$.*

(3) *$\mathscr{A}$ is reflective in $\mathscr{B}$ and contains each $\mathscr{B}$-object that is simultaneously a subobject of some $\mathscr{A}$-object and a regular quotient of some $\mathscr{A}$-object.*

Proof: Clearly (2) and (3) are equivalent since each algebraic category is uniquely (regular epi, mono)-factorizable (32.13). To show that (1) implies (2), assume that $(\mathscr{A}, U \circ E)$ is algebraic. Then E must be an algebraic functor (32.20). Hence E has a left adjoint (so that $\mathscr{A}$ is reflective in $\mathscr{B}$) and E preserves the (regular epi, mono)-factorizations in $\mathscr{A}$ (32.18). Thus $\mathscr{A}$ contains with each morphism its (regular epi, mono)-factorization in $\mathscr{B}$.

To show that (2) implies (1) it is sufficient (since the composition of algebraic functors is algebraic) to show that $\mathscr{A}$ has coequalizers and that E is algebraic. By hypothesis E preserves regular epimorphisms and since $\mathscr{A}$ is reflective in $\mathscr{B}$, E has a left adjoint. Also $\mathscr{B}$, being an algebraic category, is cocomplete (32.14), so that since $\mathscr{A}$ is reflective in $\mathscr{B}$, it must be cocomplete

(36.18). Thus $\mathcal{A}$ has coequalizers. It remains to show that E reflects regular epimorphisms. Let $g: A \to \hat{A}$ be an $\mathcal{A}$-morphism that is a regular epimorphism in $\mathcal{B}$. Since $\mathcal{B}$ is complete (32.12), we can form the congruence relation of g in $\mathcal{B}$.

$$\begin{array}{ccc} B & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[g]{} & \hat{A} \end{array}$$

Thus $(g, \hat{A})$ is the coequalizer in $\mathcal{B}$ of p and q (21.11). Now since $\mathcal{A}$ is reflective in $\mathcal{B}$, E must reflect limits (36.13). Thus the above pullback square belongs to $\mathcal{A}$. Consequently $(g, \hat{A})$ is the coequalizer in $\mathcal{A}$ of p and q. Hence E reflects regular epimorphisms. $\square$

38.2 COROLLARY

If $(\mathcal{B}, U)$ is an algebraic category and $\mathcal{A}$ is a subcategory of $\mathcal{B}$, with embedding $E: \mathcal{A} \hookrightarrow \mathcal{B}$, such that $\mathcal{A}$ is closed under the formation of subobjects in $\mathcal{B}$, then the following are equivalent:

(1) *$(\mathcal{A}, U \circ E)$ is algebraic.*

(2) *$\mathcal{A}$ is reflective in $\mathcal{B}$.*

(3) *$\mathcal{A}$ is a complete subcategory of $\mathcal{B}$.*

(4) *$\mathcal{A}$ is closed under the formation of products in $\mathcal{B}$.*

Proof: By the theorem (1) and (2) are equivalent. The equivalence of (2), (3), and (4) follows from the Characterization Theorem II (37.2). $\square$

It should be remarked that it is not always true that if $\mathcal{A}$ is an epireflective subcategory of an algebraic category $(\mathcal{B}, U)$, with embedding $E: \mathcal{A} \hookrightarrow \mathcal{B}$, then $(\mathcal{A}, U \circ E)$ is algebraic.

38.3 DEFINITION

An algebraic category $(\mathcal{B}, U)$ is called **varietal** provided that U reflects congruence relations.

38.4 EXAMPLES

The following algebraic categories are varietal: **Set**, **pSet**, **SGrp**, **Mon**, **Grp**, **Rng**, ***R*-Mod**, ***R*-Alg**, **BooAlg**, **CompT$_2$**, compact abelian groups, and (commutative) C^*-algebras [together with the unit disc functor (30H)].

The following algebraic categories are not varietal: torsion-free abelian groups, zero-dimensional compact Hausdorff spaces.

38.5 THEOREM

If $(\mathcal{B}, U)$ is varietal and $\mathcal{A}$ is a subcategory of $\mathcal{B}$ with embedding $E: \mathcal{A} \hookrightarrow \mathcal{B}$, then the following are equivalent:

(1) *$(\mathcal{A}, U \circ E)$ is varietal.*

(2) $\mathscr{A}$ is reflective in $\mathscr{B}$ and if

$$\begin{array}{ccc} A & \xrightarrow{p} & \bar{A} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ \bar{A} & \xrightarrow[f]{} & B \end{array}$$

is a pullback square in $\mathscr{B}$ where f is a regular epimorphism and A and $\bar{A}$ are $\mathscr{A}$-objects, then B is an $\mathscr{A}$-object.

Proof:

(1) ⇒ (2). Since ($\mathscr{A}$, $U \circ E$) is varietal, and hence algebraic, $\mathscr{A}$ must be reflective in $\mathscr{B}$ and $E\colon \mathscr{A} \hookrightarrow \mathscr{B}$ must preserve regular epimorphisms (38.1). If

$$\begin{array}{ccc} A & \xrightarrow{p} & \bar{A} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ \bar{A} & \xrightarrow[f]{} & B \end{array}$$

is a pullback square in $\mathscr{B}$ and f is a regular epimorphism, then (p, q) is a congruence relation for f and (f, B) is a coequalizer of (p, q) (21.16). Since U preserves and $U \circ E$ reflects congruence relations, (p, q) is a congruence relation in $\mathscr{A}$. If $(c, C) \approx Coeq(p, q)$ in $\mathscr{A}$, then

$$\begin{array}{ccc} A & \xrightarrow{p} & \bar{A} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle c} \\ \bar{A} & \xrightarrow[c]{} & C \end{array}$$

is a pullback square in $\mathscr{A}$ (and hence in $\mathscr{B}$) and c is a regular epimorphism in $\mathscr{A}$ (and hence in $\mathscr{B}$). Consequently (c, C) is a coequalizer of (p, q) in $\mathscr{B}$ (21.16). Therefore B and C are isomorphic, which implies that B belongs to $\mathscr{A}$.

(2) ⇒ *(1)*. We first will show that ($\mathscr{A}$, $U \circ E$) is algebraic by showing that it contains with each morphism, f, its (regular epi, mono)-factorization in $\mathscr{B}$ (38.1). To see this let

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

be a pullback square in $\mathscr{B}$. Since $\mathscr{A}$ is reflective in $\mathscr{B}$, P must be in $\mathscr{A}$ (36.13). Now let (c, C) be the coequalizer of p and q in $\mathscr{B}$ and let $h\colon C \to B$ be the unique $\mathscr{B}$-morphism with $f = h \circ c$. Then $f = h \circ c$ is the (regular epi, mono)-factorization of f in $\mathscr{B}$ (see the proof of Theorem 32.3) and the square

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle c} \\ A & \xrightarrow[c]{} & C \end{array}$$

is a pullback square in $\mathscr{B}$ (21.16). Thus by (2) C is an $\mathscr{A}$-object, so that h and c belong to $\mathscr{A}$. Consequently, $(\mathscr{A}, U \circ E)$ is algebraic. It remains to be shown that E reflects congruence relations. Let (p, q) be a congruence relation in $\mathscr{B}$ where each of p and q are $\mathscr{A}$-morphisms and let (c, C) be the coequalizer of p and q in $\mathscr{B}$. Then

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle c} \\ B & \xrightarrow[c]{} & C \end{array}$$

is a pullback square in $\mathscr{B}$ (21.16) so that by (2) it is a pullback square in $\mathscr{A}$. Hence (p, q) is a congruence relation in $\mathscr{A}$. □

38.6 COROLLARY

If $(\mathscr{B}, U)$ is varietal and $\mathscr{A}$ is a subcategory of $\mathscr{B}$ with embedding $E: \mathscr{A} \hookrightarrow \mathscr{B}$ and if $\mathscr{A}$ is closed under the formation of subobjects in $\mathscr{B}$, then the following are equivalent:

(1) $(\mathscr{A}, U \circ E)$ *is varietal.*

(2) $\mathscr{A}$ *is reflective in $\mathscr{B}$ and is closed under the formation of regular quotients in $\mathscr{B}$.*

(3) $\mathscr{A}$ *is closed under the formation of products in $\mathscr{B}$ and regular quotients in $\mathscr{B}$.*

Proof:

(1) $\Rightarrow$ *(2).* Clearly $\mathscr{A}$ must be reflective in $\mathscr{B}$. Suppose that $f: A \to B$ is a regular epimorphism in $\mathscr{B}$ and A is an $\mathscr{A}$-object. Let

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

be a pullback square in $\mathscr{A}$. Then by the canonical construction for pullbacks (21.3), P is the object part of a subobject of $A \times A$. Thus since $\mathscr{A}$ is reflective in $\mathscr{B}$, P must be an $\mathscr{A}$-object (36.14). Hence by the theorem, B must be an $\mathscr{A}$-object.

(2) ⇒ (3). This is immediate since reflective subcategories are closed under the formation of I-limits, for each discrete category I (36.13).
(3) ⇒ (1). By Corollary 38.2 it is clear that $(\mathscr{A}, U \circ E)$ is algebraic. That it is also varietal follows immediately from the theorem. □

A varietal subcategory of a varietal category is not necessarily closed under the formation of subobjects of regular quotients. (See Exercise 38E.)

EXERCISES

38A. Prove that a varietal category is finitary if and only if it is strongly finitary (see 22E and 32G).

38B. Show that in each finitary varietal category direct limits and finite limits commute (cf. 25B and 32G).

38C. Show that $\mathbf{CompT_2}$ is the only non-trivial† full epireflective subcategory of $\mathbf{Top_2}$ that (considered as a concrete category) is varietal.

38D. *Identities*

For any category $\mathscr{A}$, a pair of $\mathscr{A}$-morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$ with common domain and common codomain is called a **quasi-identity in** $\mathscr{A}$. It is called an **identity** provided that B is regular-projective. A quasi-identity is said to **hold** in an $\mathscr{A}$-object C if and only if $k \circ f = k \circ g$ for each morphism $k: B \to C$. If $\mathscr{F}$ is a class of quasi-identities in $\mathscr{A}$, then the full subcategory of $\mathscr{A}$ whose objects are precisely those $\mathscr{A}$-objects for which each quasi-identity in $\mathscr{F}$ holds, is denoted by $\mathscr{A}(\mathscr{F})$.

Now suppose that $(\mathscr{A}, U)$ is algebraic and $\mathscr{B}$ is a full subcategory of $\mathscr{A}$.

(a) Prove that the following are equivalent:
  (i) $\mathscr{B}$ is closed under the formation of products and subobjects.
  (ii) There exists a class $\mathscr{F}$ of quasi-identities in $\mathscr{A}$ such that $\mathscr{B} = \mathscr{A}(\mathscr{F})$.

(b) Prove that the following are equivalent:
  (i) $\mathscr{B}$ is closed under the formation of products, subobjects, and regular quotient objects.
  (ii) There exists a class $\mathscr{F}$ of identities in $\mathscr{A}$ such that $\mathscr{B} = \mathscr{A}(\mathscr{F})$.

38E. Let $(\mathscr{B}, U)$ be $\mathbf{CompT_2}$ and let A be a compact Hausdorff space with at least two points and the property that the identity on A is the only non-constant continuous self-map of A. (Such a space is called **strongly rigid.**)

(a) Prove that the only non-constant continuous maps from a power A^I to A are the projections.

(b) Show that the full subcategory $\mathscr{A}$ of $\mathscr{B}$ whose objects are the powers A^I of A satisfies condition (2) of Theorem 38.5.

(c) Conclude that if $E: \mathscr{A} \hookrightarrow \mathscr{B}$ is the embedding, then $(\mathscr{A}, U \circ E)$ is varietal.

(d) Show, however, that $\mathscr{A}$ is not closed under the formation of subobjects or regular quotients in $\mathscr{B}$.

† A subcategory of $\mathbf{CompT_2}$ is non-trivial if it has a space whose underlying set has at least two points.