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Nonlinear Existence Theorems in Nonnormable Analysis

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ABSTRACT. We obtain generalizations in nonnormable setting of the Banach fixed point theorem (parametrized version), the existence-uniqueness theorem for differential equations (parametrized version), the inverse function theorem and implicit function theorem. Our results embody new formulations of these theorems which reduce in Banach spaces to the classical versions. But in even the simplest nonnormable spaces they are significantly different from other known generalizations.

Introduction

Let us recall a typical counterexample for demonstrating failure of the usual inverse function theorem for nonnormable Fréchet spaces. Let $E = \mathcal{C}(\mathbb{R}, \mathbb{R})$, the space of continuous functions in its topology of uniform convergence on compact subsets. Define $f : E \to E$ by putting $f(x)(\xi) = x(\xi)^2$ and let $a(\xi) \geq \mu > 0$. Although the derivative of f, given by $(Df(x) \cdot h)(\xi) = 2x(\xi)h(\xi)$, is invertible at a, the map f carries every open neighborhood V of a to an image $f(V)$ whose members, being squares x^2, are all nonnegative. Thus $f(V)$ is as far from open as it can possibly be: its interior is empty. Accordingly, the inverse function theorem for E — in its classical formulation — seems as far from true as can be. And yet, the function f is 'locally' invertible at a and its inverse, the function $g(y)(\xi) = y(\xi)^{1/2}$, even looks smooth. We are taking 'locally at a' to mean in some 'quasiball' $B(a, \delta) = \{x \in E \mid a(\xi) - \delta < x(\xi) < a(\xi) + \delta\}$. Note that $B(a, \delta)$ also has empty interior. It is not hard to find numerous similar functions with inverses which are smooth looking but defined only on a domain with empty interior. The preceding situation motivated us to develop in [15] a new differentiation theory applicable to such domains with empty interior.

This work is an original contribution and will not appear elsewhere.

The main purpose of the present paper is to establish a new type of inverse function theorem, based on the mentioned differentiation theory, for which the above counterexample becomes an illustrative example. We apply this theorem to certain nonlinear differential operators whose inverse functions are not explicitly known; their values are global solutions which depend differentiably on the right hand side of the equation. A parametrized fixed point theorem is first established as general tool for existence proofs in nonnormable setting. We demonstrate its versatility by using it also for an existence-uniqueness theorem for first order differential equations with continuous dependence on the initial value.

Let us indicate briefly the heuristics of our approach. The proof of the parametrized fixed point theorem in a Banach space E uses balls $B(a,\delta)$. In Banach spaces $E = \mathcal{C}(K,\mathbb{R})$, where K is compact, such $B(a,\delta)$ can be given the two descriptions $\{x \in E \mid \|x-a\| < \delta\}$ and $\{x \in E \mid a(\xi)-\delta < x(\xi) < a(\xi)+\delta\}$. In a locally convex space $E = \mathcal{C}(W,\mathbb{R})$, where W is locally compact but not compact, the first description is no longer available, but the second is and gives 'quasiballs', which can be regarded as limits of balls. Indeed, for reasonable choices of a category $\mathcal{C}$ of continuous maps the insertion maps $\mathsf{ins}_K : K \to W$ of compact subspaces should furnish an inductive colimit diagram in $\mathcal{C}$. The contravariant functor $\mathcal{C}(-,\mathbb{R})$ (for a good choice of $\mathcal{C}$) should transform this inductive colimit to a diagram $\mathcal{C}(W,\mathbb{R}) \to \mathcal{C}(K,\mathbb{R})$ which expresses $\mathcal{C}(W,\mathbb{R})$ as a projective limit of Banach spaces. Similarly, the 'quasiball' mentioned above should be representable as a projective limit of balls in Banach spaces. If one restricts attention to limit induced maps between such 'quasiballs', certain proofs that work for balls in $\mathcal{C}(K,\mathbb{R})$ should work 'in the limit' for quasiballs in $\mathcal{C}(W,\mathbb{R})$ to give corresponding results. At first glance this discussion may appear relevant only to the special function spaces mentioned. But every Banach space is well known to be representable as a closed subspace of some $\mathcal{C}(K,\mathbb{R})$ (with K compact) and every complete locally convex space as a closed subspace of some $\mathcal{C}(W,\mathbb{R})$ (with W locally compact), hence as a projective limit of Banach spaces. So the approach should work in principle at least in all complete locally convex spaces.

1. Categorical topological preparation

We begin by recalling relevant facts about two categories of continuous maps.

1a Convergence spaces

$\mathcal{C}_c$ will denote the category of convergence spaces and continuous maps. The axioms (essentially those of [2]) regulate filter convergence: (1) every point ultrafilter $\dot{x}$ converges to x; (2) if a filter Θ converges to x, then so does every refinement of Θ; (3) if Θ is a filter whose every ultrafilter refinement converges to x, then Θ converges to x. The later (more general) axioms of [9] and [4] would serve our purposes equally well. The first pages of [1] is a convenient basic reference for $\mathcal{C}_c$.

$\mathcal{C}_c$ has initial structures (hence final structures) and constant maps. The usual category Top of topological spaces has these properties too, but lacks the canonical mapping spaces $\mathcal{C}_c(X,Y)$ of $\mathcal{C}_c$. These spaces give the exponential laws (natural isomorphisms) $\dagger_{WXY} : \mathcal{C}_c(W \times X, Y) \to \mathcal{C}(W, \mathcal{C}(X,Y))$, $\dagger(f)(w)(x) = f(w,x)$, and $\S_{WXY} : \mathcal{C}(W,\mathcal{C}(X,Y)) \to \mathcal{C}(X,\mathcal{C}(W,Y))$, $\S(f)(x)(w) = f(w)(x)$. We will also write $f^\dagger$ for $\dagger(f)$. It follows from these laws that for each $\mathcal{C}_c$-space W the functor $\mathcal{C}_c(W,-) : \mathcal{C}_c \to \mathcal{C}_c$ is right adjoint to $W \times - : \mathcal{C}_c \to \mathcal{C}_c$ and $\mathcal{C}_c(-,Y) : \mathcal{C}_c^{op} \to \mathcal{C}_c$ is right adjoint to 'itself'; more precisely to $\mathcal{C}_c(-,Y)^{op} : \mathcal{C}_c \to \mathcal{C}_c^{op}$. For the first of these adjunctions we have the evaluation map $\mathsf{eval}_X^Y : \mathcal{C}(X,Y) \times X \to Y$, $\mathsf{eval}(f,x) = f(x)$ as counit.

When X and Y are topological spaces, $\mathcal{C}_c(X,Y)$ need not be topological in general but it is so in case X is locally compact and in that case it carries the compact-open topology. Note that our 'compact' includes 'Hausdorff'. When Y becomes replaced by a Banach space F, then the compact-open topology agrees with the topology of uniform convergence on compact sets. In particular, *for compact K and Banach F, $\mathcal{C}_c(K,F)$ is a Banach space.*

We always suppose the real field $\mathbb{R}$ to carry the usual metric topology. A linear $\mathcal{C}_c$-space is a $\mathcal{C}_c$-space E on which addition $E \times E \to E$ and scalar multiplication $\mathbb{R} \times E \to E$ has been defined so as to be $\mathcal{C}_c$-maps, subject to the usual linear space axioms. Such spaces together with linear continuous maps give rise to the category $\mathsf{L}\mathcal{C}_c$. Since products in Top correspond to those in $\mathcal{C}_c$, the usual category of linear topological spaces is a full subcategory of $\mathsf{L}\mathcal{C}_c$, in fact a reflective one. The same holds for the category Lcx of locally convex spaces. Therefore a (projective) limit of Banach spaces will be essentially the same thing, whether formed in $\mathsf{L}\mathcal{C}_c$ or in Lcx. For $\mathsf{L}\mathcal{C}_c$-spaces E and F the linear hom space $[E,F]$ is the $\mathsf{L}\mathcal{C}_c$-subspace of $\mathcal{C}_c(E,F)$ formed by the linear maps. $[E,\mathbb{R}]$, the canonical dual space, is also denoted $E^\star$.

For effective analysis one needs spaces which are separated and complete enough and which form a good category. This is provided by the subcategory $\mathsf{cL}\mathcal{C}_c \subset \mathsf{L}\mathcal{C}_c$ determined by all E for which $@_E : E \to E^{\star\star}$ is a closed embedding (also by the smaller category $\mathsf{oL}\mathcal{C}_c$ of optimal spaces, see [13]). It is a reflective subcategory which contains $\mathbb{R}$ and is preserved by the functors $[E,-]$ and $\mathcal{C}_c(X,-)$. (Had we defined $\mathcal{C}_c$ via the axioms of [9] and [4], the same categories c$\mathsf{L}\mathcal{C}_c$ and o$\mathsf{L}\mathcal{C}_c$ would have resulted).

All complete locally convex spaces E are reflexive in $\mathsf{L}\mathcal{C}_c$ i.e. the canonical map $@_E : E \to E^{\star\star}$, $@(x)(u) = u(x)$, is an isomorphism (see [1] or [12]). Hence all such E are c$\mathsf{L}\mathcal{C}_c$-spaces and projective limits of such spaces are again in c$\mathsf{L}\mathcal{C}_c$. A projective limit of Banach spaces is a Fréchet space whenever the index set is countable.

1b COMPACTLY GENERATED TOPOLOGICAL SPACES

$\mathcal{C}_k$ will denote the full subcategory of Top determined by all spaces X which happen to carry the final topology induced by the family of all continuous maps $f : K \to X$ with compact domain. It is a coreflective subcategory of Top, so its final structures and its colimits coincide with those of Top while its initial structures and limits are different in general. All metrizable and all locally compact topological spaces are $\mathcal{C}_k$-spaces.

$\mathcal{C}_k$ also upholds the exponential laws mentioned in 2a; the canonical mapping spaces $\mathcal{C}_k(X, Y)$ carry the $\mathcal{C}_k$-coreflection of the compact-open topology (see e.g. [11]). On spaces $\mathcal{C}_k(X, F)$, where F is a Banach space, the compact-open topology agrees with the topology of uniform convergence on compact subsets, but the resulting space need not be a $\mathcal{C}_k$-space, even when X is locally compact; however, it is a $\mathcal{C}_k$-space in important special cases e.g. when X is locally compact and the union of an increasing sequence of compact subsets K_n in such a way that every compact $L \subset X$ is contained in some K_n. Again, *for compact K and Banach F, $\mathcal{C}_k(K, F)$ is a Banach space.*

One builds the category $\mathsf{L}\mathcal{C}_k$ of linear $\mathcal{C}_k$-spaces and the subcategory c$\mathsf{L}\mathcal{C}_k$ of closed embeddable spaces much as for $\mathcal{C}_c$ (cf. 2a). But since products in Top do not correspond to products in $\mathcal{C}_k$ in general, a linear $\mathcal{C}_k$-space is a concept quite different from a linear topological space. Every Fréchet space (hence every Banach space) is a linear $\mathcal{C}_k$-space in the obvious way. Therefore, most of the examples of $\mathsf{L}\mathcal{C}_c$-spaces mentioned below are at the same time $\mathsf{L}\mathcal{C}_k$-spaces. Every Fréchet space is known to be canonically reflexive when viewed as a linear $\mathcal{C}_k$-space (see e.g. [5] or [12]), hence a c$\mathsf{L}\mathcal{C}_k$-space.

1c PREORDER CATEGORIES

Recall that a preorder category is a small category such that for every pair of objects i and j, there is at most one morphism $i \to j$, which will be denoted ij when it exists. Thus a preordered category always arises from a preordered set $(J, \geq)$ (reflexive transitive relation $\geq$). Let us look at a few relevant examples.

(1) The set $\mathbb{N}$ of natural numbers, equipped with the relation $\geq$ in the usual sense, is a preordered set, hence determines a preorder category whose non-

identity morphisms can be visualized as follows:

$$\ldots \longrightarrow j \xrightarrow{ji} i \ldots \longrightarrow 1 \longrightarrow 0$$

(2) Let X be a topological space. The set of compact subsets K, L, $\ldots$ of X form a preordered set under the usual inclusion relation $K \subset L$. The morphisms of the corresponding preorder category correspond to insertion maps $\mathsf{ins}_{KL} : K \to L$, $\mathsf{ins}(x) = x$, where $K \subset L$.

(3) If J is a preorder category, then the opposite category is again a preorder category. The order relation and direction of arrows just becomes reversed.

(4) On any set whatever we can introduce the discrete order relation (every point relates only to itself). The corresponding preorder category has only identity morphisms. The limit of a functor with such a domain is (by definition) a cartesian product.

2. The auxiliary categories

In what follows $\mathcal{C}$ will denote one of the categories $\mathcal{C}_c$ or $\mathcal{C}_k$ and J will denote a preorder category. We will use J and $\mathcal{C}$ to construct two new 'auxiliary' categories: $\mathsf{JL}\mathcal{C}$ and $\mathsf{J}\mathcal{C}$.

2a DEFINITIONS. The symbol $[E]$, where E is an $\mathsf{L}\mathcal{C}$-space, will denote the constant functor $\mathsf{J} \to \mathsf{L}\mathcal{C}$ which carries all objects to E and all morphisms to id_E. A $\mathsf{JL}\mathcal{C}$*-space* is a triple $(E, E_{(-)}, \pi^E)$ consisting of (1) an $\mathsf{L}\mathcal{C}$-space E, (2) a functor $E_{(-)} : \mathsf{J} \to \mathsf{L}\mathcal{C}$, the *approximation functor*, which transforms every $j \in \mathsf{J}$ into a Banach space E_j, the *approximand space*, (3) a natural transformation π^E from the constant functor $[E]$ to $E_{(-)}$ which forms a limit diagram in $\mathsf{L}\mathcal{C}$. Its component at $j \in \mathsf{J}$ will be written $\pi_j = \pi_j^E : E \to E_j$.

Note that since $\mathsf{cL}\mathcal{C}$ is a reflective subcategory containing all the spaces E_j, their limit E is likewise in $\mathsf{cL}\mathcal{C}$. A $\mathsf{JL}\mathcal{C}$*-map* $u : (E, E_{(-)}, \pi^E) \to (F, F_{(-)}, \pi^F)$ between $\mathsf{JL}\mathcal{C}$-spaces is formed by an $\mathsf{L}\mathcal{C}$-map $u : E \to F$ which is induced by a natural transformation $\mathsf{u} : E_{(-)} \to F_{(-)}$ via the universal property of the natural transformation π^F. Accordingly, we have for each J-morphism $ij : i \to j$ the following commutative diagrams

$$\begin{array}{ccc} E & & \\ {\scriptstyle\pi_i}\downarrow & \searrow{\scriptstyle\pi_j} & \\ E_i & \xrightarrow[E_{ij}]{} & E_j \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{u} & F \\ {\scriptstyle\pi_j^E}\downarrow & & \downarrow{\scriptstyle\pi_j^F} \\ E_j & \xrightarrow[u_j]{} & F_j \end{array} \qquad \begin{array}{ccc} F & & \\ {\scriptstyle\pi_i}\downarrow & \searrow{\scriptstyle\pi_j} & \\ F_i & \xrightarrow[F_{ij}]{} & F_j \end{array}$$

The $\mathsf{JL}\mathcal{C}$-spaces and maps so obtained build in the obvious way a category of structured sets, denoted $\mathsf{JL}\mathcal{C}$.

2b EXAMPLES. It is well known that every complete separated locally convex space E can be represented as a projective limit of Banach spaces in the category Lcx of locally convex spaces (see e.g. §19.9(1) in [8]). Since Lcx is a full reflective subcategory of $\mathsf{L}\mathcal{C}_c$, this projective limit can just as well be regarded as one in $\mathsf{L}\mathcal{C}_c$. Therefore every complete separated $E \in \mathsf{Lcx}$ can be structured to become a $\mathsf{JL}\mathcal{C}_c$-space in at least one way — usually in several different ways. Let us look more closely at a few special cases. The first two will be used in later examples.

(1) Let F be a Banach space. Let U be open in $\mathbb{R}^n$ with K_j $(j \in \mathsf{N})$ a covering of U formed by an increasing sequence of compact subspaces such that $K_j \subset \text{int}K_{j+1}$ (interior). We construct a $\mathsf{JL}\mathcal{C}$-space as follows. Let J be the preorder category $(\mathsf{N}, \geq)$ (see 1c). Let $E = \mathcal{C}_c(U, F)$ and let $E_{(-)} : \mathsf{J} \to \mathsf{cL}\mathcal{C}_c$ be the functor obtained by putting $E_j = \mathcal{C}_c(K_j, F)$, $E_{ji} : E_j \to E_i$ the restriction map $\phi \mapsto \phi|K_i$, where of course $K_i \subset K_j$. Let $\pi^E : E \to E_j$ likewise be the restriction map. We claim $(E, E_{(-)}, \pi^E)$ is a $\mathsf{JL}\mathcal{C}$-space. It is clear from the facts in 1a that $E = \mathcal{C}_c(U, F)$ is a Fréchet space and $E_j = \mathcal{C}_c(K_j, F)$ a Banach space. It is readily seen that $E_{(-)}$ is a functor and π a natural transformation. To see that π provides a limit for $E_{(-)}$ is the main task. The crucial topological fact at work is that the family of insertion maps

$$(*) \quad \mathsf{ins}_j : K_j \to U \ (j \in \mathsf{J})$$

is a final epifamily in $\mathcal{C}_c$. We omit the simple verification, based on local compactness of U. The proof that π provides the stated limit can be completed in several ways. Let us indicate an argument which exploits properties of the functor $\mathcal{C}_c(-, F) : \mathcal{C}_c^{op} \to \mathsf{cL}\mathcal{C}_c$. The restriction maps π_j^E and E_{ji}, expressed in terms of the inclusion maps $\mathsf{ins}_j : K_j \to U$ and $\mathsf{ins}_{ij} : K_i \to K_j$, become $\pi_j^E = \mathcal{C}_c(\mathsf{ins}_j, F)$ and $E_{ji} = \mathcal{C}_c(\mathsf{ins}_{ji}, F)$. Moreover, $E_{(-)} = \mathcal{C}_c(-, F) \circ \mathsf{K}$, where $\mathsf{K} : \mathsf{J} \to \mathcal{C}_c^{op}$ is the functor which carries $ji : j \to i$ to the insertion map $\mathsf{ins}_{ij} : K_i \to K_j$. From the finality of the family (*) it is easily seen that this family actually constitutes a colimit in $\mathcal{C}_c$ for $\mathsf{K}^{op} : \mathsf{J}^{op} \to \mathcal{C}_c$, hence a limit for $\mathsf{K} : \mathsf{J} \to \mathcal{C}_c^{op}$. The right adjoint $\mathcal{C}_c(-, F) : \mathcal{C}_c^{op} \to \mathsf{cL}\mathcal{C}_c$ preserves limits. So it follows at once that the family $\pi_j = \mathcal{C}_c(\mathsf{ins}_j, F) : E \to E_j$ $(j \in \mathsf{J})$ furnishes the required limit.

For a simple example of a $\mathsf{JL}\mathcal{C}_c$-map, take $A : E \to E$ given by the components $A_j : E_j \to E_j$, $A_j \cdot h = 2x_j h$, $(h \in E)$, where $x \in E$ remains fixed and $x_j = \pi_j(x)$.

(2) An interesting generalization of the preceding example can be obtained by taking $E = \mathcal{C}_c^r(U, F)$ $(r \in \mathsf{N})$ and $E_j = \mathcal{C}_c^r(K_j, F)$, the usual Banach space

of r-times continuously differentiable maps, with the remaining data as before. To avoid complications we here suppose the K_j to be convex, hence primary domains hence tangential domains, as required for the differentiation theory of [15] (see 2a in [14] and 3e in [15]). It follows via 4e in [15] that the family (*) is final for the category $\mathcal{C}_c^r$, hence a colimit and the remaining argument proceeds as before.

(3) Consider the following variation on the theme of example (2). Specialize $U = \Omega \stackrel{\text{def}}{=} (-1,1) \subset F$ with the covering family K_j now a strictly increasing sequence of compact subintervals. Choose a point $z \in K_0 \subset K_j$. Take E to be the $\mathsf{L}\mathcal{C}_c$-subspace $\mathcal{C}_c^r(\Omega, F)_z$ of $\mathcal{C}_c^r(\Omega, F)$ formed by all functions x such that $x^{(k)}(z) = 0$ for $k < r$ and similarly E_j consists of all functions in $\mathcal{C}_c^r(K_j, F)$ for which these derivatives vanish at z. The remaining data are defined as before. The verification that one has the data for a $\mathsf{JL}\mathcal{C}_c$-space is left as an exercise. It is somewhat easier to do the case $r = 0$ first i.e. the corresponding variation on (1). By representing the subspaces formed in the definition as 'natural' equalizer diagrams, the argument from that point on is purely categorical.

In the preceding three examples we could replace $\mathcal{C}_c$ throughout by $\mathcal{C}_k$ and only very minor adaptations will be needed: the function spaces in question carry the same topology regardless of whether $\mathcal{C}_c$ or $\mathcal{C}_k$ is used. In the following example however, where we generalize (1) in a different direction, we cannot put $\mathcal{C}_c$ in the role of $\mathcal{C}_k$.

(4) Let U be an open subset of an infinite dimensional Banach space E, let F also be a Banach space and let J be the preorder category formed by all compact $K \subset U$, ordered by the relation $L \supset K$. Thus J is opposite to the category whose morphisms are the insertion maps $\mathsf{ins}_{KL} : K \to L$ (see 1c(2)(3)). Put $E = \mathcal{C}_k(U,F)$, $E_K = \mathcal{C}_k(K,F)$, $E_{LK} = \mathcal{C}_k(\mathsf{ins}_{KL}, F)$ and $\pi_K^E = \mathcal{C}_k(\mathsf{ins}_K, F)$, the latter two again amounting to restriction maps. Much as in (1), we thus have the data $(E, E_{(-)}, \pi^E)$ for a $\mathsf{JL}\mathcal{C}_k$-space. The crucial topological fact at work is again that the family of insertions $\mathsf{ins}_K : K \to U$ $(K \in \mathsf{J})$ constitute a colimit in $\mathcal{C}_k$. And this is where it fails for $\mathcal{C}_c$: it is not a colimit in that category. If it were one, then the family $\mathcal{C}_c(\mathsf{ins}_j, F) : \mathcal{C}_c(U,F) \to \mathcal{C}_c(K_j, F)$ would form a limit in $\mathsf{L}\mathcal{C}_c$, hence the domain would carry the initial $\mathcal{C}_c$-structure induced by the family and since the codomains are all topological, so would be the domain. But since U is not locally compact, $\mathcal{C}_c(U,F)$ cannot be topological (see [17]).

(5) Every cartesian product $E = \prod_{j\in J} E_j$ in $\mathsf{L}\mathcal{C}$ of Banach spaces can be interpreted as a $\mathsf{JL}\mathcal{C}$-space: the index set is taken to be a discrete preorder (see 1c(4)) and $\pi_j : E \to E_j$ the cartesian projection.

(6) Every Banach space E can be interpreted as a $\mathsf{JL}\mathcal{C}$-space $(E, E_{(-)}), \pi^E)$ in a trivial way: take $\mathsf{J} = \{0\}$ $E_0 = E$, $\pi_0 = \mathsf{id}_E$.

2c PROPOSITION. *The category* JL$\mathcal{C}$ *has finite products and the obvious forgetful functor* JL$\mathcal{C}$ $\to$ L$\mathcal{C}$ *preserves them.*

PROOF. We omit the tedious, straight forward proof.

2d DEFINITIONS. Proceeding analogously to 2a, we define a J$\mathcal{C}$-space to be a triple $(X, X_{(-)}, \pi^X)$ consisting of (1) an $\mathcal{C}$-space X, (2) a functor $X_{(-)} : \mathsf{J} \to \mathcal{C}$, the *approximation functor*, which transforms every $j \in \mathsf{J}$ into an *approximand space* X_j which is a $\mathcal{C}$-subspace of a Banach space, (3) a natural transformation π^X from the constant functor $[X]$ to $X_{(-)}$ which furnishes a limit diagram in $\mathcal{C}$. Its component at j will be written $\pi_j = \pi_j^X : X \to X_j$. A J$\mathcal{C}$-map $f : (X, X_{(-)}, \pi^X) \to (Y, Y_{(-)}, \pi^Y)$ between J$\mathcal{C}$-spaces is a $\mathcal{C}$-map $f : X \to Y$ which is induced by a natural transformation $\mathsf{f} : X_{(-)} \to Y_{(-)}$ via the universal property of π^Y. These spaces and maps build in the obvious way a category of structured sets, denoted J$\mathcal{C}$. If the f above is a natural $\mathcal{C}$-embedding, then the induced map f will be called a J$\mathcal{C}$*-embedding*; and if such embedding takes the form $\mathsf{f} = \mathsf{ins}$, a natural insertion with components $\mathsf{ins}_j(x) = x$, then we have a J$\mathcal{C}$*-subspace.* Thus subspaces $X_j \subset E_j \in$ JL$\mathcal{C}$ give rise to J$\mathcal{C}$-subspaces when $E_{ji}(X_j) \subset X_i$.

Since the underlying $\mathcal{C}$-space functor $\mathsf{U} : \mathsf{L}\mathcal{C} \to \mathcal{C}$ preserves limits, it induces in the obvious way a similar underlying functor JL$\mathcal{C}$ $\to$ J$\mathcal{C}$ which preserves embeddings, subspaces and finite products.

2e PROPOSITION. J$\mathcal{C}$ *has finite products and the obvious forgetful functor* J$\mathcal{C}$ $\to$ $\mathcal{C}$ *preserves them.*

PROOF. The tedious but straight forward proof is again omitted.

2f DEFINITION. Suppose now that $E = (E, E_{(-)}, \pi^E)$ is a given JL$\mathcal{C}$-space, that $X = (X, X_{(-)}, \pi^X)$ is a J$\mathcal{C}$-subspace and let $a \in X$. Put $a_j = \pi_j(a)$. If each approximand X_j is an open ball $B(a_j, \epsilon) \stackrel{\text{def}}{=} \{w \in E_j \mid \|w - a_j\| < \epsilon\}$ $(j \in \mathsf{J})$, then the limit space X will be denoted $B(a, \epsilon)$ and called the *quasiball* in E with center a and radius ϵ. Let us emphasize that the same ϵ is used for all j. The definition of quasiball just given differs from that used temporarily in the introductory remarks. The next proposition shows that the two descriptions coincide.

2g PROPOSITION. *Let* $E = C_c^r(\mathbb{R}, F)$ *be the* JL$\mathcal{C}$*-space described in example* 2b, $a \in E$ *and* $0 < \delta \in \mathbb{R}$. *Then the quasiball* $B(a, \delta)$ *consists of all* $x \in E$ *such that* $\max_{k \le r} |x^{(k)}(\xi) - a^{(k)}(\xi)| < \delta$ *for all* $\xi \in \mathbb{R}$. *A similar result holds for the spaces* $E = C_c^r(U, F)$ *and* $E = C_c^r(\Omega, F)_z$ *of* 2b.

PROOF. By restricting π_j^E to the mentioned subspace for each j, one obtains a limit for the $B(a_j, \delta)$, as a direct verification shows.

2h EXAMPLE. Let $\mathsf{J} = (\mathbb{N}, \geq)$ and define the $\mathsf{JL}\mathcal{C}_c$-space $(E, E_{(-)}, \pi^E)$ by putting $E = \mathbb{R}$, $E_j = \mathbb{R}$ equipped with the norm $\|\xi\|_j = j|\xi|$ and $\pi_j(\xi) = \xi$. Then every quasiball $B(0, \delta)$ is degenerate i.e consists of a single point.

This example shows that where quasiballs are to serve as domains for $\mathcal{C}^1$-maps, one has to restrict to $\mathsf{JL}\mathcal{C}$-spaces in which the foregoing pathology does not occur. But we can also learn something positive from this example: when a quasiball is degenerate with respect to a given $\mathsf{JL}\mathcal{C}$-structure, it may be possible to choose another such structure (in the present case e.g. by redefining the norm on E_j as $\|\xi\|_j = |\xi|$) which yields the same underlying $\mathsf{L}\mathcal{C}$-space while having nice quasiballs.

3. A parametrized fixed point theorem

The theorem to follow generalizes to $\mathsf{JL}\mathcal{C}$-spaces and quasiballs a long known theorem for Banach spaces and open balls (see e.g. 10.1.1 in [3]), namely that for a given parametrized family $c(w, -)$ of contraction maps there is a corresponding parametrized family of fixed points $f(w)$ which vary continuously with the parameter w. We will produce a $\mathsf{J}\mathcal{C}$-map f whose values $f(w)$ are fixed points for a given $\mathsf{J}\mathcal{C}$-parametrized family of $\mathsf{J}\mathcal{C}$-contractions $c(w, -)$. The proof for the $\mathsf{J}\mathcal{C}$-version departs from certain key steps of the Banach space proof (rather than the Banach space result itself).

3a THEOREM. *Suppose given the quasiballs $U = B(a, \alpha) \subset E$, $V = B(b, \beta) \subset F$, a scalar $0 \leq \rho < 1$ and a $\mathsf{J}\mathcal{C}$-map $c : U \times V \to F$ such that*

$$\|c_j(w, y) - c_j(w, z)\| \leq \rho\|y - z\| \text{ and } \|c_j(w, b_j) - b_j\| < \beta(1 - \rho)$$

hold for all w in U_j, y and z in V_j, $(j \in \mathsf{J})$. Then there exists a $\mathsf{J}\mathcal{C}$-map $f : U \to V$ such that $f(w) = c(w, f(w))$ and $f(w)$ is the only fixed point of $c(w, -)$ $(w \in U)$.

PROOF. Fix $j \in \mathsf{J}$. Define the sequence $f_{jn} : U_j \to F_j$ $(n \in \mathbb{N})$ of $\mathcal{C}$-maps inductively by putting

$$f_{j0}(w) = b_j, \quad f_{j,k+1}(w) = c_j(w, f_{jk}(w)) \quad (w \in U_j, \ k \in \mathbb{N}).$$

Then, as in the cited classical proof, one obtains by induction that

$$\begin{aligned} &\|f_{j,m+1}(w) - f_{jm}(w)\| = \|c_j(w, f_{jm}(w)) - c_j(w, f_{j,m-1}(w))\| \leq \\ &\leq \rho\|f_{jm}(w) - f_{j,m-1}(w)\| \leq \ldots \leq \rho^m\|c_j(w, b_j) - b_j\| \ (w \in U_j). \end{aligned}$$

Moreover, $\|f_{jn}(w)-b_j\| \le \|c_j(w,b_j)-b_j\|(\rho^{n-1}+\rho^{n-2}+\ldots+1) < \beta$. We thus have a sequence of bounded F_j-valued $\mathcal{C}$-maps which is uniformly Cauchy hence uniformly convergent on U_j to a $\mathcal{C}$-map $f_j : U_j \to V_j$ satisfying $c_j(w, f_j(w)) = f_j(w)$ and $f_j(w)$ is the only fixed point of $c_j(w,-)$. The identity satisfied by f_j can be expressed as the following commutative diagram:

DIAGRAM (j)

$$\begin{array}{ccc} U_j & \xrightarrow{(\mathrm{id}, f_j)} & U_j \times V_j \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle c_j} \\ V_j & \xrightarrow[\mathrm{ins}_j]{} & F_j \end{array}$$

CLAIM 1. There is a natural transformation $f : U_{(-)} \to V_{(-)}$ between the approximation functors of the given quasiballs having the maps f_j as its components.

To establish Claim 1 we have to return to the sequence f_{jn} which defined f_j and note from its inductive construction that we have for $j \in \mathrm{J}$, $m \in \mathbb{N}$ the commutative rectangle

DIAGRAM (j,m)

$$\begin{array}{ccc} U_j & \xrightarrow{(\mathrm{id}, f_{jm})} & U_j \times V_j \\ {\scriptstyle f_{j,m+1}}\downarrow & & \downarrow{\scriptstyle c_j} \\ V_j & \xrightarrow[\mathrm{ins}_j]{} & F_j \end{array}$$

We will need to establish that

DIAGRAM (i,j,n)

$$\begin{array}{ccccc} U_j & \xrightarrow{f_{in}} & V_i & \xrightarrow{\mathrm{ins}_i} & F_i \\ {\scriptstyle U_{ij}}\downarrow & & \downarrow{\scriptstyle V_{ij}} & & \downarrow{\scriptstyle F_{ij}} \\ U_j & \xrightarrow[f_{jn}]{} & V_j & \xrightarrow[\mathrm{ins}_j]{} & F_j \end{array}$$

always commutes. Now the rectangle on the right commutes by definition of a $\mathrm{J}\mathcal{C}$-subspace (see 2d). We use induction to show that the one on the left also commutes. For Diagram $(i,j,0)$ this holds by definition. Choose m and suppose Diagram (i,j,m) commutes. By using the commutative Diagrams (i,m) and (j,m) we can express Diagram $(i,j,m+1)$ equivalently as the commutative diagram

$$\begin{array}{ccccc} U_i & \xrightarrow{(\mathsf{id}, f_{im})} & U_i \times V_i & \xrightarrow{c_i} & F_i \\ {\scriptstyle U_{ij}} \downarrow & & \downarrow {\scriptstyle U_{ij} \times V_{ij}} & & \downarrow {\scriptstyle F_{ij}} \\ U_j & \xrightarrow[(\mathsf{id}, f_{jm})]{} & U_j \times V_j & \xrightarrow[c_j]{} & F_j \end{array}$$

We conclude that Diagram $(i, j, m+1)$ also commutes, hence by induction, Diagram (i, j, n) commutes for all i, j, n. It follows at once that we have the natural transformation f asserted by Claim 1, hence a $\mathsf{J}\mathcal{C}$-map $f : U \to V$. Moreover, from Diagram (j) we have the equation $\mathsf{c} \circ (\mathsf{id}, f) = \mathsf{ins} \circ f$ of natural transformations. In other words, we have a $\mathsf{J}\mathcal{C}$-map $f : U \to V$ such that $c(w, f(w)) = f(w)$. The stated uniqueness follows at once from the corresponding uniqueness present in every component together with the fact that the underlying set functors on $\mathcal{C}$ and $\mathsf{L}\mathcal{C}$ preserve limits.

Banach spaces E and F can be regarded as $\mathsf{JL}\mathcal{C}$-spaces as described in 2b(6) and in that case the last theorem clearly reduces to the classical result.

4. A parametrized existence-uniqueness theorem

For curves into Banach spaces there are effective results affirming existence and uniqueness of solutions to first order initial value problems (see e.g. [3]). But already for Fréchet-spaces these results fail (in their classical formulation). Such theorems that have appeared in nonnormable setting are, not surprisingly, much more complicated than the Banach space version. The theorem closest to the one derived below, but still significantly different from it, is that of [16]. We begin by introducing an appropriate $\mathsf{J}\mathcal{C}$ version of Lipschitz maps.

4a DEFINITION. Let $g : U \times V \to F$ be a $\mathsf{J}\mathcal{C}$-map whose domain is a product of two quasiballs $U \subset E$ and $V \subset F$. Such g is called *$\mathsf{J}\mathcal{C}$-Lipschitz in its second variable* if there exist positive constants μ and ν such that the relations

$$\|g_j(w, h) - g_j(w, k)\| \leq \mu \|h - k\| \text{ and } \|g_j(w, h)\| \leq \nu$$

hold for all $j \in \mathsf{J}$, all $w \in U_j$ and all $h, k \in V_j$.

A further preliminary is concerned with the familiar reduction of an initial value problem to an integral equation. For our purpose we need to show that the problem $u'(\tau) = g(\tau, u(\tau))$ with $u(\alpha) = z$ reduces to the problem $u(\tau) = c(z, u)(\tau) \stackrel{\text{def}}{=} z + \int_\alpha^\tau g(\lambda, u(\lambda))d\lambda$, where g and c are $\mathsf{J}\mathcal{C}$-maps. The simple proposition to follow paves the way for the formal verification of this.

4b PROPOSITION. *A functor* $\mathsf{G} : \mathsf{c}\mathcal{LC} \to \mathsf{c}\mathcal{LC}$ *lifts to a functor* $\mathsf{G} : \mathsf{J}\mathcal{LC} \to \mathsf{J}\mathcal{LC}$ *whenever it preserves Banach spaces and also preserves limits. Moreover, a natural transformation* $\mathsf{u} : \mathsf{G} \to \mathsf{H}$ *between two such functors in* $\mathsf{c}\mathcal{LC}$ *induces, for every* $\mathsf{J}\mathcal{LC}$*-space* $(E, E_{(-)}, \pi^E)$*, a natural transformation between the approximation functors of the two lifted* $\mathsf{J}\mathcal{LC}$*-spaces on* $\mathsf{G}E$ *and* $\mathsf{H}E$ *and then every component* u_E *lifts to a* $\mathsf{J}\mathcal{LC}$*-map* $\mathsf{G}E \to \mathsf{H}E$.

PROOF. It is clear that G transforms the data of one $\mathsf{J}\mathcal{LC}$-space $(E, E_{(-)}, \pi^E)$ into the data $(\mathsf{G}E, \mathsf{G} \circ E_{(-)}, \mathsf{G}\pi^E)$ for another. The required verifications are straight forward.

The key theorem in the categorical calculus of vector valued curves (see theorem 2a in [15]) affirms existence of a natural isomorphism 'average value' with components

$$\mathsf{av}_{\Lambda E} : \mathcal{C}(\Lambda, E) \to \mathsf{ad}\mathcal{C}(\Lambda \times \Lambda, E) \subset \mathcal{C}(\Lambda \times \Lambda, E)$$

in $\mathsf{c}\mathcal{LC}$, where Λ is a non-empty real interval and $\mathsf{ad}\mathcal{C}(\Lambda \times \Lambda,)$ a certain natural subspace of $\mathcal{C}(\Lambda \times \Lambda, E)$. The integral of a $\mathcal{C}$-curve $f : \Lambda \to E$ is defined in terms of av by the formula

$$\textstyle\int_\alpha^\beta f(\lambda)d\lambda = (\beta - \alpha)\mathsf{av}(f)(\alpha, \beta).$$

All the basic properties of such integrals are simple consequences of the mentioned fact that av is a natural isomorphism. For the present purposes we will need the following specialized result.

4c PROPOSITION. *Let* $\Lambda \subset \mathbb{R}$ *be an interval and choose* $\alpha \in \Lambda$. *Then there is a natural transformation ('antiderivative at* α*') in* $\mathsf{c}\mathcal{LC}$ *with components*

$$\mathsf{a}(\alpha)_E : \mathcal{C}(\Lambda, E) \to \mathcal{C}(\Lambda, E), \; \mathsf{a}(\alpha)(f)(\tau) = \textstyle\int_\alpha^\tau f(\lambda)d\lambda.$$

such that $\mathsf{a}(\alpha)'(\tau) = f(\tau)$. *Moreover, if* Λ *is compact and* E *is a* $\mathsf{J}\mathcal{LC}$*-space, then* $\mathsf{a}(\alpha)_E$ *underlies a* $\mathsf{J}\mathcal{LC}$*-map.*

PROOF. Form the composition $\mathsf{a}(\alpha) = \mathsf{e}(\alpha) \circ \mathsf{av}$ where $\mathsf{e}(\alpha)$ is the natural transformation with components $\mathsf{e}(\alpha)_E : \mathcal{C}(\Lambda \times \Lambda, E) \to \mathcal{C}(\Lambda, E)$ given by $\mathsf{e}(\alpha)(f)(\tau) = (\tau - \alpha)\mathsf{av}(f)(\alpha, \tau)$. If Λ is compact, then the functors $\mathcal{C}(\Lambda, -)$ and $\mathcal{C}(\Lambda \times \Lambda, -) : \mathsf{c}\mathcal{LC} \to \mathsf{c}\mathcal{LC}$ preserve Banach spaces and they preserve limits because they are right adjoints. So proposition 4b applies.

4d THEOREM. *Let* $\Omega_\epsilon = B(\alpha, \epsilon) \subset \mathbb{R}$ *and* $B(b, \beta) \subset E$ *be quasiballs and* $g : \Omega_\epsilon \times W \to E$ *a* $\mathsf{J}\mathcal{C}$*-map which is* $\mathsf{J}\mathcal{C}$*-Lipschitz in its second variable. Then* *the initial value problem*

$$(*) \qquad u'(\tau) = g(\tau, u(\tau)) \text{ with } u(\tau) = z \in B(b, \beta/2)$$

has a unique local solution which is a $\mathsf{J}\mathcal{C}$*-function of the initial value* z.

The conclusion, more explicitly stated, reads as follows: There exists a compact ball $\Lambda = \overline{\Omega_\gamma} \subset \Omega_\epsilon$ and a $\mathsf{J}\mathcal{C}$-map $v : B(b, \beta/2) \to \mathcal{C}(\Lambda, W)$ such that $u = v(z) : \Lambda \to W$ is on Λ the unique solution to problem (*).

PROOF. In view of the hypotheses there exist positive constants μ and ν such that

$$\|g_j(\tau, h) - g_j(\tau, k)\| \leq \mu\|h - k\| \text{ and } \|g_j(\tau, h)\| \leq \nu$$

hold for all $j \in \mathsf{J}$, all $\tau \in \Omega_\epsilon$ and all $h, k \in W_j$. Choose γ so that the following hold:

$$0 < \gamma < \epsilon \text{ and } \gamma\mu < 1 \text{ and } \gamma < \beta/(2\nu + 2\beta\mu).$$

Put $\Lambda = \overline{\Omega_\gamma}$. Define for $F \in \mathsf{cL}\mathcal{C}$ the $\mathcal{C}$-map $c_F : B(b, \beta/2) \times \mathcal{C}(\Lambda, W) \to \mathcal{C}(\Lambda, F)$ by putting $c(z, u)(\tau) = z + \int_\alpha^\tau g(\lambda, u(\lambda))d\lambda$. Since c is obtained by adding a constant function to $\mathsf{a}(\alpha)_F \circ g^\dagger$, it follows from propositions 4b and 4c that c_E is a $\mathsf{J}\mathcal{C}$-map whenever E underlies a $\mathsf{JL}\mathcal{C}$-space, moreover that $c(z, u)'(\tau) = g(\tau, u(\tau))$. We can now establish without difficulty that

$$\sup_{\tau\in\Lambda} \|c_j(z_j, h)(\tau) - c_j(z_j, k)(\tau)\| \leq \sup_{\tau\in\Lambda} \| \textstyle\int_\alpha^\tau [g_j(\lambda, h(\lambda)) - g_j(\lambda, k(\lambda))]d\lambda\|$$
$$\leq \gamma\mu \sup_{\lambda\in\Lambda} \|h(\lambda) - k(\lambda)\|$$

Also, using $[b]$ to denote the constant function with value b and using the relations assumed for γ, we have

$$\|c_j(z_j, [b])(\tau)\| = \|z_j + \textstyle\int_\alpha^\tau g_j(\lambda, b)d\lambda\| \leq \beta/2 + \gamma\nu < \beta(1 - \mu\gamma).$$

Putting $\rho = \gamma\mu$ and restating in terms of the norm of $\mathcal{C}(\Lambda, E_j)$, we obtain

$$\|c_j(z, h) - c_j(z, k)\| \leq \rho\|h - k\| \text{ and } \|c_j(z, [b])\| \leq \beta(1 - \rho).$$

Thus the parametrized fixed point theorem 3a applies when we put $U = B(b, \beta/2)$ and $V = \mathcal{C}(\Lambda, W) = B([b], \beta) \subset \mathcal{C}(\Lambda, E)$. We conclude existence of a $\mathsf{J}\mathcal{C}$-map $f : U \to V$ such that $f(z) = c(z, f(z))$, unique among functions satisfying this identity. So it follows from what was established for c that $f(z)'(\tau) = c(z, f(z))'(\tau) = g(\tau, f(z)(\tau))$, as required.

In the special case of Banach spaces we again recover the classical result.

5. Prelude to the inverse function theorem

In this section we establish results needed for proving and illustra inverse function theorem in the next section. The first of these (prop ..., is of independent interest, forming part of the categorical theory of differential calculus [15]: it is valid for any category $\mathcal{C}$ satisfying the calculus axioms of [15]. A special case of this result (for Fréchet spaces) can be found in [7], derived there with different technique.

We recall briefly the basic concepts of the differentiation theory [15]. Let Λ be a nondegenerate real interval (a fundamental domain in the scalar field) and take a $\mathcal{C}$-subspace $U \subset E \in \mathsf{cL}\mathcal{C}$. A $\mathcal{C}$-map $p : \Lambda \to U$ is called a curve; such a curve p is called a *path* if there exists a (necessarily unique) curve $p' : \Lambda \to E$ such that $p(\beta) - p(\alpha) = \int_\alpha^\beta p'(\tau)d\tau$ holds identically in α and β. A *tangential domain* in E is a $\mathcal{C}$-subspace $U \subset E$ such that at every $x \in U$ the tangent vectors $p'(\alpha)$, formed by paths with $p(\alpha) = x$, generate the linear $\mathcal{C}$-space E. In case $E \in \mathsf{L}\mathcal{C}_c$ is locally convex, the latter condition is satisfied whenever these tangent vectors have a linear span weakly dense in E. A $\mathcal{C}$-map $f : U \to F$ on such a tangential domain U is called a C^1-map if there exists a (necessarily unique) $\mathcal{C}$-map $Df : U \to [E, F]$ such that $f(p(\beta)) - f(p(\alpha)) = \int_\alpha^\beta Df(p(\tau)) \cdot p'(\tau)d\tau$ holds identically in p, α and β. Such f is called a C^r-map ($r \geq 1$) if its derivative Df is a C^{r-1}-map. The derivative $Df : U \to [E, F]$ and its transpose the differential $df = (Df)^\ddagger : U \times E \to F$, $df(x, h) = Df(x) \cdot h$, determine each other. The usual formulas of calculus hold on the basis of these definitions; see [15] for more details. By the transpose of $f : X \times Y \to Z$ is meant the map $f^\dagger : X \to \mathcal{C}(Y, Z)$ (see 1a) while $g : X \to \mathcal{C}(Y, Z)$ has the transpose $g^\ddagger : X \times Y \to Z$, $g^\ddagger(x, y) = g(x)(y)$ i.e. $\ddagger$ is the inverse of $\dagger$.

5a PROPOSITION. *Let E and F be $\mathsf{cL}\mathcal{C}$-spaces and U a tangential domain in E. Let $Q : U \to [E, F]$ and $R : U \to [F, E]$ be $\mathcal{C}$-maps such that for every $x \in U$ the maps $Q(x) : E \to F$ and $R(x) : F \to E$ are inverses to each other. Then Q is a C^r-map if and only if R is a C^r-map ($1 \leq r \leq \infty$). In fact, when Q is a C^r-map the derivative of R is given by*

(1) $\quad DR(x) \cdot h = S(x) \cdot h \stackrel{\text{def}}{=} -R(x) \circ (DQ(x) \cdot h) \circ R(x).$

PROOF. Suppose Q is C^1. To show R is C^1 we have to verify that for any path $p : \Lambda \to U$ the following holds identically:

(2) $\quad R(p(\beta)) - R(p(\alpha)) = \int_0^1 S(p(\tau) \cdot p'(\tau))d\tau.$

Define the $\mathcal{C}$-map $q : \Lambda \to [E, F]$ by putting $q(\tau) = DQ(p(\tau)) \cdot p'(\tau)$. Then we can form also the $\mathcal{C}$-map $\mathsf{av}(q) : \Lambda \times \Lambda \to [E, F]$ such that $(\beta - \alpha)\mathsf{av}(q)(\alpha, \beta) = \int_\alpha^\beta q(\tau)d\tau$. Now construct the $\mathcal{C}$-map $A : \Lambda \times \Lambda \to [F, E]$ by putting $A(\alpha, \beta) = -R(p(\beta)) \circ \mathsf{av}(q)(\alpha, \beta) \circ R(p(\alpha))$. It follows from the linearity of $R(p(\alpha))$ that $(\beta-\alpha)A(\alpha, \beta) = -R(p(\beta))\circ(\beta-\alpha)\mathsf{av}(q)(\alpha, \beta)\circ R(p(\alpha)) = -R(p(\beta))\circ[Q(p(\beta)) - Q(p(\alpha))] \circ R(p(\alpha)) = R(p(\beta)) - R(p(\alpha))$. We conclude from this (see theorem 2d(6) of [15]) that $R \circ p$ is a path with $(R \circ p)'(\tau) = A(\tau, \tau) = -R(p(\tau)) \circ q(\tau) \circ R(p(\tau))$. It follows at once that (2) holds, thus that R is $\mathcal{C}^1$. For the general statement, fix $r \geq 1$ and assume, for induction on r, that the implication $Q \in \mathcal{C}^k \Rightarrow R \in \mathcal{C}^k$ holds for all $1 \leq k \leq r$. Suppose then that $Q \in \mathcal{C}^{r+1}$. We have to show that the differential $dR : U \times E \to [F, E]$ is a $\mathcal{C}^r$-map. But formula (1) allows it to be expressed as a composition of $\mathcal{C}^r$-maps as follows. Using $\mathsf{pro}_U : U \times E \to U$ we form the composition $R \circ \mathsf{pro}_U : U \times E \to [F, E]$, hence the $\mathcal{C}^r$-map $S \overset{\text{def}}{=} (-R \circ \mathsf{pro}_U, dQ, R \circ \mathsf{pro}_U) : U \times E \to [F, E] \times [E, F] \times [F, E]$. The map $\mathsf{comp} : [F, E] \times [E, F] \times [F, E] \to [F, E]$, $\mathsf{comp}(u, v, w) = w \circ v \circ u$, is multilinear hence a $\mathcal{C}^\infty$ map. So we can express $dR = \mathsf{comp} \circ S$, a composition of $\mathcal{C}^r$-maps, as required.

5b PROPOSITION. *Suppose E and F are $\mathsf{JL}\mathcal{C}$-spaces and $U \subset E$ a tangential domain in E which is also a $\mathsf{J}\mathcal{C}$-subspace of E. Let $f : U \to F$ be a $\mathsf{J}\mathcal{C}$-map such that $f_j \in \mathcal{C}^1(U_j, F_j)$ for all $j \in \mathsf{J}$. Then $f \in \mathcal{C}^1(U, F)$ and the differential $df : U \times E \to F$ is a $\mathsf{J}\mathcal{C}$-map with components $df_j : U_j \times E_j \to F_j$.*

PROOF. We begin by showing that the given differentials $df_j : U_j \times E_j \to F_j$ are natural in j i.e. that $F_{ji} \circ df_j = df_i \circ (U_{ji} \times E_{ji})$. For every path $p : \Lambda \to U_j$ we have $\int_\alpha^\beta F_{ji} \cdot df_j(p(\tau), p'(\tau))d\tau = F_{ji} \cdot \int_\alpha^\beta df_j(p(\tau), p'(\tau))d\tau = F_{ji} \cdot f_j(p(\beta)) - F_{ji} \cdot f_j(p(\alpha)) = (f_i \circ E_{ji})(p(\beta)) - (f_i \circ E_{ji})(p(\alpha))$. On the other hand, using the linearity of U_{ji}, $\int_\alpha^\beta df_j(U_{ji} \cdot p(\tau), U_{ji} \cdot p'(\tau)) = (f_i \circ U_{ji})(p(\beta)) - (f_i \circ U_{ji})p(\alpha)$. Since U_{ji} is just a restriction of E_{ji}, the two integrals agree for all choices of α and β. Therefore the integrands agree. By transposing, we see that the compositions $\mathsf{eval}(-, p(\tau)) \circ (F_{ji} \circ df_j)^\dagger$ and $\mathsf{eval}(-, p(\tau)) \circ (df_i \circ (U_{ji} \times E_{ji}))^\dagger$, as maps $E_j \to F_i$, agree for all paths p. Since the evaluations $\mathsf{eval}(-, p(\tau))$ constitute a monofamily while the tangent vectors $p'(\tau)$ generate E_j, we conclude $F_{ji} \circ df_j = df_i \circ (U_{ji} \times E_{ji})$ as required for the stated naturality. It follows that there is a limit induced $\mathsf{J}\mathcal{C}$-map $df_J : U \times E \to F$, linear in the E-variable. To show that $df_J = df$, the differential of f, it is enough to consider a path $p : \Lambda \to U$ and verify that $\int_\alpha^\beta df_J(p(\tau), p'(\tau))d\tau = f(p(\beta)) - f(p(\alpha))$. Composing with the relevant projections π_j, we have $\pi_j^F \int_\alpha^\beta df_J(p(\tau), p'(\tau))d\tau = \int_\alpha^\beta \pi_j^F df_J(p(\tau), p'(\tau))d\tau = \int_\alpha^\beta df_(\pi_j^U p(\tau), \pi_j^E p'(\tau))d\tau = f_j(\pi_j(p(\beta))) - f_j(\pi_j(p(\alpha))) = \pi_j f(p(\beta)) - \pi_j f(p(\alpha)) = \pi_j[f(p(\beta)) - f(p(\alpha))]$. After canceling the monofamily (π_j), we obtain the required equation.

5c PROPOSITION. *Let $E = C_c^r(\Omega, \mathbb{R})$ be the* JL$\mathcal{C}$*-space described in* 2b(2)*. Then every quasiball in E is a tangential domain. The same holds for the space $E = C_c^r(\Omega, \mathbb{R})_0$ (see* 2b(3)*).*

PROOF. We have seen in 2g that a quasiball $B(a, \delta)$ consists of all x such that $|x^{(k)}(\xi) - a^{(k)}(\xi)| < \delta$ for all ξ and all $k = 0, 1, \ldots, r$. Fix $x \in B(a, \delta)$ and $h \in E$ arbitrarily. Take a C_c^∞-function $\psi_j : \mathbb{R} \to [0, 1]$ of compact support which has the constant value 1 on K_j. Let μ_j be the maximum value of $|x^{(k)}(\xi) - a^{(k)}(\xi)|$ for ξ in the support of ψ_j and $0 \leq k \leq r$. Note that $\mu_j < \delta$. Choose $\lambda_j > 0$ so that the absolute value of $\lambda_j \psi_j h$ and of its first r derivatives is always below $\delta - \mu_j$. Then $x + \lambda_j h \psi_j$ lies in $B(a, \delta)$. The path $p(\tau) = x + \tau \lambda_j h \psi_j$ $(0 \leq \tau \leq 1)$ therefore takes values in $B(a, \delta)$ and $p'(0) = \lambda_j h \psi_j$. Since $h \psi_j$ converges to h in E as $j \to \infty$, it follows that the positive multiples of tangent vectors to $B(a, \delta)$ at x form a dense subset of E. This is sufficient for $B(a, \delta)$ to be tangential in E (see 3b and 3c in [15]). A similar argument works in the corresponding space of 2b(3).

6. Inverse function and implicit function theorems

In all that follows, E and F are assumed to be cL$\mathcal{C}$-spaces, $U \subset E$ a tangential domain, $a \in U$ and $f \in C^1(U, F)$.

The inverse function theorem to be established here differs significantly from previous versions in nonnormable setting (cf. [7], [6], [18]) in that open neighborhoods become replaced by tangential 'quasineighborhoods' which need not be open and may even have empty interior. The justification and motivation for this was given in the introduction. The restrictions to be placed on the function to be inverted will largely be expressed in terms of ad hoc JL$\mathcal{C}$-structures on the spaces concerned. Since every complete locally convex space can carry some JL$\mathcal{C}$-structure, our conditions are not all that restrictive on the spaces; rather, it is the map which is required to behave itself. We will now describe how it should.

6a DEFINITIONS. For Banach spaces E and F, Ban(E, F) will denote the usual Banach space of linear continuous maps $E \to F$. It is well known and easy to see that, as cL$\mathcal{C}$-space, Ban(E, F) carries a structure finer than that of the canonical space $[E, F]$; in general, if $f \in C^1(V, F)$, its derivative Df is a $\mathcal{C}$-map on V into $[E, F]$, which need not be continuous as a map into Ban(E, F); when it is, we will say Df is *norm-continuous.*

Suppose the $\mathsf{L}\mathcal{C}$-spaces E and F have been structured to become $\mathsf{JL}\mathcal{C}$-spaces. A *quasineighborhood* of $a \in E$ means a $\mathsf{J}\mathcal{C}$-subspace $W \subset E$ such that for every $j \in \mathsf{J}$ the approximand space W_j is a neighborhood of a_j in E_j in the usual sense.

In general, a quasiball $B(a, \delta)$ in E may consist of a single point, as example 2h shows. Mindful of this, we define a $\mathsf{JL}\mathcal{C}$-structure on E to be *differentially compatible* if every quasiball is a tangential domain.

To say $f : U \to F$ is *calm near* a means there exists a quasiball $B(a, \delta) \subset U$ such that f is a $\mathsf{J}\mathcal{C}$-map $B(a, \delta) \to F$ with components $f_j : B(a_j, \delta) \to F_j$ such that the following hold: (1) Df_j is norm-continuous; (2) there exists $\mu > 0$ such that $\|Df_j(a_j)\| < \mu$ ($j \in \mathsf{J}$); (3) for every $\epsilon > 0$ there exists $\eta > 0$ such that for all $x \in B(a_j, \eta)$ and all $j \in \mathsf{J}$ we have $\|Df_j(x) - Df_j(a_j)\| < \epsilon$.

To say $Df(a)$ is *equiinvertible* means $A \stackrel{\text{def}}{=} Df(a) : E \to F$ is a $\mathsf{JL}\mathcal{C}$-isomorphism such that $\|A_j^{-1}\| \le \kappa < \infty$ holds for all $j \in \mathsf{J}$.

Let us emphasize that the constants δ, μ and κ appearing above are required to work for all $j \in \mathsf{J}$ at the same time. The norm used is of course that of $\mathsf{Ban}(E_j, F_j)$.

6b PROPOSITION. *Suppose $f : B(a, \delta) \to F$ to be a $\mathsf{J}\mathcal{C}$-map whose components f_j are C^1-maps with $\sup_j \|Df_j(a_j)\| < \infty$. Then the following implications hold:*

(a) *all f_j are C^2-maps and $\nu \stackrel{\text{def}}{=} \sup_j \sup_x \|D^2 f_j(x)\| < \infty$,*

$\Downarrow$

(b) *Df_j is Lipschitz into $\mathsf{Ban}(E_j, F_j)$ with one Lipschitz constant for all j,*

$\Downarrow$

(c) *f is calm near a.*

PROOF. We have (a)$\Rightarrow$ $\|Df_j(x) - Df_j(w)\| = \| \int_0^1 D^2 f_j(w + \theta(x - w)) \cdot (x - w)\| \le \nu\|x - w\| \Rightarrow$ (b) $\Rightarrow$ (c).

6c EXAMPLES.

(1) Let $E = F = \mathcal{C}_c(\mathbb{R}, \mathbb{R})$, define $f : E \to E$ by $f(x) = x^2$ and choose $a \in E$ to be the constant function $a(\xi) = 1$ for all ξ. Then $(Df(x) \cdot h)(\xi) = 2x(\xi)h(\xi)$ and $(D^2 f(x) \cdot h \cdot k)(\xi) = 2h(\xi)k(\xi)$. By choosing as $\mathsf{JL}\mathcal{C}$-structures for E and F that described in 2b(1), it follows readily that 6b(a) is satisfied, hence f is calm near a.

(2) Let $E = \mathcal{C}_c^1(\Omega, \mathbb{R})_0$ and $F = \mathcal{C}_c(\Omega, \mathbb{R})$ be the $\mathsf{JL}\mathcal{C}_c$-spaces described in 2b (see (1) and (3)). Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^1-map such that ϕ' and ϕ'' are bounded and define $f : E \to F$ by

$$f(x)(\tau) = x'(\tau) + \phi(x(\tau)).$$

Then f is calm near any point $a \in E$, as a straight forward verification shows.

6d INVERSE FUNCTION THEOREM. *Suppose $f \in C^1(U,F)$ is such that there exist* J*LC-structures for E and F, with that of E differentially compatible, such that f is calm near $a \in U$ and $Df(a)$ is equiinvertible. Then there exists a tangential quasineighborhood V of a such that $W = f(V)$ is a tangential quasineighborhood of $f(a)$ and $f : V \to W$ has a C^1-inverse $g : W \to V$ with every Dg_j norm-continuous. Moreover, if f is a C^r-map, then so is g $(1 \leq r \leq \infty)$.*

PROOF. Choose $0 < \rho < 1$ once and for all. By calmness of f there exists $\delta > 0$ such that $f : B(a, \delta) \to F$ is a JC-map and there exists $\mu > 0$ such that

$$\|Df_j(a_j)\| < \mu \ (j \in \mathrm{J})$$

By equiinvertibility of $Df(a)$ there exists κ such that

$$\|A_j^{-1}\| \leq \kappa \text{ where } A_j \stackrel{\text{def}}{=} Df_j(a_j).$$

Choose $\alpha_1 > 0$ such that

$$\|Df_j(z) - Df_j(a_j)\| < \rho/\kappa \text{ for all } z \in B(a_j, \alpha_1).$$

This choice is possible by calmness of f near a.

CLAIM 1. *For all $j \in \mathrm{J}$ and all $w, x \in B(a_j, \alpha_1)$ we have* $\|x - w - A_j^{-1}(f_j(x) - f_j(w))\| \leq \rho\|x - w\|$.

Proof 1. Put $z(\theta) = w + \theta(x - w)$. Then the left side can be expressed $\|x - w - A_j^{-1} \int_0^1 Df_j(z(\theta))d\theta \cdot (x - w)\| = \|x - w - A_j^{-1} \int_0^1 [Df_j(z(\theta)) - A_j]d\theta \cdot (x - w) - (x - w)\| \leq \|A_j^{-1}\| \cdot \|Df_j(z(\theta)) - Df_j(a_j)\| \cdot \|x - w\| \leq \kappa \cdot (\rho/\kappa) \cdot \|x - w\|$. So claim 1 holds.

Choose $\beta > 0$ so that $0 < \kappa\beta < \alpha_1(1 - \rho)$. Put $b = f(a)$ and $c(y, x) = x + A^{-1} \cdot (y - f(x))$.

CLAIM 2. *There exists a unique* J*C-map $g : B(b, \beta) \to B(a, \alpha_1)$ such that $g(y)$ is the unique fixed point of $c(y, -) : B(a, \alpha_1) \to E$. Moreover, $f(g(y)) = y$.*

Proof 2. It is clear from its definition that $c : B(b, \beta) \times B(a, \alpha_1) \to E$ is a JC-map. For all $j \in \mathrm{J}$, all $y \in B(b_j, \beta)$ and all $w, x \in B(a_j, \alpha_1)$ we have $\|c_j(y, x) - c_j(y, w)\| = \|x - w - A_j^{-1}(f_j(x)) - f_j(w)\| \leq \rho\|x - w\|$, by claim 1. Moreover, $\|c_j(y, a_j) - a_j\| = \|A_j^{-1}(y - f(a_j))\| < \kappa\beta < \alpha_1(1 - \rho)$. So the fixed point theorem 3a applies to c and provides a JC-map $g : B(b, \beta) \to B(a, \alpha_1)$, unique among maps such that $c(y,x) = x$ implies $x = g(y)$. Algebraic manipulation shows the equation $g(y) = c(y, g(y))$ to be equivalent to $y = f(g(y))$. So claim 2 holds.

Now choose $\alpha > 0$ so that $\alpha \leq \alpha_1$ and $(\mu + \rho/\kappa)\alpha < \beta$. Put $V = B(a, \alpha)$ and $W_j = f_j(V_j)$ ($j \in \mathsf{J}$). We have $W_j \subset B(b_j, \beta)$ because $\|f_j(x) - b_j\| = \| \int_0^1 Df_j(a_j + \theta(x-a_j))\cdot(x-a_j)d\theta\| \leq \|[Df_j(z) - Df_j(a_j) + Df_j(a_j)]\|\cdot\|x - a_j\| \leq (\mu + \rho/\kappa)\alpha < \beta$. Let W denote the $\mathsf{J}\mathcal{C}$-subspace of $B(b, \beta)$ determined by the spaces W_j ($j \in \mathsf{J}$).

Claim 3. *The restricted map $f : V \to W$ is a $\mathsf{J}\mathcal{C}$-isomorphism with inverse $g : W \to V$. Moreover, W_j is open in F_j for each j.*

Proof 3. Since $f_j(V_j) = W_j$ it is clear that $f(V) \subset W$. Direct calculation shows that $c(f(x), x) = x$ for all $x \in V$. It follows by claim 2 that $g(f(x)) = x$ and $f(g(y)) = y$ for $y \in W$. Note that $W_j = g_j^{-1}(V_j)$, so it is must be open in F_j by continuity of g_j. So claim 3 holds.

Define $\Phi : V \times V \times E \to F$ by putting $\Phi(w, x, h) = \int_0^1 Df(w + \theta(x - w))d\theta \cdot h$. Then the transposed map $\Phi^\dagger : V \times V \to [E, F]$ is a difference factorizer of $f : V \to F$ (see 4c in [15]).

Claim 4. *$\Phi : V \times V \times E \to F$ is a $\mathsf{J}\mathcal{C}$-map with components Φ_j such that $\Phi_j^\dagger : V_j \times V_j \to [E_j, F_j]$ lifts to a $\mathcal{C}$-map into $\mathsf{Ban}(E_j, F_j)$ whose values $\Phi_j^\dagger(w, x)$ are $\mathsf{L}\mathcal{C}$-isomorphisms $E_j \to F_j$. Hence for each $(w, x) \in V \times V$ the map $\Phi(w, x, -) : E \to F$ is a $\mathsf{JL}\mathcal{C}$-isomorphism.*

Proof 4. That Φ is again a $\mathsf{J}\mathcal{C}$-map follows readily from the naturality of the integral: $F_{ji} \int_0^1 = \int_0^1 F_{ji}$ and so on. Take $(w, x) \in V_j \times V_j$ and put $z(\theta) = w + \theta(x - w)$. By calmness we have $Df_j : V_j \to \mathsf{Ban}(E_j, F_j)$ continuous, so the same holds for $\Phi_j^\dagger : V_j \times V_j \to \mathsf{Ban}(E_j, F_j)$ (see 4c in [15]). Moreover, by choice of α, $\|\Phi_j^\dagger(w, x) - A_j\| = \| \int_0^1 [Df_j(z(\theta)) - Df_j(a_j)]d\theta\| \leq \rho/\kappa \leq \|A_j^{-1}\|^{-1}$. It follows, by a well known criterion of Banach space theory, that $\Phi(w, x, -)$ is close enough to the invertible operator A_j to be invertible itself. Since the maps $\Phi_j(w, x, -)$ are natural in j, so are their inverses. Hence $\Phi^\dagger(w, x)$ is a $\mathsf{JL}\mathcal{C}$-isomorphism as claimed.

Claim 5. *W is a tangential domain in F.*

Proof 5. In view of claim 4, the derivative $Df(x) = \Phi(x, x, -)$ is invertible for every $x \in V$. Therefore W, the image $f(V)$, is tangential by proposition 4i in [15].

CLAIM 6. *$g : W \to V$ is a C^1-map, with $Dg(y) = Df(g(y))^{-1}$.*

Proof 6. By proposition 5b it is enough to show that every component $g_j : W_j \to V_j$ is a C^1-map with the derivative indicated. Since W_j is open in F_j, every $y \in W_j$ has an open ball neighborhood $B_y = B(y, \delta_y) \subset W_j$. We will use the localization theorem (see 4e of [15]) which requires that we show $g_j : B_y \to E_j$ is a C^1-map with derivative $Dg_j(y) = Df_j(g_j(y))^{-1}$ for each y. Since the balls B_y are convex, we can apply the characterization of C^1-maps in terms of difference factorizers (4c in [15]). Accordingly, our task is to exhibit a $\mathcal{C}$-map $\Psi : B_y \times B_y \to [F_j, E_j]$ such that $g(y_2) - g(y_1) = \Psi(y_1, y_2) \cdot (y_2 - y_1)$. Using claim 4, we construct Ψ as the following composition.

$$
\begin{array}{ccccc}
B_y \times B_y & & \xrightarrow{\quad\Psi\quad} & & [F_j, E_j] \\
{\scriptstyle g \times g}\big\downarrow & & & & \big\uparrow{\scriptstyle \mathsf{ins}} \\
V_j \times V_j & \xrightarrow[\dagger(\Phi_j)]{} & \mathsf{Inv}(E_j, F_j) & \xrightarrow[\mathsf{inv}]{} & \mathsf{Ban}(F_j, E_j)
\end{array}
$$

In this construction, $\mathsf{Inv}(E_j, F_j)$ denotes the open subset of $\mathsf{Ban}(E_j, F_j)$ formed by all invertibles, $\mathsf{inv}(u) = u^{-1}$, and $\mathsf{ins}(v) = v$. These maps are all well known to be $\mathcal{C}$-maps. Put $x_1 = g(y_1)$, $x_2 = g(y_2)$. Since $\Phi_j^\dagger$ is a difference factorizer for f_j, we have $\Phi_j^\dagger(x_1, x_2) \cdot (x_2 - x_1) = f_j(x_2) - f_j(x_1) = y_2 - y_1$. Hence $g(y_2) - g(y_1) = \Phi_j^\dagger(x_1, x_2)^{-1} \cdot (y_2 - y_1) = (\mathsf{inv} \circ \Phi_j^\dagger \circ (g \times g)(y_1, y_2)) \cdot (y_2 - y_1) = \Psi(y_1, y_2) \cdot (y_2 - y_1)$, as required. Moreover, since Ψ factored through $\mathsf{Ban}(F_j, E_j)$, the derivative Dg_j is norm-continuous.

CLAIM 7. *If f is a C^r-map, then so is g.*

Proof 7. We proceed by induction on $r = 1, 2, \ldots$. The case $r = 1$ is dealt with by claim 6. Choose $q \geq 1$ and suppose claim 7 is valid for $r \leq q$. Define $Q : V \to [E, F]$ and $R : V \to [F, E]$ by $Q(x) = Df(x)$, $R(x) = Df(x)^{-1}$. If f is a C^{q+1}-map, then Q and g are C^q-maps. So by proposition 5a, R is a C^q-map. Hence, in view of claim 6, $Dg = R \circ g$ is a C^q-map, which means g is a C^{q+1}-map, as required. This ends the proof of the inverse function theorem.

In the above theorem, the quasineighborhood V could be chosen to be a quasiball $B(a, \delta)$, as the proof shows. Its image $W = f(V)$ cannot be expected to contain a quasiball $B(f(a), \epsilon)$, even when f is linear. This is illustrated by the map $f : \mathcal{C}_c(\mathbb{R}, \mathbb{R}) \to \mathcal{C}_c(\mathbb{R}, \mathbb{R})$, $f(x) = \psi x$ where $\psi(\xi) = \exp(-\xi^2)$. Thus in general a quasineighborhood of a is not a set containing a quasiball centered on a.

In the special case when E and F are Banach spaces, one could employ the trivial JL$\mathcal{C}$-structure in which all $E_j = E$ (see 2b(6)). Then the calmness requirement becomes simply the demand that the derivative be norm-continuous; equiinvertibility reduces to ordinary invertibility. In order to see that the above inverse function theorem reduces here to a result equivalent to the classical inverse function theorem of the Fréchet calculus, we need only the assurance that the maps are Fréchet-C^1. But C^1-maps are clearly Gateaux-differentiable and Gateaux-differentiability combined with norm-continuity of the derivative implies Fréchet-C^1, as is well known.

Our concept of C^1-map agrees on open domains of complete locally convex spaces with that used in [7]. The inverse function theorem for Banach spaces derived in [7] as prelude to the Nash-Moser theorem requires the map f to be C^2 and to that small extent it is a weaker result (for Banach spaces) than the one derived here. Note in this connection that $f \in C^2 \Rightarrow Df$ is norm-continuous $\Rightarrow f \in C^1$ (cf. 5b).

The counterexample of the introduction becomes an example to illustrate the preceding theorem. Most of the required verifications have already been done in 5c and 6c; the remaining ones are straight forward. We now give a nontrivial application of the theorem to a situation where the inverse function is not known in advance.

6e PROPOSITION *Let* $f : E = C_c^1(\Omega, \mathbb{R})_0 \to C_c(\Omega, \mathbb{R}) = F$ *be the nonlinear differential operator* $f(x) = x' + \phi \circ x$ *described in* 6c(2). *Let* $v \in E$ *and* $w \in F$ *be such that* $f(v) = w$ *i.e.* $v'(\tau) + \phi(v(\tau)) = w(\tau)$. *Then there exists a tangential quasineighborhood* V *of* v *in* E *and a tangential quasineighborhood* W *of* w *in* F *such that for every* $y \in W$ *the equation* $x'(\tau) + \phi(x(\tau)) = y(\tau)$ *has a unique solution* $x = g(y) \in V$ *and the solution map* $g : W \to V$ *is* C^1.

PROOF. Differentiation gives $Df(x) \cdot h = h' + (\phi' \circ x)h$. The equation $Df(x) \cdot h = k$ is thus nothing but the elementary initial value problem

$$h'(\tau) + \phi'(x(\tau))h(\tau) = k(\tau) \text{ with } h(0) = 0.$$

This problem has a well known solution, namely

$$h(\tau) = e^{-P(\tau)} \textstyle\int_0^\tau e^{P(\theta)} k(\theta) d\theta \text{ where } P(\tau) = \textstyle\int_0^\tau \phi'(x(\lambda)) d\lambda.$$

By using the naturality of the integral $\int_0^\tau$ (see section 3), it is readily verified that $Df(v)$ is equiinvertible and we have already seen in 6c(2) that f is calm near v and via 5c that the JL$\mathcal{C}$-structure is differentially compatible. So the inverse function theorem applies to give the stated conclusion.

Note that the values of the guaranteed inverse function are global solutions on the given domain Ω, in contrast to the local solutions of 4d. It would be of interest to know to what extent one could replace Ω by an unbounded domain and moreover to obtain a corresponding application for first order systems of equations. One should also experiment with other choices of JL$\mathcal{C}$-structures. Another line of further investigation would be to determine what class of nonlinear partial differential operators could be addressed by the new inverse function theorem.

6g IMPLICIT FUNCTION THEOREM. *Let $f \in C^1(U \times V, G)$ and suppose that E, F and G can be given* JL$\mathcal{C}$*-structures such that those of E and F are differentially compatible with f calm near a point $(a,b) \in U \times V$ and $\partial_2 f(a,b) : F \to G$ equiinvertible. Then there exists a quasiball T centered on a and a unique C^1-map $g : T \to V$ such that $f(x, g(x)) = f(a,b)$ holds for all $x \in T$. Moreover, if f is a C^r-map, then so is g.*

PROOF. This is derived quickly from the above inverse function theorem along familiar lines: the argument as presented e.g. in [10] for the Banach space situation requires only obvious adaptations to apply here. One can also derive this theorem independently via the fixed point theorem.

References

[1] Ernst Binz, Continuous convergence on $C(X)$, Lecture Notes in Math. 469 (1975).

[2] G. Choquet, Convergences, Ann. Univ. Grenoble (=Ann. Inst. Fourier) 23 (1948) 57-112.

[3] J. Dieudonné, Foundations of Modern Analysis (Academic Press) (1969).

[4] H.R. Fischer, Limesräume, Math. Ann. 137 (1959) 269-303.

[5] A. Frölicher and H. Jarchow, Zur Dualitätstheorie compakt erzeugter und lokalkonvexer Vektorräume, Comment. Math. Helv. 47 (1972) 289-310.

[6] R.A. Graff, Elements of non-linear functional analysis, Mem. Amer. Math. Soc. 16 (1978) Number 206.

[7] R.S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (1982) 65-222.

[8] G. Köthe, Topological vector spaces I (Springer) (1969).

[9] H.J. Kowalsky, Limesräume und Komplettierung 1954, Math. Nachr. 12 (1954) 301-340.

[10] S. Lang, Real analysis (Addison-Wesley) (1983).

[11] L.D. Nel, Cartesian closed coreflective hulls, Quaestiones Math. 2 (1977) 269-283.

[12] L.D. Nel, Convenient topological algebra and reflexive objects, Categorical Topology, Lecture Notes in Math. (Springer) 719 (1979) 259-276.

[13] L.D. Nel, Categorical differential calculus for infinite dimensional spaces, Cahiers Topologie Géom. Différentielle Catégoriques 29 (1988) 257-286.

[14] L.D. Nel, Categorical differential calculus and Banach-Steinhaus, Categorical Topology (Prague Conference proceedings) (World Scientific) (1989) 149-162.

[15] L.D. Nel, Infinite dimensional calculus allowing non-convex domains with empty interior, Monatsh. Math. 110 (1990) 145-166.

[16] V.A. Netes, An existence and uniqueness theorem for the solution of an ordinary differential equation in a locally convex space (Russian), Mat. Zametki 25 (1979) 891-898. English Translation: Math. Notes 25 (1979) 460-463.

[17] F. Schwarz, Topological continuous convergence, Manuscripta Math. 49 (1984) 79-89.

[18] S. Yamamuro, A theory of differentiation in locally convex spaces, Mem. Amer. Math. Soc. 17 (1979) Number 212.

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