Category Theory at Work
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Contravariant Hom-Functors and Monadicity

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ABSTRACT. Let $\mathfrak{C}$ be a complete category. Then $H : \mathfrak{C}^{op} \longrightarrow$ Set is a right adjoint if and only if up to natural isomorphism, it is a hom-functor $\mathfrak{C}(-,A)$, for some $\mathfrak{C}$-object A. In this context, we consider the questions of knowing, which Eilenberg-Moore categories of algebras are induced in Set and in $\mathfrak{C}$ by the objects of $\mathfrak{C}$, i.e. which categories arise as categories of the form $\mathrm{Set}^{\mathbb{L}}$ or $\mathfrak{C}^{\mathbb{T}}$ for the monads $\mathbb{L}$ in Set and $\mathbb{T}$ in $\mathfrak{C}$ whose functor parts are $L = \mathfrak{C}(A^{-},A)$ and $T = A^{\mathfrak{C}(-,A)}$, for some $\mathfrak{C}$-object A. We show that, under mild conditions on A, $\mathfrak{C}^{\mathbb{T}}$ is equivalent to Set^{op}, and so to the category CABool of complete atomic boolean algebras. Furthermore, CABool is, up to equivalence, a reflective, and in general non full, subcategory of $\mathfrak{C}$, whenever there exists no unnatural isomorphism of powers of A, in the sense of V. Trnkova [24]. An approach to the characterization of $\mathrm{Set}^{\mathbb{L}}$ is given, via closure operators. Finally, we characterize the categories of algebras induced in $\mathfrak{C}$ and $\mathrm{Set}^{\mathbb{L}}$ by the corresponding comparison adjunction, assuming that $\mathfrak{C}$ is also well-powered and that A satisfies an injectivity condition

0. Introduction

To each object A of a complete category $\mathfrak{C}$ we can associate a contravariant functor $P = A^{-} : \mathrm{Set} \longrightarrow \mathfrak{C}$ which assigns to each set X the power object A^X and to each function $f : X \longrightarrow Y$ the unique $\mathfrak{C}$-morphism $Pf : A^Y \longrightarrow A^X$ such that $p_x \cdot Pf = p_{f(x)}$, for each $x \in X$, where $(p_x)_{x \in X}$ and $(p_y)_{y \in Y}$ are the corresponding projections. Furthermore, for $H = \mathfrak{C}(-,A)$, the function $\gamma_X : X \longrightarrow HPX = \mathfrak{C}(A^X, A)$ which sends $x \in X$ to the projection p_x and the unique $\mathfrak{C}$-morphisms $\delta_C : C \longrightarrow PHC = A^{\mathfrak{C}(C,A)}$ such that $P_f \cdot \delta_C = f$, for each $f \in \mathfrak{C}(C,A)$, are universal morphisms from X to H and from C to P. Hence, we have a contravariant adjunction which induces the monads $\mathbb{L} = (\mathfrak{C}(A^{-}, A), \gamma, H\delta P)$ in Set and $\mathbb{T} = (A^{\mathfrak{C}(C,A)}, \delta, P\gamma H)$ in $\mathfrak{C}$ that will be called the monads induced by A in Set an in $\mathfrak{C}$, respectively.

This work is an original contribution and will not appear elsewhere.

Section 1 and 2 are concerned with the questions of characterizing the categories of algebras that are induced in $\mathfrak{C}$ and in Set by each object of a fixed category $\mathfrak{C}$. We show that. under mild conditions on A. the category $\mathfrak{C}^{\mathbb{T}}$ is equivalent to the category CABool of complete atomic boolean algebras and derive some consequences of the non existence of unnatural isomorphisms of powers of A. In Section 2 we deal with the characterization of $\mathrm{Set}^{\mathbb{L}}$. This question was raised in [11] for $\mathfrak{C}$ = Top, the category of topological spaces and continuous maps, and some special cases were studied there. Namely, it was proved that the categories of boolean algebras, of complete atomic boolean algebras and of frames are amongst the categories of algebras induced by topological spaces. Indeed, they are induced by the two point discrete space, the two point indiscrete space and the Sierpiński space, respectively. In all the examples given in [11] the topological space A satisfies an injectivity condition. We prove that. when A is an object of a complete well-powered and co-well-powered category $\mathfrak{C}$, this condition implies the following: The Eilenberg-Moore factorization of $\mathfrak{C}(-,A) : \mathfrak{C}^{op} \longrightarrow \mathrm{Set}$ can be obtained via the corresponding factorization of its restriction U to the dual of the category AC(A) of absolutely closed spaces of the reflective hull $\mathfrak{A}$ of {A} in $\mathfrak{C}$. Moreover, $\left(AC(A)^{op}, U\right)$ is an algebraic category over Set. Although these facts were very useful in the special situations we referred to above, they are far from giving a general procedure for the characterization of the categories of algebras induced by right adjoints $H : \mathfrak{C}^{op} \longrightarrow \mathrm{Set}$ from the dual of a fixed complete category $\mathfrak{C}$. We may ask what is the situation like if we replace the category of sets by a monadic category over Set. Right adjoints from cocomplete categories into such categories were studied in [7] by P. Freyd and into categories of relational systems in [16] by A. Pultr, who called our attention to these references.

Given a right adjoint $K : \mathfrak{C}^{op} \longrightarrow \mathrm{Set}^{\mathbb{R}}$, we have that, up to natural isomorphism, $H^{\mathbb{R}}.\ K = \mathfrak{C}(-,A)$ for some $\mathfrak{C}$-object A, $H^{\mathbb{R}}$ denoting the forgetful functor. In the last section, we consider the case where $H^{\mathbb{R}}.\ K$ is the Eilenberg-Moore factorization of $\mathfrak{C}(-,A)$ and so $\mathbb{R} = \mathbb{L}$. Then we show that the injectivity of A with respect to a class of morphisms we will define there, is equivalent to some close relations between the reflective hull of $\{A\}$ in $\mathfrak{C}$ and the categories of algebras induced in $\mathfrak{C}$ and in $\mathrm{Set}^{\mathbb{L}}$ by the adjunction associated with the comparison functor K.

Throughout this paper we shall assume that $\mathfrak{C}$ is a complete and well-powered category. Hence. the reflective hull $\mathfrak{A}$ of $\{A\}$ in $\mathfrak{C}$ is, for each $\mathfrak{C}$-object A, its limit-closure and A is an extremal cogenerator in $\mathfrak{A}$ ([23] ,6.1).

Let $\mathfrak{M}$ be the full subcategory of $\mathfrak{C}^2$, where 2 denote the category with two objects and one morphism from one to another, with objects the extremal monomorphisms of $\mathfrak{C}$. For each subcategory $\mathfrak{B}$ of $\mathfrak{C}$ the $\mathfrak{B}$-closure operator ([3], [5]), originally defined in [18] when $\mathfrak{C}$ = Top, is the functor $[\ ]_{\mathfrak{B}} : \mathfrak{M} \longrightarrow \mathfrak{M}$, which assigns to each extremal monomorphism m the generalized pullback of the equalizers of all pairs (f,g) such that $f \cdot m = g \cdot m$, cod f = cod g being a $\mathfrak{B}$-object. Since $\mathfrak{C}$ has (epi, extremal mono)-factorizations of morphisms ([9], 34.5), we consider the extension of $[\ ]_{\mathfrak{B}}$ to $\mathfrak{C}^2$, also denoted by $[\ ]_{\mathfrak{B}}$ or simply by $[\ ]$, assigning to $f = m.e$ in $\mathfrak{C}$, the $\mathfrak{B}$-closure $[m]$ of m, where m.e is the (epi, extremal mono)-factorization of f.

If $U : \mathfrak{C} \longrightarrow \mathfrak{K}$ has a left adjoint F with unit η and counit ε, let $\mathbf{T}$ be the monad induced by this adjunction in $\mathfrak{K}$, $\Phi : \mathfrak{C} \longrightarrow \mathfrak{K}^{\mathbf{T}}$ be the comparison functor and $Fix(T,\eta)$ be the full subcategory with objects all X in $\mathfrak{K}$ for which η_X is an isomorphism.

We shall make use of the following facts:

0.1 Proposition([2],3.3 Cor.7). The counit ε is pointwise a regular epimorphism if and only if ε_X is the coequalizer of the pair $\left(\varepsilon_{FUX}, FU\varepsilon_X\right)$, for every X in $\mathfrak{C}$.

0.2 Proposition. If $\mathfrak{C}$ has coequalizers then the comparison functor Φ has a left adjoint.

Proof: It follows from Th. 1 in [6]. ∎

0.3 Proposition. For a monad $\mathbf{T} = (T, \eta, \mu)$ in $\mathfrak{K}$ the following assertions are equivalent:

(i) μ is an isomorphism.
(ii) $\eta_T = T\eta$.
(iii) η_T is an isomorphism.
(iv) $\mathfrak{K}^{\mathbf{T}}$ is concretely isomorphic to $Fix(T,\eta)$.

A monad $\mathbf{T}$ is called **idempotent** if it satisfies one, and so all, of the conditions (i) - (iv) in 0.3.

1. Categories of algebras induced in $\mathfrak{C}$ by $\mathfrak{C}$-objects

Let us consider the adjunction $H \dashv P(\delta, \gamma^{op})$ and the monad $\mathbf{T} = \left(A^{\mathfrak{C}(-,A)}, \delta, P\gamma H\right)$ induced in $\mathfrak{C}$ by a $\mathfrak{C}$-object A.

If A is the Sierpiński space, i.e., A is the two point set with one non-trivial open subset, then a description of $Top^{\mathbf{T}}$ is given in [21] (See also [17]). In this case, the $\mathbf{T}$-algebras are exactly the T_0-injective spaces which admit an L-subbase, in the terminology used there. That is, these are exactly the spaces that admit a $\mathbf{T}$-structure map, which, furthermore, is unique. If A is neither the Sierpiński space nor the one point space, topological spaces can have more than one $\mathbf{T}$-structure map ([20], 1.5 (b)). However, topological spaces do not induce the same diversity of categories of algebras over Top as they do over Set. Indeed, if A is a space with at least two points then $A^{-} : Set^{op} \longrightarrow Top$ is weakly monadic, i.e. the comparison functor $\Phi : Set^{op} \longrightarrow Top^{\mathbf{T}}$ is an equivalence.

Let us analyse the same question in the general context we start with. We denote by $\left(M, \Phi, \alpha, \beta^{op}\right) : \mathfrak{C}^{\mathbf{T}} \longrightarrow Set^{op}$ the comparison adjunction. For $(B,\xi) \in \mathfrak{C}^{\mathbf{T}}$, we have that $M(B,\xi)$, being the equalizer object of the morphisms

$$\gamma_{HB}, (H\xi)^{op} : \mathfrak{C}(B,A) \longrightarrow \mathfrak{C}\left(A^{\mathfrak{C}(B,A)}, A\right)$$

which are defined by $\gamma_{HB}(f) = p_f$ and $(H\xi)^{op}(f) = f \cdot \xi$, is the set $\left\{f \in \mathfrak{C}(C,A) \;\middle|\; p_f = f \cdot \xi\right\}$.

We recall that, for each $\mathbf{T}$-algebra (B,ξ) and for each set X, $\alpha_{(B,\xi)}$ and β_X are the unique $\mathbf{T}$-morphism and the unique function which satisfies the equalities $\alpha_{(B,\xi)} \cdot \xi = \Phi\, e_{(B,\xi)}$ and $e_{\Phi X} \cdot \beta_X = \gamma_X$, where $e_{(B,\xi)} = eq\left(\gamma_{HB}, (H\xi)^{op}\right)$ and $e_{\Phi X}$ is defined analogously (See [6], Th. 1).

From now on we shall assume that A is not preterminal (i.e. that there exist at least two parallel morphims in $\mathfrak{C}$ with codomain A) and that the set $\mathfrak{C}(1,A)$ is non-empty, 1 being the terminal object.

If (B,ξ) is a $\mathbf{T}$-algebra then $M(B,\xi)$ is a monosource in $\mathfrak{C}$ and the $\mathbf{T}$-morphisms $f : (B,\xi) \longrightarrow (B',\xi')$ are exactly the $\mathfrak{C}$-morphisms $f : B \longrightarrow B$ such that $g f \in M(B,\xi)$ whenever $g \in M(B',\xi')$ ([20],1.4). The fact that $\mathfrak{C}(1,A) \neq \emptyset$ is important to show that $B = 1$ if $M(B,\xi) = \emptyset$

1.1 Theorem ([20] , 2.1). The comparison functor $\Phi : \mathrm{Set}^{op} \longrightarrow \mathfrak{C}^{\mathbf{T}}$ is an equivalence.

Proof: Since A is not preterminal in $\mathfrak{C}$, then it is easy to prove that $\gamma_X : X \longrightarrow \mathfrak{C}\left(A^X, A\right)$ is a regular monomorphism, for each set X. Thus, by 0.1, γ_X is the equalizer of the pair $\left(\gamma_{HPX}, \left(HP\gamma_X^{op}\right)^{op}\right)$. Hence, by definition of β, β_X is an isomorphism, for each set X.

If $M(B,\xi) \neq \emptyset$ then $e_{(B,\xi)}$ is a split monomorphism in Set and so $\alpha_{(B,\xi)}$ is a split epimorphism, since $\alpha_{(B,\xi)} \cdot \xi = \Phi\, e_{(B,\xi)}$. If $M(B,\xi) = \emptyset$ then B as well as $\Phi M(B,\xi) = A^{\emptyset}$ are the terminal object of $\mathfrak{C}$, and so $\alpha_{(B,\xi)}$ is an isomorphism.

For all $f \in M(B,\xi)$, we have that

$$\hat{p}_f \cdot \alpha_{(B,\xi)} \cdot \xi = p_f = f \cdot \xi$$

where $\left(\hat{p}_f\right)_{f \in M(B,\xi)}$ and $\left(p_g\right)_{g \in \mathfrak{C}(B,A)}$ are the projections. Since ξ is an epimorphism, $\hat{p}_f \cdot \alpha_{(B,\xi)} = f$, for all $f \in M(B,\xi)$, and this implies that $\alpha_{(B,\xi)}$ is a monomorphism since $M(B,\xi)$ is a monosource. Thus $\alpha_{(B,\xi)}$ is an isomorphism for every $\mathbf{T}$-algebra (B,ξ), since it is simultaneously a monomorphism and a split epimorphism.■

It is well-known that CABool is dually equivalent to Set (See e.g. [13], VI 4.6 (a) and (b); [20] , 2.2). From this fact and 1.1 we conclude the following:

1.2 Theorem ([20] , 2.3) The category CABool is monadic, up to equivalence, over all categories which have arbitrary powers of a non preterminal object A and at least one morphism from the terminal object to A.

Since $\mathrm{Top}^{\mathbf{T}}$, when A is the Sierpiński, space, is, up to isomorphism, a reflective subcategory of Top, one may ask what happens when we consider an arbitrary category $\mathfrak{C}$ and a $\mathfrak{C}$-object A satisfying our general assumptions.

In a category, an isomorphism between two powers of a non terminal object, indexed by sets with different cardinalities, is called an **unnatural isomorphism** (See [24]).

1.3 Theorem. If there exists no unnatural isomorphism between any two powers of the $\mathfrak{C}$-object A then $\mathfrak{C}^{\mathbf{T}}$ is, up to equivalence, a reflective and, in general, non-full subcategory of $\mathfrak{C}$.

Proof: By 1.1, $\alpha_{(B,\xi)}$ is a $\mathbf{T}$-isomorphism for each $\mathbf{T}$-algebra (B,ξ). Let ξ and Θ be two $\mathbf{T}$-structure morphisms on B. Then $P^{\mathbf{T}}\alpha_{(B,\Theta)} \cdot P^{\mathbf{T}}\alpha^{-1}_{(B,\xi)}$ is an isomorphism in $\mathfrak{C}$ from $A^{M(B,\xi)}$ to $A^{M(B,\Theta)}$. Then, there is a bijection between the sets $M(B,\xi)$ and $M(B,\Theta)$ and so (B,ξ) and (B,Θ) are isomorphic $\mathbf{T}$-algebras. Thus a skeleton of $\mathfrak{C}^{\mathbf{T}}$ is, up to isomorphism, a reflective (and in general non-full) subcategory of $\mathfrak{C}$.■

The identification of $\mathfrak{C}^{\mathbf{T}}$ with CABool, established in 1.1, as well as 1.3 tell us that any non preterminal object A of a category $\mathfrak{C}$ which admits arbitrary powers of A, no unnatural isomorphisms between any two such powers and at least one morphism from the terminal object to A, lives in a reflective subcategory of $\mathfrak{C}$ that looks like the category of complete atomic boolean algebras. In addition, it has a monadic embedding, though not full in general.

2. Categories of algebras induced in Set by $\mathfrak{C}$-objects

Let $\mathfrak{C}$ be a complete, well-powered and co-well-powered category.

An object B of a subcategory $\mathfrak{B}$ of $\mathfrak{C}$ is said to be **$\mathfrak{B}$-absolutely closed** if every $\mathfrak{C}$-extremal monomorphism $m : B \longrightarrow B'$, with $B' \in \mathfrak{B}$, is a regular monomorphism in $\mathfrak{B}$ ([4]). We remark that a similar notion had already appeared in [12].

Let AC(A) be the category of $\mathcal{A}$-absolutely closed objects of the reflective hull $\mathcal{A}$ of $\{A\}$ in $\mathcal{C}$ and Q denote the class of extremal monomorphisms in $\mathcal{C}$ between $\mathcal{A}$-objects, i.e. Q = ExtMono($\mathcal{C}$) ∩ Mor($\mathcal{A}$).

We recall that a concrete category $(\mathcal{B},\ U : \mathcal{B} \longrightarrow \mathrm{Set})$ is **algebraic** if $\mathcal{B}$ has coequalizers and U is a right adjoint functor which preserves and reflects regular epimorphisms ([9] , 32.1).

2.1 Theorem. If A is Q-injective then $(\mathcal{A}^{op},\ U)$ is an algebraic category, U being the restriction of the functor $\mathcal{C}(-,A)$ to $\mathcal{A}^{op}$. Furthermore, $\mathcal{A}$ = AC(A) and the Eilenberg-Moore factorization of the functor $\mathcal{C}(-,A)$ can be obtained, up to isomorphism, via the corresponding factorization of U.

Proof: By [9] 32.21 (2) ⇔ (3), $(\mathcal{A}^{op},\ U)$ is an algebraic category if and only if A is a regular cogenerator (also called a coseparator) and it is regular injective in $\mathcal{A}$.

If m ∈ ExtMono($\mathcal{C}$) has codomain in $\mathcal{A}$ and m = [m] . c is its factorization through the $\mathcal{A}$-closure of m, then [m] belongs to $\mathcal{A}$ since it is the equalizer of a pair of morphisms in $\mathcal{A}$. Then, the injectivity of A with respect to Q enables us to prove that c is an $\mathcal{A}$-epimorphism. Indeed, given two morphisms f, g in $\mathcal{A}$ such that f c = g c, with f ≠ g, we may assume that its codomain is A, since A is a cogenerator in $\mathcal{A}$, and so conclude that there exist two morphisms f' and g' such that f'[m] = f and g'[m] = g. By definition of [m], then f'm = g' . m implies that f'[m] = g'[m] and so that f = g. Hence, c is an isomorphism if m ∈ ExtMono ($\mathcal{A}$) and so the classes ExtMono($\mathcal{A}$) and RegMono($\mathcal{A}$) coincide. Thus A, being an extremal cogenerator in its reflective hull $\mathcal{A}$, is a regular cogenerator in $\mathcal{A}$. Furthermore, since RegMono($\mathcal{A}$) = ExtMono($\mathcal{A}$) ⊂ Q, A is regular injective in $\mathcal{A}$. We have therefore concluded that $(\mathcal{A}^{op},\ U)$ is algebraic. Now, since U reflects regular epimorphisms and A is Q-injective then Q = Reg Mono ($\mathcal{A}$) and this implies that AC(A) = $\mathcal{A}$.

Let $H = H^{\mathbf{L}} . K$ be the Eilenberg-Moore factorization of the functor $H = \mathcal{C}(-,A) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$. If a subcategory $\mathcal{B}$ of $\mathcal{C}$ contains the images of all sets by the left adjoint P of H, then we can consider P co-restricted to $\mathcal{B}$ obtaining, in this way, the left adjoint to the restriction H' of H to $\mathcal{B}^{op}$. It is easily seen that this adjunction induces in Set the same monad than the former one does. Furthermore, if $\mathcal{B}$ is reflective in $\mathcal{C}$ with reflector R and embedding E, and $H' = H^{\mathbf{L}} . K'$ is its Eilenberg-Moore factorization then, for each C ∈ $\mathcal{C}$,

$$H^{\mathbb{L}} K' R^{op}(C) = H^{\mathbb{L}}\left(\mathfrak{C}\ (R(C),\ A),\ H\ \delta_C^{op}\right) = \mathfrak{C}(R(C),\ A) \cong \mathfrak{C}(C,A) = H^{\mathbb{L}}KC$$

and the reflection $r : Id_{\mathfrak{C}} \xrightarrow{\bullet} ER$ brings out a natural isomorphism between K and $K' . R^{op}$. Observing that the reflective hull of $\{A\}$ in $\mathfrak{C}$ is the smallest reflective subcategory $\mathbb{B}$ satisfying the above conditions and that it coincides with AC(A), then the proof is concluded.■

For $\mathfrak{C}$ = Top, the fact that the Eilenberg-Moore factorization of $\mathfrak{C}(-,A)$ can be obtained via the corresponding factorization of its restriction to $\mathfrak{A}$. was a useful tool in [11] to characterize the category $Set^{\mathbb{L}}$ of $\mathbb{L}$-algebras. In all the examples considered there $\mathfrak{A}$ = AC(A). The coincidence of these two categories was analysed in [19] where we considered the subcategory of $\mathfrak{E}$-absolutely closed objects of the epireflective hull $\mathfrak{E}$ of $\{A\}$ in Top. Also, it is easy to prove that the $\mathfrak{E}$-absolutely closed objects are exactly the $\mathfrak{A}$-absolutely closed objects, using the fact that an object of $\mathfrak{C}$ belongs to $\mathfrak{E}$ if and only if it is an extremal subobject of some power of A.

The T_0-spaces which are neither T_1 nor sober, are examples of objects that are not Q-injective, with respect to the corresponding class Q. Indeed, the epireflective hull of each such space A is Top_0, because there does not exist any epireflective subcategory of Top strictly between Top_0 and Top_1 ([8], Prop. 1.1). Observing that AC(A) = Sob, the full subcategory of sober spaces which are exactly the Top_0-absolutely closed spaces, and that AC(A) is strictly contained in $\mathfrak{A}$, the conclusion follows from 2.1 .

To our knowledge there is not known any way to decide, given a monadic category over Set, i.e. an equationally presentable category of algebras with free algebras ([15] Chap 1), when there does or does not exist a topological space inducing it. For that we would need to know if the categories of algebras induced by topological spaces have some particular behaviour, for example in terms of Mal'sev conditions. Then, it could be interesting to know if the properties of the topological spaces are, in some way, reflected by those of the corresponding category of algebras.

In this context, it is natural to make a first distinction between the topological spaces which induce a finitary theory and those inducing an infinitary one. Though we have not got a complete answer to this question, we remark that the two point discrete space induces a finitary theory and it is easily proved that this is false for each topological space A, which is not a T_1-space. Indeed, the theory $\mathbb{L}$ is finitary if for each set X and each $h \in LX = \mathrm{Top}\left(A^X, A\right)$ there exist a natural number n, a function $f : n \longrightarrow X$ and a continuous map $t \in Ln = \mathrm{Top}\left(A^n, A\right)$, such that $t . Pf = h$ ([15] , 1.4.18). If A has an indiscrete subspace $\{a,b\}$ then $h : A^{\mathbf{N}} \longrightarrow A$, defined by

$$h\left[\left(a_n\right)\right] = \begin{cases} a & \text{if } a_n = a \quad \text{for exactly one } n \in \mathbf{N} \\ b & \text{otherwise} \end{cases}$$

is a continuous function which does not satisfy the above conditions. The same holds with

$$h\left[\left(a_n\right)\right] = \begin{cases} b & \text{if } a_n = b \quad \text{for all } n \in \mathbf{N} \\ a & \text{otherwise} \end{cases}$$

if $\{a\}$ but not $\{b\}$ is an open subset of the subspace $\{a,b\}$.

3. Monads induced by the comparison adjunction

The comparison functor $K : \mathfrak{C}^{op} \longrightarrow \mathrm{Set}^{\mathbb{L}}$ has a left adjoint, by 0.2, since $\mathfrak{C}$ is assumed to be complete. Let us denote by $\left(N, K , \sigma , \rho^{op}\right) : \mathrm{Set}^{\mathbb{L}} \longrightarrow \mathfrak{C}^{op}$ the comparison adjunction and by $\mathbf{S} = \left(N K , \sigma , K \rho^{op} N\right)$ and $\mathbf{S}' = \left(NK, \rho , (N\sigma K)^{op}\right)$ the monads it induces in $\mathrm{Set}^{\mathbb{L}}$ and in $\mathfrak{C}$, respectively.

We recall that, for each $\mathbb{L}$-algebra (X,ξ), $N(X,\xi)$ is the equalizer object of the $\mathfrak{A}$-morphisms $(P\xi)^{op}$ and δ_{PX} and so that $N(X,\xi)$ belongs to $\mathfrak{A}$. We denote the equalizer by $e_{(X,\xi)}$.

3.1 Lemma. For each $\mathcal{C}$-object C, $\left[\delta_C\right]_{\mathcal{A}} = e_{KC}$.

Proof: Since the closure operators relative to the subcategories $\{A\}$ and $\mathcal{A}$ coincide ([8], 2.1), i.e. $[\ ]_{\{A\}} = [\ ]_{\mathcal{A}}$, it is enough to prove that $\left[\delta_C\right]_{\{A\}} = e_{KC}$ and this is essentially lemma 1 in [14].∎

3.2 Lemma ([22], 2.2). If A is injective with respect to the class $D = \left\{\left[\delta_C\right]_{\mathcal{A}} \mid C \in \mathcal{C}\right\}$ then ρ_C is an $\mathcal{A}$-epimorphism, for each $C \in \mathcal{C}$.

Proof: For each object C of $\mathcal{C}$, ρ_C is the unique $\mathcal{C}$-morphism such that $e_{KC} \cdot \rho_C = \delta_C$. If f and g are morphisms in $\mathcal{A}$ such that $f \cdot \delta_C = g \cdot \delta_C$, we may assume, without loss of generality, that its codomain is A, because A is a cogenerator of its reflective hull $\mathcal{A}$. The D-injectivity of A enables us to prove, in a complete analogous way we did in the proof of 2.1, that ρ_C is an $\mathcal{A}$-epimorphism.∎

3.3 Theorem ([22], 4.1) For an object A of $\mathcal{C}$ the following are equivalent:

(i) A is D-injective.

(ii) The comparison functor K induces an equivalence between $\mathcal{A}^{op}$ and $\left(Set^{\mathbf{L}}\right)^{\mathbf{S}}$.

(iii) $\mathcal{A}$ and $\mathcal{C}^{\mathbf{S}'}$ are concretely isomorphic over $\mathcal{C}$.

Proof: (i) ⇒ (ii) As we already observed, $N(X,\xi)$ is an $\mathcal{A}$-object for every $\mathbf{L}$-algebra (X,ξ). This implies that N is left adjoint to the restriction K_1 of K to $\mathcal{A}^{op}$ and that this adjunction induces in $Set^{\mathbf{L}}$ the same monad as the former one does.

For each $\mathcal{A}$-object B, δ_B is an extremal monomorphism in $\mathcal{A}$, because B is an extremal subobject of some power of A in its reflective hull. But, by 3.2, ρ_B is an epimorphism in $\mathcal{A}$ and so, from the equality $\delta_B = \left[\delta_B\right] \cdot \rho_B$, we conclude that ρ_B is an isomorphism. Hence $K\rho^{op}N$ is an isomorphism and, consequently, $\mathbf{S}$ is an idempotent monad (0.3 (i)). It is now clear that $K_1 : \mathfrak{A}^{op} \longrightarrow \mathrm{Fix}(KN, \sigma) \cong (\mathrm{Set}^{\mathbb{L}})^{\mathbf{S}}$ is an equivalence.

(ii) ⇒ (iii) If K_1 is an equivalence then ρ_B is an isomorphism for each B in $\mathfrak{A}$. Then ρ_{NK} is a natural isomorphism and so S' is an idempotent monad (0.3 (iii)). The functor $G : \mathfrak{A} \longrightarrow \mathfrak{C}^{S'} = \mathrm{Fix}(NK, \rho)$, defined on objects by $G(B) = (B, \rho_B^{-1})$, is a concrete isomorphism over $\mathfrak{C}$ in the sense that $U^{S'} . G = E$, where E is the embedding of $\mathfrak{A}$ in $\mathfrak{C}$ and $U^{S'} : \mathfrak{C}^{S'} \longrightarrow \mathfrak{C}$ is the forgetful functor.

(iii) ⇒ (i) For each $\mathfrak{C}$-object C, ρ_C is the reflection of C in $\mathfrak{A}$. Indeed, NKC is an $\mathfrak{A}$-object and the fact that $\rho_C : C \longrightarrow NKC \cong U^{S'} F^{S'} C$, where $F^{S'}$ denotes the left adjoint to $U^{S'}$, is universal from C to the embedding $E : \mathfrak{A} \longrightarrow \mathfrak{C}$ is a straightforward consequence of the hypothesis.

Let $f : NKC \longrightarrow A$ be a $\mathfrak{C}$-morphism. Then, by definition of δ, $p_g . \delta_C = g$ for $g = f . \rho_C$. Since $p_g . [\delta_C] . \rho_C = f . \rho_C$ and ρ_C is the reflection morphism, then $p_g . [\delta_C] = f$ and so A is D-injective.∎

By 3.3 (i) ⇒ (ii), $\mathfrak{A}$ is dually equivalent to a monadic category $(\mathrm{Set}^{\mathbb{L}})^{\mathbf{S}}$ over a complete and well-powered category $\mathrm{Set}^{\mathbb{L}}$, because monadic functors create limits. Thus we have the following:

3.4 Corollary. If A is D-injective then $\mathfrak{A}$ is a cocomplete and cowell-powered category.

This was the main aim of [22].

It is known that if $\mathfrak{C}$ is also co-well-powered then it has (epi, extremal mono) - factorizations of sources and this implies the existence of certain colimits in $\mathfrak{C}$ ([10] , 1.1) and so in its full reflective subcategories.

In the recent book [1] it is proved that every cocomplete and co-well-powered category with a generator is complete and well-powered (Th. 12.13). By its dual, a stronger result than 3.4 is obtained. Indeed, it follows that each object of a complete and well-powered category has a cocomplete and co-well-powered reflective hull. The only advantage of the extra condition seems to be the explicit description of the colimits in the reflective hull.

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