Category Theory at Work
H. Herrlich, H.-E. Porst (eds.)

289–306

Universal Algebraic Completions of Right Adjoint Functors

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Abstract

We introduce a general concept of an algebraic category, the (**E**, **M**)-algebraic categories (**A**, U) over an (**E**, **M**)-category **X**, i.e. U is a right adjoint and lifts (**E**, **M**)-factorizations uniquely. We prove some basic properties of (**E**, **M**)-categories. Given any right adjoint functor $U : \mathbf{A} \to \mathbf{X}$ where **X** is an (**E**, **M**)-category with enough **E**-projectives. we construct an (**E**, **M**)-algebraic category $(\widehat{\mathbf{A}}, \widehat{U})$ over **X** together with a comparison functor $E : (\mathbf{A}, U) \to (\widehat{\mathbf{A}}, \widehat{U})$ and prove that E is universal. We prove that $(\widehat{\mathbf{A}}, \widehat{U})$ inherits the rank of (**A**, U). We give a full embedding and reflectivity criterion for E. Some examples of $(\widehat{\mathbf{A}}, \widehat{U})$ over **Set** are determined.

0 Introduction

Category Theory answers the question what the *algebraicity* and the *topologicity* of classes of structures is. Whereas topologicity is described by the categorical notion of a *topological* (i.e., *initial structures* admitting) category (for a survey see Herrlich [4] or [1]), there are several competitive categorical concepts of algebraicity, e.g. *monadic, varietal, algebraic, regular, essentially algebraic categories* (or *functors*) (for a survey see [1] or [12]). In this paper, we deal with a more general notion of Herrlich's concept of a *regular* functor [3]. A functor $U : \mathbf{A} \to \mathbf{X}$ is called *regular* provided that U is a right adjoint, **X** has (regular epi, monosource)-factorizations and U uniquely lifts (regular epi, monosource)-factorizations. (Note that the property **X** *has (regular epi, monosource)-factorizations* is a categorical version of the *Homomorphism Theorem* for algebras). This axiomatic concept of a regular functor is justified by the following remarkable *Representation Theorem* essentially due to Linton [9] (see also [12: I.4]): A functor $U : \mathbf{A} \to \mathbf{Set}$ is regular iff $(\mathbf{A}, U)$ is concretely isomorphic to a quasivariety (i.e., to an implicationally defined class) of universal algebras of

This work is an original contribution and will not appear elsewhere.

a possibly infinitary or rankless type, considered as a concrete category over **Set**. Now, by extending the (regular epi, monosource)-factorization structure to any $(\mathbf{E},\mathbf{M})$-*factorization structure* on $\mathbf{X}$, we introduce the concept of an $(\mathbf{E},\mathbf{M})$-*algebraic category* $(\mathbf{A},U)$ over $\mathbf{X}$, i.e. $U:\mathbf{A}\to\mathbf{X}$ is a right adjoint, $\mathbf{X}$ is an $(\mathbf{E},\mathbf{M})$-category and U uniquely lifts $(\mathbf{E},\mathbf{M})$-factorizations (see 2.1). This concept makes decisive use of the fundamental categorical tool of *factorization structures*; moreover, there are natural examples of $(\mathbf{E},\mathbf{M})$-algebraic categories which are not covered by one of the above mentioned notions of algebraicity (see 2.2 (b)).

One of the main topics in Category Theory is the construction and investigation of *topological completions* of (concrete) categories, e.g., the *universal initial completion*, the *MacNeille completion*, the *cartesian closed topological hull* etc. (see, e.g., Herrlich [5], [6]). The main goal of this paper is to provide $(\mathbf{E},\mathbf{M})$-*algebraic completions* of categories $(\mathbf{A},U)$ over $\mathbf{X}$. We show that they exist provided that $U:\mathbf{A}\to\mathbf{X}$ is a right adjoint and $\mathbf{X}$ is an $(\mathbf{E},\mathbf{M})$-category with enough $\mathbf{E}$-projective objects. We construct an $(\mathbf{E},\mathbf{M})$-algebraic category $(\hat{\mathbf{A}},\hat{U})$ over $\mathbf{X}$ together with a comparison functor $E:(\mathbf{A},U)\to(\hat{\mathbf{A}},\hat{U})$. Construction and verification of $(\hat{\mathbf{A}},\hat{U})$ require more effort than topological completions. The main property we prove is that E is *universal*, i.e., each functor $D:(\mathbf{A},U)\to(\mathbf{B},V)$ over $\mathbf{X}$ where $(\mathbf{B},V)$ is $(\mathbf{E},\mathbf{M})$-algebraic factorizes in a unique way through E. So, we may speak about E as the *universal* $(\mathbf{E},\mathbf{M})$-*algebraic completion*. The investigation of algebraic properties preserved under this completion may be of interest, in particular we show that the rank of $(\mathbf{A},U)$ is carried over to $(\hat{\mathbf{A}},\hat{U})$. Unlike topological completions, $E:(\mathbf{A},U)\to(\hat{\mathbf{A}},\hat{U})$ need not be full (but it is *partially* full). We characterize internally those $(\mathbf{A},U)$ for which E is full and a full reflective embedding, resp., and we obtain an internal characterization of those categories $(\mathbf{A},U)$ over $\mathbf{X}$ which are fully and reflectively embeddable into a regular category over $\mathbf{X}$, thus getting an *algebraic* counterpart of the well-known *topological* result that a category $(\mathbf{A},U)$ over $\mathbf{X}$ is *solid* (formerly called *semi-topological*) iff it is fully and reflectively embeddable into a topological category over $\mathbf{X}$ (see Hoffmann [8], Tholen [14]). Thus, the essence of this paper is to supplement topological by algebraic completions and to demonstrate similarities and differences in methods and results.

1 Terminology

We assume that the reader is familiar with the basic concepts and results of Category Theory. In particular, we use notions and notations as they occur in [1] which the reader is referred to if he/she wants to look up a notion or a result.

1.1 *Categories over* X

A *category over (a base category)* $\mathbf{X}$ is a pair $(\mathbf{A},U)$ where $U:\mathbf{A}\to\mathbf{X}$ is a (not necessarily faithful) functor. A *functor* $D:(\mathbf{A},U)\to(\mathbf{B},V)$ *over* $\mathbf{X}$ between categories over $\mathbf{X}$ is a functor $D:\mathbf{A}\to\mathbf{B}$ with $U=VD$.

1.2 (E, M)-*factorization structures ([1: 15.])*

Let **X** be an (E, M)-*category* (i.e. E is a class of **X**-morphisms, M a conglomerate of **X**-monosources, E and M are closed under composition with **X**-isomorphisms such that **X** has (E, M)-factorizations of sources and has the unique (E, M)-diagonalization property ([1: 15.1]). Then we use the following facts (see [1: 15.], [7]): E ∩ M = {**X**-isomorphisms}. E consists only of **X**-epi(morphism)s. E is closed under composition. M is closed under composition of **X**-sources. If f, g are **X**-morphisms with fg defined and $fg \in$ E, then $f \in$ E. If $(f_i)_I$ is an **X**-source, J a subclass of I such that $(f_i)_J$ belongs to M, then also $(f_i)_I$ belongs to M. Each regular epi in **X** belongs to E. **X** has coequalizers.

1.3 *Projective objects and right adjoint functors*

Let E be a class of **X**-morphisms. An **X**-object X is said *to have an* E-*projective cover* provided that there exists a morphism $e : P \to X$ in E such that P is an E-projective object. **X** is said *to have enough* E-*projectives* provided that each **X**-object has an E-projective cover.

A very useful tool is the following

Lemma. *Let U : **A** → **X** be a functor with a left adjoint F and* E *be a class of* **X**-*morphisms. Then the following conditions hold:*

(a) for each E-*projective object X FX is a U^{-1}[E]-projective object,*

(b) if E *satisfies the cancellation property "$fg \in$ E $\Rightarrow f \in$ E" (see 1.2) and if* **X** *has enough* E-*projectives, then for each* **A**-*object A there exists an* E-*projective object P and a U^{-1}[E]-morphism $e : FP \to A$, hence* **A** *has enough U^{-1}[E]-projectives.*

1.4 *Categories with rank (Herrlich [2])*

Let α be a regular cardinal and **I** be the category which is induced by a partially ordered set $(I, \leq)$. $(I, \leq)$ is called α-filtered provided that each subset J of I with card $J < \alpha$ has an upper bound in $(I, \leq)$. For an α-filtered $(I, \leq)$ the colimit of a diagram $D : \mathbf{I} \to \mathbf{A}$ is called α-filtered. A category (**A**, U) over **X** is said *to have a rank* $\leq \alpha$ provided that $U : \mathbf{A} \to \mathbf{X}$ sends each α-filtered colimit in **A** to an episink in **X**.

2 Algebraic Categories

As outlined in Chapter 0, we introduce the fairly general and flexible notion of an (E, M)-*algebraic category* over **X**, using arbitrary factorization structures (E, M) on **X**. But note that in this paper M consists only of **X**-monosources (see 1.2). We collect some basic properties of (E, M)-algebraic categories.

Definition 2.1. A functor $U : \mathbf{A} \to \mathbf{X}$ or a category $(\mathbf{A}, U)$ over $\mathbf{X}$ is called $(\mathbf{E}, \mathbf{M})$-*algebraic* provided that the following conditions are satisfied:

(ALG 0) $\mathbf{X}$ is an $(\mathbf{E}, \mathbf{M})$-category

(ALG 1) U is a right adjoint

(ALG 2) U *lifts* $(\mathbf{E}, \mathbf{M})$-*factorizations uniquely*, i.e., for every $\mathbf{A}$-source $(f_i : A \to A_i)_I$ and every $(\mathbf{E}, \mathbf{M})$-factorization $(UA \xrightarrow{e} X \xrightarrow{m_i} UA_i)_I$ of $(Uf_i)_I$ there exists exactly one factorization $(A \xrightarrow{\overline{e}} \overline{X} \xrightarrow{\overline{m}_i} A_i)_I$ of $(f_i)_I$ in $\mathbf{A}$ with $U\overline{e} = e$ and $(U\overline{m}_i)_I = (m_i)_I$.

Remark 2.2.

(a) The *regular* functors in the sense of Herrlich [3] are just the (regular epi, monosource)-algebraic functors. (In [1], regular functors are more distinctly called *regularly algebraic.*)

(b) $(\mathbf{E}, \mathbf{M})$-algebraic categories occur quite naturally in Topological Algebra. If $\mathbf{A}$ is any quasivariety and $\mathbf{A}(\mathbf{Top})$ the associated category of topological $\mathbf{A}$-algebras, then the forgetful functor $U : \mathbf{A}(\mathbf{Top}) \to \mathbf{Top}$ need not be regular (see Richter [12: III.2.13], Nel [10]), but U is (surjective, initial point separating)-algebraic.

Proposition 2.3. *Let* $U : \mathbf{A} \to \mathbf{X}$ *be* $(\mathbf{E}, \mathbf{M})$-*algebraic. Then the following conditions hold:*

(a) U *is faithful,*

(b) U *creates isomorphisms, hence reflects isomorphisms and identities.*

PROOF.

(a) Let ε be the counit belonging to the right adjoint U. It is sufficient to show that each ε_A, A $\mathbf{A}$-object, is an $\mathbf{A}$-epi (see [1: 19.14]). Since each $U\varepsilon_A$ is a retraction, hence a regular epi, it belongs to $\mathbf{E}$ (see 1.2). Thus, it is sufficient to prove that U reflects $\mathbf{E}$-morphisms into $\mathbf{A}$-epis: Let $e : A \to B$ be in $\mathbf{A}$ with $Ue \in \mathbf{E}$ and $(p, q) : B \rightrightarrows C$ a pair in $\mathbf{A}$ with $pe = qe$, hence $UpUe = UqUe$, so $Up = Uq$, since Ue is an $\mathbf{X}$-epi (see 1.2). Therefore, $(e, (p, q, id_B))$ and $(e, (q, p, id_B))$ are factorizations of (pe, qe, e) so that $(Ue, (Up, Uq, id_{UB})) = (Ue, (Uq, Up, id_{UB}))$ is an $(\mathbf{E}, \mathbf{M})$-factorization of (Upe, Uqe, Ue) (note that $(Up, Uq, id_{UB}) \in \mathbf{M}$ by 1.2). From (ALG 2) it follows that $(e, (p, q, id_B)) = (e, (q, p, id_B))$, hence $p = q$.

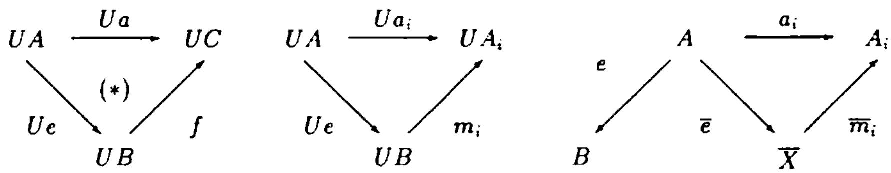
(b) We prove firstly that U reflects identities: Let $f : A \to B$ be in $\mathbf{A}$ with $Uf = id_X$ where $X = UA = UB$. (f, id_B) and (id_A, f) are factorizations of f for which $(Uid_A, Uf) = (Uf, Uid_B) = (id_X, id_X)$ is an $(\mathbf{E}, \mathbf{M})$-factorization of Uf. From (ALG 2) it follows that $f = id_A$. Now, let $f : UA \to X$ be an $\mathbf{X}$-isomorphism with inverse g. Then (f, g) is an $(\mathbf{E}, \mathbf{M})$-factorization of Uid_A. Therefore there exist $\overline{f} : A \to \overline{X}$ and $\overline{g} : \overline{X} \to A$ in $\mathbf{A}$ with $\overline{g}\overline{f} = id_A$, $U\overline{f} = f$ and $U\overline{g} = g$. Hence, $\overline{f}$ is a section in $\mathbf{A}$. Since $U\overline{f} = f$ is an $\mathbf{X}$-epi and U is faithful by (a), $\overline{f}$ is an $\mathbf{A}$-epi, hence an $\mathbf{A}$-isomorphism. Let $h : A \to B$

be in **A** with $Uh = f$. Then, for the inverse $k : \overline{X} \to A$ of $\overline{f}$, $a := hk : \overline{X} \to B$ is in **A** with $h = a\overline{f}$ and $Ua = id_X$, hence $a = id_{\overline{X}}$ and $h = \overline{f}$, i.e., the uniqueness of an $h : A \to B$ in **A** with $Uh = f$ is proved. □

The reflection of isomorphisms is a typically *algebraic* property which distinguishes the (**E**, **M**)-algebraic from the topological functors. Also the following useful property is *algebraic*; from it it follows that (**E**, **M**)-algebraic functors $U : \mathbf{A} \to \mathbf{X}$ lift the (**E**, **M**)-factorization structure from **X** to **A** in a canonical way.

Proposition 2.4. *Every* (**E**, **M**)*-algebraic functor* $U : \mathbf{A} \to \mathbf{X}$ *reflects* **E***-morphisms into* U*-final morphisms.*

PROOF. Let $e : A \to B$ be in **A** with $Ue \in \mathbf{E}$, $a : A \to C$ in **A** and $f : UB \to UC$ in **X** such that the left diagram



commutes.

Let $(m_i : UB \to UA_i)_I$ be the source of all U-structured morphisms with domain UB such that for each $i \in I$ the composition $m_i Ue$ is an **A**-morphism $a_i : A \to A_i$. There exists $j \in I$ with $B = A_j$ and $id_{UB} = m_j$. Hence $(m_i)_I \in \mathbf{M}$ by 1.2 and $Ua_j = Ue$, hence $a_j = e$ by 2.3. The (**E**, **M**)-factorization $(Ue, (m_i)_I)$ of $(Ua_i)_I$ is lifted by U to a factorization $(\overline{e}, (\overline{m}_i)_I)$ of $(a_i)_I$. Hence (e, id_B) and $(\overline{e}, \overline{m}_j)$ are factorizations of a_j and they must be equal by $U\overline{m}_j = m_j = id_{UB}$ and (ALG 2). Thus $e = \overline{e}$. Since $(*)$ commutes, there exists $k \in I$ with $m_k = f$ and $A_k = C$. Hence we have for $\overline{m}_k : B \to A_k$ that $U\overline{m}_k = m_k = f$ holds, hence $a = \overline{m}_k e$, since U is faithful. □

Proposition 2.5. *If* $U : \mathbf{A} \to \mathbf{X}$ *is* (**E**, **M**)*-algebraic, then* **A** *is a* $(U^{-1}[\mathbf{E}], U^{-1}[\mathbf{M}])$*-category.*

PROOF. That **A** has $(U^{-1}[\mathbf{E}], U^{-1}[\mathbf{M}])$-factorizations is obvious from (ALG 2). The diagonalization property follows immediately from (ALG 0), from 2.4 and the faithfulness of U (2.3). □

Proposition 2.6. *Every* (**E**, **M**)*-algebraic functor* $U : \mathbf{A} \to \mathbf{X}$

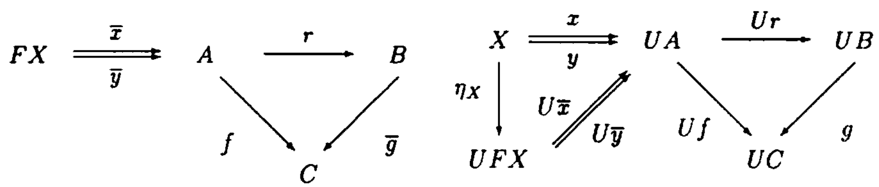
(a) reflects regular epimorphisms,

(b) creates limits,

(c) detects colimits.

PROOF.

(a) Let F be the left adjoint of U and η the unit. Let $r : A \to B$ be in **A** and $(x, y) : X \rightrightarrows UA$ be a pair in **X** such that Ur is a coequalizer of (x, y) in **X**. There exist **A**-morphisms $\overline{x}, \overline{y} : Fx \rightrightarrows A$ with $x = (U\overline{x})\eta_X$ and $y = (U\overline{y})\eta_X$.



Because of the universality of η_X, r equalizes $(\overline{x}, \overline{y})$. Now, let $f : A \to C$ be in **A** also equalizing $(\overline{x}, \overline{y})$. Then Ur equalizes (x, y). Since Ur is the coequalizer of (x, y), there exists $g : UB \to UC$ in **X** with $Uf = gUr$. Since Ur is a regular epi, it belongs to **E** (1.2). By 2.4, r is U-final. Hence there exists $\overline{g} : B \to C$ in **A** with $f = \overline{g}r$.

(b) and

(c) follow immediately from 2.3, 2.5, 1.2 and [1: 23.11]. (Observe that (**E**, **M**)-algebraic categories are essentially algebraic as defined in [1: 23.1].) □

3 Algebraic Completions: Construction

Our objective is to algebraically complete categories $(\mathbf{A}, U)$ over **X**, i.e., to construct an (**E**, **M**)-algebraic category $(\hat{\mathbf{A}}, \hat{U})$ over **X** together with a *comparison* functor $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$.

Construction of $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$. Assume that the category $(\mathbf{A}, U)$ over **X** satisfies the following conditions:

(I) **X** is an (**E**, **M**)-category and has enough **E**-projectives,

(II) $U : \mathbf{A} \to \mathbf{X}$ has a left adjoint F with unit η.

Under these conditions the construction of an (**E**, **M**)-algebraic completion $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ is possible, as we now demonstrate.

Let $(UA \xrightarrow{e} X \xrightarrow{m_i} UA_i)_I$ be an (**E**, **M**)-factorization of $(Uf_i)_I$ for some **A**-source $(f_i : A \to A_i)_I$. If there is a functor $E : (\mathbf{A}, U) \to (\mathbf{B}, V)$ over **X** where $(\mathbf{B}, V)$ is (**E**, **M**)-algebraic, then there exists a unique **B**-*algebra structure* $\overline{X}$ over X such that e and $(m_i)_I$ lift to a **B**-morphism $\overline{e} : EA \to \overline{X}$ and a **B**-source $(m_i : \overline{X} \to EA_i)_I$, resp., with $V\overline{e} = e$ and $(V\overline{m}_i)_I = (m_i)_I$. Thus, it is obvious to consider such $e : UA \to X$ as the candidates for the objects of the (**E**, **M**)-algebraic category $\hat{\mathbf{A}}$ over **X** we want to construct.

Definition 3.1.

(a) If $(UA \xrightarrow{e} X \xrightarrow{m_i} UA_i)_I$ is an $(\mathbf{E}, \mathbf{M})$-factorization such that all $m_i e$ are $\mathbf{A}$-morphisms $A \to A_i$ $(i \in I)$, then the pair (A, e) is called an $(\mathbf{E}, \mathbf{M})$-*structure quotient (with respect to* U*)*.

(b) A *map* $f : (A, e) \to (A', e')$ between $(\mathbf{E}, \mathbf{M})$-structure quotients is an $\mathbf{X}$-morphism $f : X \to X'$ such that there exists an $\mathbf{A}$-morphism $a : A \to A'$ so that the diagram

$$\begin{array}{ccc} UA & \xrightarrow{Ua} & UA' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ X & \xrightarrow{f} & X' \end{array}$$

commutes.

Let $e : UA \to X$ be an $(\mathbf{E}, \mathbf{M})$-structure quotient. By (I), (II) and 1.3 there exists a $U^{-1}[\mathbf{E}]$-projective cover $a : P \to A$. Then $eUa : UP \to X$ is also an $(\mathbf{E}, \mathbf{M})$-structure quotient, and the following Lemma shows that it is sufficient for the construction of $\hat{\mathbf{A}}$ to consider $(\mathbf{E}, \mathbf{M})$-structure quotients $e : UP \to X$ with P being a $U^{-1}[\mathbf{E}]$-projective object.

Lemma 3.2. *Let* $e : UA \to X$ *and* $e' : UA' \to X$ *be* $(\mathbf{E}, \mathbf{M})$-*structure quotients and* $a : A \to A'$ *an* $\mathbf{A}$-*morphism with* $e = e'Ua$. *Let* $E : (\mathbf{A}, U) \to (\mathbf{B}, V)$ *be a functor over* $\mathbf{X}$ *into an* $(\mathbf{E}, \mathbf{M})$-*algebraic* $(\mathbf{B}, V)$. *Then, for every* V-*lift* $\bar{e} : EA \to \overline{X}$ *of* $e : UA = VDA \to X$ *and* $\bar{e}' : EA' \to \overline{X}'$ *of* $e' : VDA' \to X$, *the identity* $\overline{X} = \overline{X}'$ *holds.*

PROOF. Since $\bar{e} : EA \to \overline{X}$ is V-final by 2.4, it follows from $id_X V\bar{e} = e = e'Ua = V\bar{e}'VEa = V(\bar{e}'Ea)$ that there exists $b : \overline{X} \to \overline{X}'$ in $\mathbf{B}$ with $Vb = id_X$. Thus, by 2.3, $b = id_{\overline{X}}$, hence $\overline{X} = \overline{X}'$. □

The Lemma shows that two different $(\mathbf{E}, \mathbf{M})$-structure quotients with the same codomain X may induce identical *algebraic structures* on X. This means for our construction of $\hat{\mathbf{A}}$ that we have to identify certain $(\mathbf{E}, \mathbf{M})$-structure quotients. In view of the Lemma, the identification is given by the following equivalence relation:

Definition 3.3. Two $(\mathbf{E}, \mathbf{M})$-structure quotients $e : UP \to X$ and $e' : UP' \to X$ are said to be *equivalent* provided that there exist $\mathbf{A}$-morphisms $a : P \to P'$ and $a' : P' \to P$ such that the diagram

$$\begin{array}{ccc} UP & & \\ {\scriptstyle Ua}\downarrow\uparrow{\scriptstyle Ua'} & \searrow^{e} & \\ & & X \\ & \nearrow_{e'} & \\ UP' & & \end{array}$$

commutes, i.e., $e = e'Ua$ and $e' = eUa'$.

The equivalence class of $e : UP \to X$ is denoted by $[P, e]$.

$(\widehat{\mathbf{A}}, \widehat{U})$:

The objects of $\widehat{\mathbf{A}}$ are the equivalence classes $[P, e]$ of $(\mathbf{E}, \mathbf{M})$-structure quotients (P, e) where P is a $U^{-1}[\mathbf{E}]$-projective object. $f : [P, e] \to [P', e']$ is an $\widehat{\mathbf{A}}$-morphism (resp., an $\widehat{\mathbf{A}}$-identity) provided that $f : (P, e) \to (P', e')$ is a map (resp., $f = id_X$ if (P, e) and (P', e') are equivalent). The composition of $\widehat{\mathbf{A}}$ is adopted from $\mathbf{X}$. $\widehat{U} : \widehat{\mathbf{A}} \to \mathbf{X}$ is defined by $\widehat{U}[P, e] = X =$ codomain of e and $\widehat{U}f = f$. Hence, $\widehat{U}$ is faithful.

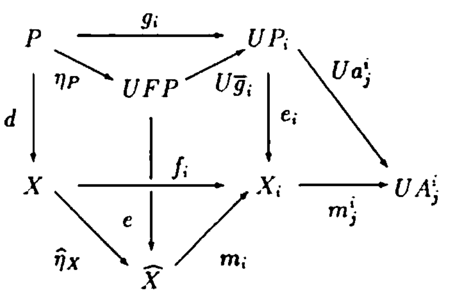
$E : (\mathbf{A}, U) \to (\widehat{\mathbf{A}}, \widehat{U})$:

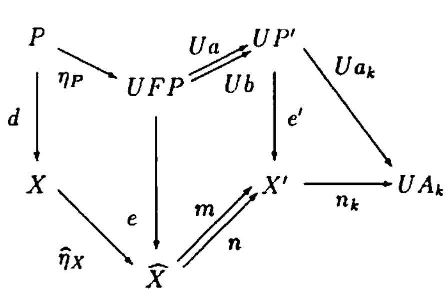
For any $\mathbf{A}$-object A, there exists a $U^{-1}[\mathbf{E}]$-projective cover $d : P \to A$ (see 1.3). Since the singleton source (Uid_A) belongs to $\mathbf{M}$ (see 1.2), the pair (P, Ud) is an $(\mathbf{E}, \mathbf{M})$-structure quotient. Put $EA = [P, Ud]$. This object assignment $Ob\mathbf{A} \to Ob\widehat{\mathbf{A}}$ is well-defined, since if there exists another $U^{-1}[\mathbf{E}]$-projective cover $d' : P' \to A$, then (P, Ud) and (P', Ud') are equivalent. For any $\mathbf{A}$-morphism $f : A \to A'$, Uf is an $\widehat{\mathbf{A}}$-morphism $EA \to EA'$. Thus a morphism assignment $Mor\,\mathbf{A} \to Mor\,\widehat{\mathbf{A}}$ is defined by $Ef = Uf$. Obviously, $E : (\mathbf{A}, U) \to (\widehat{\mathbf{A}}, \widehat{U})$ is a functor over $\mathbf{X}$.

Now we prove the claim that $(\widehat{\mathbf{A}}, \widehat{U})$ is $(\mathbf{E}, \mathbf{M})$-algebraic. To prove in particular (ALG 1) we make heavy use of the assumptions (I) and (II).

Theorem 3.4. *$(\widehat{\mathbf{A}}, \widehat{U})$ is $(\mathbf{E}, \mathbf{M})$-algebraic.*

Proof. (ALG 1) for $\widehat{U}$: Let X be an $\mathbf{X}$-object and $(f_i : X \to X_i = \widehat{U}[P_i, e_i])_I$ be the $\widehat{U}$-source of all $\widehat{U}$-structured morphisms with domain X. By (I), there exists an $\mathbf{E}$-projective cover $d : P \to X$. Hence, for each $i \in I$ there exists $g_i : P \to UP_i$ in $\mathbf{X}$ with $f_i d = e_i g_i$ and $\overline{g}_i : FP \to P_i$ in $\mathbf{A}$ with $g_i = U\overline{g}_i \eta_P$.

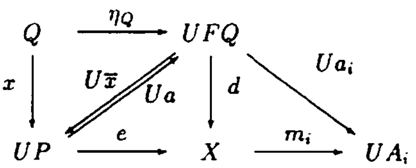




There exists the $(\mathbf{E}, \mathbf{M})$-factorization $(UFP \xrightarrow{e} \widehat{X} \xrightarrow{m_i} X_i)_I$ of the $\mathbf{X}$-source $(e_i U\overline{g}_i : UFP \to X_i)_I$. Therefore there exists the diagonal $\widehat{\eta}_X : X \to \widehat{X}$ in $\mathbf{X}$ with $e\eta_P = \widehat{\eta}_X d$ and $f_i = m_i \widehat{\eta}_X$ for all $i \in I$. Since for each e_i there exists a U-structured source $(m^i_j)_j$ in $\mathbf{M}$ such that the compositions $m^i_j e_i$ are $\mathbf{A}$-morphisms $a^i_j : P_i \to A^i_j$, the composition $(m^i_j m_i)_{i,j}$ belongs to $\mathbf{M}$ (see 1.2), and we have $m^i_j m_i e = m^i_j e_i U\overline{g}_i = Ua^i_j U\overline{g}_i = U(a^i_j \overline{g}_i)$ for all i, j, i.e., e is an $(\mathbf{E}, \mathbf{M})$-structure quotient and $[FP, e]$ even an $\widehat{\mathbf{A}}$-object, since FP is a $U^{-1}[\mathbf{E}]$-projective object by 1.3. Hence all m_i are $\widehat{\mathbf{A}}$-morphisms $m_i : [FP, e] \to [P_i, e_i]$. It remains to show that $\widehat{\eta}_X$ is a $\widehat{U}$-epi (since then $\widehat{\eta}_X$ is $\widehat{U}$-universal): Let $m, n : [FP, e] \rightrightarrows [P', e']$ be $\widehat{\mathbf{A}}$-morphisms with $\widehat{U}m\widehat{\eta}_X = \widehat{U}n\widehat{\eta}_X$. Then there exist $a, b : FP \rightrightarrows P'$ with $me = e'Ua$ and

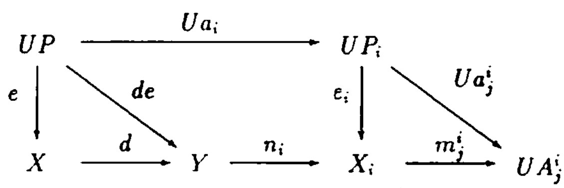
$ne = e'Ub$ and a U-structured source $(n_k : X' \to UA_k)_k$ in **M** such that the n_ke' are **A**-morphisms $a_k : P' \to A_k$. We have $U(a_ka)\eta_P = Ua_kUa\eta_P = n_ke'Ua\eta_P = n_kme\eta_P = n_km\hat{\eta}_Xd = n_k\hat{U}m\hat{\eta}_Xd = n_k\hat{U}n\hat{\eta}_Xd = \cdots = U(a_kb)\eta_P$, hence $a_ka = a_kb$ for all k. So we obtain $n_kme = n_ke'Ua = Ua_kUa = U(a_ka) = U(a_kb) = \cdots = n_kne$ for all k, hence $m = n$, since e is an epi (see 1.2) and $(n_k)_k$ is a monosource.

Next we prove that $\hat{U}$ reflects identities: Let X be an **X**-object and $id_X : [P,e] \to [P',e']$ be an $\hat{\mathbf{A}}$-morphism. By 1.3, there exists a U^{-1}[**E**]-projective cover $r : FQ \to P'$ where Q is **E**-projective. Since FQ is U^{-1}[**E**]-projective, r is a retraction. Hence, (P',e') and $(FQ,e'Ur)$ are equivalent, i.e., $[P',e'] = [FQ,d]$ for $d = e'Ur$. There exists $a : P \to FQ$ in **A** with $dUa = e$. Because of the **E**-projectivity of Q there exist $x : Q \to UP$ with $ex = d\eta_Q$ and hence $\overline{x} : FQ \to P$ in **A** with $x = U\overline{x}\eta_Q$.



There exists a U-structured source $(m_i)_I$ in **M** such that each m_id is an **A**-morphism $a_i : FQ \to A_i$. For $b_i = a_ia$ we have $m_ie = Ub_i$. Thus, $Ua_i\eta_Q = m_id\eta_Q = m_iex = (Ub_i)x = U(b_i\overline{x})\eta_Q$, hence $a_i = b_i\overline{x}$, therefore $m_ieU\overline{x} = U(b_i\overline{x}) = Ua_i = m_id$ for all $i \in I$. Hence $eU\overline{x} = d$, since $(m_i)_I$ is a monosource. Thus, $[P,e] = [FQ,d]$ and the $\hat{\mathbf{A}}$-morphism id_X is the $\hat{\mathbf{A}}$-identity of $[P,e]$.

(ALG 2) for $\hat{U}$: Let $(\hat{U}[P,e] = X \xrightarrow{d} Y \xrightarrow{n_i} X_i = \hat{U}[P_i,e_i])_I$ be an (**E**, **M**)-factorization such that all n_id are $\hat{\mathbf{A}}$-morphisms $[P,e] \to [P_i,e_i]$. Hence, for each $i \in I$ there exist $a_i : P \to P_i$ in **A** with $e_iUa_i = n_ide$ and a U-structured source $(m^i_j)_j$ in **M** such that each $m^i_je_i$ is an **A**-morphism $a^i_j : P_i \to A^i_j$.



(P,de) is an (**E**, **M**)-structure quotient, since $de \in$ **E** and $(m^i_jn_i)_{i,j} \in$ **M** (see 1.2) and $m^i_jn_ide = m^i_je_iUa_i = U(a^i_ja_i)$ for all i,j. Hence $d : [P,e] \to [P,de]$ and all $n_i : [P,de] \to [P_i,e_i]$ are $\hat{\mathbf{A}}$-morphisms. Now, let $[Q,c]$ be in $\hat{\mathbf{A}}$ with $\hat{U}[Q,c] = Y$ such that $d : [P,e] \to [Q,c]$ and all $n_i : [Q,c] \to [P_i,e_i]$ are $\hat{\mathbf{A}}$-morphisms. Then there exists $a : P \to Q$ in **A** with $de = cUa$, i.e., $id_Y : [P,de] \to [Q,c]$ is an $\hat{\mathbf{A}}$-morphism. We have seen that $\hat{U}$ reflects identities. Thus, $[P,de] = [Q,c]$, and the uniqueness of the $\hat{U}$-lift of $(n_id)_I$ is proved. □

The comparison functor $E : (\mathbf{A},U) \to (\hat{\mathbf{A}},\hat{U})$ has two useful and interesting properties. We show that E, although not necessarily full, is *partially* full and that E sends U-free objects with an **E**-projective base to $\hat{U}$-free objects.

Proposition 3.5.

(a) $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ *is full on* $U^{-1}[\mathbf{E}]$*-projective objects, i.e. for each* $U^{-1}[\mathbf{E}]$*-projective object* P *and each* $\mathbf{A}$*-object* A, $E : Hom_{\mathbf{A}}(P, A) \to Hom_{\hat{\mathbf{A}}}(EP, EA)$ *is onto.*

(b) $EFP \cong \hat{F}P$ *for each* $\mathbf{E}$*-projective object* P *(*$\hat{F}$ *denotes the left adjoint of* $\hat{U}$*).*

PROOF.

(a) Let $f : EP \to EA$ be in $\hat{\mathbf{A}}$ where P is $U^{-1}[\mathbf{E}]$-projective. By definition of E we may write $EA = [Q, Ud]$ and $EP = [P, Uid_P]$. So, there exists $a : P \to Q$ in $\mathbf{A}$ with $f = fUid_P = UdUa = U(da)$. Hence $g = da : P \to Q \to A$ is in $\mathbf{A}$ with $Eg = Ug = f$.

(b) Let P be an $\mathbf{E}$-projective object. The construction of $\hat{\eta}_X : X \to \hat{U}[FP, e] = \hat{U}\hat{F}X$ (see Proof of 3.4) shows that for $X = P$ we can choose $d = id_P$. So we have $\hat{\eta}_P = e\eta_P$, even $\hat{\eta}_P = \hat{U}e\eta_P$, since because of $eid_{UFP} = eU(id_{FP})$, 1.3 and the definition of E, e is an $\hat{\mathbf{A}}$-morphism $EFP \to [FP, e] = \hat{F}P$. Since $\eta_P : P \to UFP = \hat{U}EFP$ is $\hat{U}$-structured and $\hat{\eta}_P : P \to \hat{U}\hat{F}P$ is $\hat{U}$-universal, there exists $h : \hat{F}P \to EFP$ in $\hat{\mathbf{A}}$ with $\eta_P = \hat{U}h\hat{\eta}_P$. The composition $he : EFP \to EFP$ is in $\hat{\mathbf{A}}$. Since FP is $U^{-1}[\mathbf{E}]$-projective, there exists $a : FP \to FP$ in $\mathbf{A}$ with $Ea = he$ by (a). Hence $Ua\eta_P = \hat{U}Ea\eta_P = \hat{U}(he)\eta_P = \hat{U}hUe\eta_P = \hat{U}h\hat{\eta}_P = \eta_P$, hence $a = id_{FP}$ and $he = Ea = id_{EFP}$. Thus, e is a section and, by 1.2, an epimorphism, hence an isomorphism in $\hat{\mathbf{A}}$. □

4 Algebraic Completions: Universality

We prove that $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ is a universal construction and that $(\hat{\mathbf{A}}, \hat{U})$ inherits the algebraic property *having a rank* from $(\mathbf{A}, U)$.

Definition 4.1. A functor $E : (\mathbf{A}, U) \to (\mathbf{B}, V)$ over $\mathbf{X}$ into an $(\mathbf{E}, \mathbf{M})$-algebraic $(\mathbf{B}, V)$ is called a *universal* $(\mathbf{E}, \mathbf{M})$*-algebraic completion* (of $(\mathbf{A}, U)$) provided that for every functor $D : (\mathbf{A}, U) \to (\mathbf{C}, W)$ over $\mathbf{X}$ into an $(\mathbf{E}, \mathbf{M})$-algebraic $(\mathbf{C}, W)$ there exists exactly one functor $\overline{D} : (\mathbf{B}, V) \to (\mathbf{C}, W)$ over $\mathbf{X}$ such that the diagram

$$\begin{array}{ccc} (\mathbf{A}, U) & \xrightarrow{D} & (\mathbf{C}, W) \\ {\scriptstyle E}\downarrow & \nearrow {\scriptstyle \overline{D}} & \\ (\mathbf{B}, V) & & \end{array}$$

commutes.

Theorem 4.2. *Let* $U : \mathbf{A} \to \mathbf{X}$ *be a right adjoint and* $\mathbf{X}$ *an* $(\mathbf{E}, \mathbf{M})$*-category with enough* $\mathbf{E}$*-projectives. Then* $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ *is a universal* $(\mathbf{E}, \mathbf{M})$*-algebraic completion.*

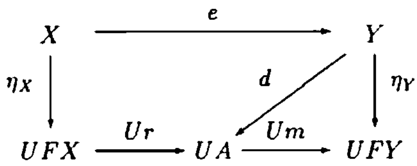
PROOF. Let $D : (\mathbf{A}, U) \to (\mathbf{B}, V)$ be a functor over $\mathbf{X}$ into an $(\mathbf{E}, \mathbf{M})$-algebraic $(\mathbf{B}, V)$. Let $e : UP \to X$ and $e' : UP' \to X$ be equivalent $(\mathbf{E}, \mathbf{M})$-structure quotients, i.e., $[P, e] = [P', e'] \in Ob\widehat{\mathbf{A}}$. Then $e : VDP \to X$ and $e' : VDP' \to X$ are equivalent $(\mathbf{E}, \mathbf{M})$-structure quotients with respect to V. The V-lifts $\overline{e} : DP \to \overline{X}$ and $\overline{e}' : DP' \to \overline{X}'$ of $e : VDP \to X$ and $e' : VDP' \to X$, resp., exist with $\overline{X} = \overline{X}'$ by 3.2. Therefore, $\overline{D}[P, e] = \overline{X}$ yields a well-defined object assignment $\overline{D} : Ob\widehat{\mathbf{A}} \to Ob\mathbf{B}$. Let $f : [P, e] \to [P', e']$ be an $\widehat{\mathbf{A}}$-morphism, and let $\overline{e} : DP \to \overline{X}$ and $\overline{e}' : DP' \to \overline{X}'$ be the $\overline{D}[P, e]$ and $\overline{D}[P', e']$ defining $\mathbf{B}$-morphisms. There exists $a : P \to P'$ in $\mathbf{A}$ with $fe = e'Ua$.

$$\begin{array}{ccc} UP = VDP & \xrightarrow{\;Ua = VDa\;} & UP' = VDP' \\ e = \Big\downarrow V\overline{e} & (*) & e' = \Big\downarrow V\overline{e}' \\ X = V\overline{X} & \xrightarrow{\;\;f\;\;} & X' = V\overline{X}' \end{array}$$

By 2.4, $\overline{e}$ is V-final. Hence there exists exactly one $\mathbf{B}$-morphism $\overline{f} : \overline{X} \to \overline{X}'$ with $V\overline{f} = f$. Thus, $\overline{D}f = \overline{f}$ yields a morphism assignment $\overline{D} : Mor\,\widehat{\mathbf{A}} \to Mor\,\mathbf{B}$. By definition of $\overline{D}$, $V\overline{D} = \widehat{U}$ holds for $\widehat{\mathbf{A}}$-objects and $\widehat{\mathbf{A}}$-morphisms. It follows that $\overline{D} : \widehat{\mathbf{A}} \to \mathbf{B}$ is a functor, since for an $\widehat{\mathbf{A}}$-identity $id_{[P,e]}$ with $e : UP \to X$ and for $\widehat{\mathbf{A}}$-morphisms $f : [P, e] \to [P', e']$ and $g : [P', e'] \to [P'', e'']$ we have $V\overline{D}id_{[P,e]} = \widehat{U}id_{[P,e]} = id_X$, hence $\overline{D}id_{[P,e]} = id_{\overline{D}[P,e]}$ (since V reflects identities by 2.3) and we have $V\overline{D}(gf) = \widehat{U}(gf) = \widehat{U}g\widehat{U}f = V\overline{D}gV\overline{D}f = V(\overline{D}g\overline{D}f)$, hence $\overline{D}(gf) = \overline{D}g\overline{D}f$ (since V is faithful by 2.3). By definition of E and $\overline{D}$, $D = \overline{D}E$ holds for all $\mathbf{A}$-objects. Hence, for every $\mathbf{A}$-morphism $f : A \to B$, we have $VDf = Uf = \widehat{U}Ef = V\overline{D}Ef$, hence $Df = \overline{D}Ef$ by the faithfulness of V. Thus, $D = \overline{D}E$ holds on $\mathbf{A}$. Finally, let $G : (\widehat{\mathbf{A}}, \widehat{U}) \to (B, V)$ be a functor over $\mathbf{X}$ with $D = GE$. Then, for every $\widehat{\mathbf{A}}$-object $[P, e]$, $e : EP \to [P, e]$ is an $\widehat{\mathbf{A}}$-morphism (see Proof of 3.5(b)). Hence $Ge : GEP = G[P, e]$ coincides with the $\overline{D}[P, e]$ defining $\mathbf{B}$-morphism $\overline{e} : DP \to \overline{X}$. Thus, $G = \overline{D}$ holds for all $\widehat{\mathbf{A}}$-objects, and if $f : [P, e] \to [P', e']$ is an $\widehat{\mathbf{A}}$-morphism, then $VGf = \widehat{U}f = V\overline{D}f$, hence $Gf = \overline{D}f$ by the faithfulness of V. Thus, $G = \overline{D}$ holds on $\widehat{\mathbf{A}}$. □

Remark 4.3. As an application, we show that the algebraic property *having a rank* $\leq \alpha$ (see 1.4) is stable under the formation of a universal $(\mathbf{E}, \mathbf{M})$-algebraic completion. *Having a rank* $\leq \alpha$ is a categorical notion for the property of a class of algebras having the arities of their (proper) operations bounded by the regular cardinal α. We need the following

Lemma. *If* $U : \mathbf{A} \to \mathbf{X}$ *has a left adjoint* F *with unit* η *and* $\mathbf{X}$ *is an* $(\mathbf{E}, \mathbf{M})$*-category such that* $\mathbf{A}$ *is a* $(U^{-1}[\mathbf{E}], U^{-1}[\mathbf{M}])$*-category, then* $F[\mathbf{E}] \subset U^{-1}[\mathbf{E}]$. (PROOF. For any $e : X \to Y$ in $\mathbf{E}$, there exist $r : FX \to A$ in $U^{-1}[\mathbf{E}]$ and $m : A \to FY$ in $U^{-1}[\mathbf{M}]$ with $Fe = mr$. By the diagonalization property for $(\mathbf{E}, \mathbf{M})$, there exists the diagonal $d : Y \to UA$ in $\mathbf{X}$ making the diagram



commutative.

So, there exists $\bar{d} : FY \to A$ in $\mathbf{A}$ with $d = U\bar{d}\eta_Y$. Hence $U(\bar{d}Fe)\eta_X = U\bar{d}UFe\eta_X = U\bar{d}\eta_Y e = de = Ur\eta_X$, hence $\bar{d}Fe = r$. Together with $Fe = mr$ it follows that $\bar{d}mr = r$ and $Fe = m\bar{d}Fe$, hence $\bar{d}m = id_A$ and $m\bar{d} = id_{FY}$, since r is an $\mathbf{A}$-epi, e an $\mathbf{X}$-epi (see 1.2) hence Fe is an $\mathbf{A}$-epi. So, m is an $\mathbf{A}$-isomorphism. Since $U^{-1}[\mathbf{M}]$ is closed under $\mathbf{A}$-isomorphisms, $Fe = mr$ shows that $Fe \in U^{-1}[\mathbf{M}]$.)

Theorem 4.4. *Let $U : \mathbf{A} \to \mathbf{X}$ be faithful and a right adjoint, $\mathbf{X}$ an $(\mathbf{E}, \mathbf{M})$-category with enough $\mathbf{E}$-projectives and let $\mathbf{X}$ have coproducts. Then $(\hat{\mathbf{A}}, \hat{U})$ has a rank $\leq \alpha$, if $(\mathbf{A}, U)$ has a rank $\leq \alpha$.*

Proof. Preliminaries: Let $\mathbf{I}$ be the category induced by an α-filtered partially ordered set $(I, \leq)$. For any categories $\mathbf{C}, \mathbf{D}$ and functors $V : \mathbf{C} \to \mathbf{D}$, let $[\mathbf{I}, \mathbf{C}]$ denote the functor category and $[\mathbf{I}, V] : [\mathbf{I}, \mathbf{C}] \to [\mathbf{I}, \mathbf{D}]$ denote the obvious functor (defined for diagrams $D : \mathbf{I} \to \mathbf{C}$ by $[\mathbf{I}, V]D = VD$). Let I denote the discrete category induced by the set I. For a diagram $D : \mathbf{I} \to \mathbf{X}$, the restriction $|D|$ of D to the subcategory I of $\mathbf{I}$ gives a functor $|\ | : [\mathbf{I}, \mathbf{X}] \to [I, \mathbf{X}]$. Since $\mathbf{X}$ has coproducts, it is known that for each diagram $G : I \to \mathbf{X}$ there exists a $|\ |$-universal $[I, \mathbf{X}]$-morphism $G \to |\bar{G}|$ where $\bar{G} : \mathbf{I} \to \mathbf{X}$ is defined on $\mathbf{I}$-objects by $\bar{G}i = \amalg(Gk | k \leq i)$ (see, e.g. [13: 17.1.6]). Moreover, $|\ |$ is $(\mathbf{E}, \mathbf{M})$-algebraic, since obviously for the powers $\mathbf{E}^I$ and $\mathbf{M}^I$, $[I, \mathbf{X}]$ is an $(\mathbf{E}^I, \mathbf{M}^I)$-category, and it is easy to check that $|\ |$ lifts $(\mathbf{E}^I, \mathbf{M}^I)$-factorizations uniquely. Hence, by 2.5, $[\mathbf{I}, \mathbf{X}]$ is an $(\mathbf{E}^\mathbf{I}, \mathbf{M}^\mathbf{I})$-category where $\mathbf{E}^\mathbf{I} = |\ |^{-1}[\mathbf{E}^I]$ and $\mathbf{M}^\mathbf{I} = |\ |^{-1}[\mathbf{M}^I]$.

Now, let $(b_i : Di \to B)_I$ be an α-filtered colimit of a diagram $D : \mathbf{I} \to \hat{\mathbf{A}}$ in $\hat{\mathbf{A}}$. The functor $[\mathbf{I}, \hat{U}] : [\mathbf{I}, \hat{\mathbf{A}}] \to [\mathbf{I}, \mathbf{X}]$ has a left adjoint, namely $[\mathbf{I}, \hat{F}]$ where $\hat{F}$ is the left adjoint of $\hat{U}$, also $|\ | : [\mathbf{I}, \mathbf{X}] \to [I, \mathbf{X}]$ has a left adjoint, say L. Thus, the composition $|[\mathbf{I}, \hat{U}]| : [\mathbf{I}, \hat{\mathbf{A}}] \to [I, \mathbf{X}]$ of $[\mathbf{I}, \hat{U}]$ with $|\ |$ has $[\mathbf{I}, \hat{F}]L$ as a left adjoint. So, since D is an $[\mathbf{I}, \hat{\mathbf{A}}]$-object, by 1.3 there exist an $\mathbf{E}^I$-projective object P in $[I, \mathbf{X}]$ and an $[\mathbf{I}, \hat{\mathbf{A}}]$-morphism $e : [\mathbf{I}, \hat{F}]L(P) = \hat{F}LP \to D$ in $|[\mathbf{I}, \hat{U}]|^{-1}[\mathbf{E}^I]$. By 2.5, $\hat{\mathbf{A}}$ is a $(\hat{U}^{-1}[\mathbf{E}], \hat{U}^{-1}[\mathbf{M}])$-category, hence $[\mathbf{I}, \hat{\mathbf{A}}]$ is a $(\hat{U}^{-1}[\mathbf{E}]^\mathbf{I}, \hat{U}^{-1}[\mathbf{M}]^\mathbf{I})$-category in the sense of the above Preliminaries. It is straightforward to check that $|[\mathbf{I}, \hat{U}]|^{-1}[\mathbf{E}^I] = \hat{U}^{-1}[\mathbf{E}]^\mathbf{I}$. Thus, $e : \hat{F}LP \to D$ belongs to $\hat{U}^{-1}[\mathbf{E}]^\mathbf{I}$. For convenience, put $\bar{P} = LP$. By assumption, $\mathbf{X}$ has coproducts and, by 1.2, coequalizers. Hence $\mathbf{X}$ is cocomplete. Therefore, the colimit $(x_i : \bar{P}i \to X)_I$ of $\bar{P} : \mathbf{I} \to \mathbf{X}$ exists in $\mathbf{X}$. It is routine to check that X is the coproduct object of the $\bar{P}i$, $i \in I$ (using that by definition $\bar{P}i = \amalg(Pk | k \leq i)$). Since all Pk, $k \in I$, are $\mathbf{E}$-projective and coproducts of $\mathbf{E}$-projective objects are $\mathbf{E}$-projective, $\bar{P}i$ and X are $\mathbf{E}$-projective. Since the left adjoint $\hat{F} : \mathbf{X} \to \mathbf{A}$

of $\widehat{U}$ preserves colimits, $(\widehat{F}x_i : \widehat{F}\overline{P}i \to \widehat{F}X)_I$ is an α-filtered colimit of $\widehat{F}\overline{P} : \mathbf{I} \to \widehat{\mathbf{A}}$ in $\widehat{\mathbf{A}}$. By 3.5 we have $\widehat{F}\overline{P}i \cong EF\overline{P}i$ and $\widehat{F}X \cong EFX$. Thus, we may assume that $(\widehat{F}x_i : EF\overline{P}i \to EFX)_I$ is an α-filtered colimit of $\widehat{F}\overline{P}$ in $\widehat{\mathbf{A}}$. By 3.5, for each $i \in I$ there exists an $\mathbf{A}$-morphism $a_i : F\overline{P}i \to FX$ with $Ea_i = \widehat{F}x_i$. Now, $(a_i : F\overline{P}i \to FX)_I$ is an α-filtered colimit of $F\overline{P}$ in $\mathbf{A}$. This follows immediately from the faithfulness of E (implied by the assumed faithfulness of U), the $U^{-1}[\mathrm{E}]$-projectivity of FX (see 1.3) and from 3.5. Since $\mathbf{X}$ is cocomplete, $\widehat{\mathbf{A}}$ is cocomplete by 2.6. So, it is well-known that the canonical embedding (the diagonal functor) $\Delta : \widehat{\mathbf{A}} \to [\mathbf{I}, \widehat{\mathbf{A}}]$ has a left adjoint, namely the colimit functor colim : $[\mathbf{I}, \widehat{\mathbf{A}}] \to \widehat{\mathbf{A}}$. For the above $e : EF\overline{P} \to D$ put $c = \mathrm{colim}(e)$. Then the following diagrams commute:

$$
\begin{array}{ccc}
F\overline{P}_i \\
\Big\downarrow{\scriptstyle a_i} \\
FX
\end{array}
\qquad
\begin{array}{ccc}
EF\overline{P}_i & \xrightarrow{\;e_i\;} & D_i \\
{\scriptstyle \widehat{F}x_i}\Big\downarrow & (*) & \Big\downarrow{\scriptstyle b_i} \\
EFX & \xrightarrow{\;c\;} & B
\end{array}
\qquad
\begin{array}{ccc}
UF\overline{P}_i & \xrightarrow{\;\widehat{U}e_i\;} & \widehat{U}D_i \\
{\scriptstyle Ua_i}\Big\downarrow & (**) & \Big\downarrow{\scriptstyle \widehat{U}b_i} \\
UFX & \xrightarrow{\;\widehat{U}c\;} & \widehat{U}B
\end{array}
$$

Recall from above that $[\mathbf{I}, \widehat{\mathbf{A}}]$ is a $(\widehat{U}^{-1}[\mathrm{E}]^{\mathbf{I}}, \widehat{U}^{-1}[\mathrm{M}]^{\mathbf{I}})$-category, and it is straightforward to check that $\Delta^{-1}[\widehat{U}^{-1}[\mathrm{E}]^{\mathbf{I}}] = \widehat{U}^{-1}[\mathrm{E}]$ and $\Delta^{-1}[\widehat{U}^{-1}[\mathrm{M}]^{\mathbf{I}}] = \widehat{U}^{-1}[\mathrm{M}]$ (using that $I \neq \phi$). Hence, by the Lemma in 4.3 (applied to Δ and colim), $c = \mathrm{colim}(e) \in \widehat{U}^{-1}[\mathrm{E}]$, hence $\widehat{U}c \in \mathrm{E}$ and therefore $\widehat{U}c$ is an $\mathbf{X}$-epi (see 1.2). By assumption on U, $(Ua_i : UF\overline{P}i \to UFX)_I$ is an episink in $\mathbf{X}$. It follows from the commutativity of $(**)$ that $(\widehat{U}b_i : \widehat{U}Di \to \widehat{U}B)_I$ is also an episink in $\mathbf{X}$. □

Example 4.5. $(\mathbf{A}, U)$ need not have a rank $\leq \alpha$ whenever $(\widehat{\mathbf{A}}, \widehat{U})$ has one: Consider the category $\mathbf{A}$ of complete separated uniform spaces and uniformly continuous maps with the usual forgetful functor $U : \mathbf{A} \to \mathbf{Set}$. The discrete uniform spaces are the surjective-projectives in $\mathbf{A}$. Thus, by the construction of $(\widehat{\mathbf{A}}, \widehat{U})$, the universal (surjective, point separating)-algebraic completion coincides up to isomorphism over $\mathbf{Set}$ with $(\mathbf{Set}, id_{\mathbf{Set}})$. Trivially, $(\mathbf{Set}, id_{\mathbf{Set}})$ has a rank $\leq \alpha$ for each regular α. But we show that $(\mathbf{A}, U)$ has a rank $\leq \alpha$ for no regular α : Let α be regular. Then α is α-filtered with respect to the well-ordering of α. For each $\beta < \alpha$, consider the interval $D\beta = [0, \beta]$. The topology on $D\beta$ induced by the well-ordering of $D\beta$, is compact and separated. Hence, this topology admits exactly one uniformity on $D\beta$ which is complete. Therefore, there exists a diagram $D : \alpha \to \mathbf{A}$ given by $\beta \mapsto D\beta$ (where α is considered as the category induced by the well-ordered set α). The embeddings $D\beta \hookrightarrow \alpha + 1$, $\beta < \alpha$, into the uniform completion $\alpha + 1$ of α form an α-filtered colimit of D in $\mathbf{A}$, which is not sent by U to an episink = covering sink in $\mathbf{Set}$ because $\cup\{D\beta | \beta < \alpha\} = \alpha$.

5 Algebraic Completions: Full Embedding and Reflectivity Criteria

Proposition 5.1. *Let $U : \mathbf{A} \to \mathbf{X}$ be a right adjoint and $\mathbf{X}$ an $(\mathbf{E}, \mathbf{M})$-category with enough $\mathbf{E}$-projectives. Then the following conditions are equivalent:*

(a) the universal $(\mathbf{E}, \mathbf{M})$-algebraic completion $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ is a full embedding

(b) there exists a full embedding $D : (\mathbf{A}, U) \to (\mathbf{B}, V)$ over $\mathbf{X}$ into an $(\mathbf{E}, \mathbf{M})$-algebraic $(\mathbf{B}, V)$

(c) U is amnestic and reflects $\mathbf{E}$-morphisms into U-final morphisms.

PROOF. (a) $\Rightarrow$ (b) trivial.

(b) $\Rightarrow$ (c) By 2.3, V is amnestic. Also D, since D is a full embedding. Hence $U = VD$ is amnestic. Now, let $e : A \to B$ be in $U^{-1}[\mathbf{E}]$, $f : UB \to UC$ in $\mathbf{X}$ with B, C in $\mathbf{A}$ such that fUe is an $\mathbf{A}$-morphism $a : A \to C$. Hence $VDa = Ua = fUe = fVDe$. By 2.4 applied to V, De is V-final, thus f is a $\mathbf{B}$-morphism $g : DB \to DC$. Since D is full, there exists $h : B \to C$ in $\mathbf{A}$ with $Eh = g$. Hence $Uh = VDh = Vg = f$ and therefore $U(he) = fUe = Ua$. Hence $he = a$, since U is faithful (because D and V (by 2.3) are faithful and $U = VD$).

(c) $\Rightarrow$ (a) Assume that E is full and faithful. Then, for each A and B in $\mathbf{A}$ with $EA = EB$, there exist $a : A \to B$ and $b : B \to A$ in $\mathbf{A}$ with $Ea = id_{EA} = Eb$, hence $E(ba) = Eid_A$ and $E(ab) = Eid_B$ and therefore $ba = id_A$ and $ab = id_B$, i.e., a is an $\mathbf{A}$-isomorphism with $Ua = id_{UA}$, hence $a = id_A$ and therefore $A = B$. Thus, E is injective on $\mathbf{A}$-objects. Now we prove that E is in fact full and faithful: Let A, B be in $\mathbf{A}$ and $f : EA = [P, Ue] \to [Q, Ud] = EB$ be in $\hat{\mathbf{A}}$. Hence there exists $a : P \to Q$ in $\mathbf{A}$ with $fUe = UdUa = U(da)$. Since $Ue \in \mathbf{E}$, e is U-final. Hence there exists $g : A \to B$ in $\mathbf{A}$ with $Ug = f$, hence $Eg = f$. In order to show that U is faithful, it is sufficient to show that each $U^{-1}[\mathbf{E}]$-morphism is an $\mathbf{A}$-epi, as the Proof of 2.3 (a) shows. Let $e : A \to B$ be in $U^{-1}[\mathbf{E}]$ and $a, b : B \rightrightarrows C$ be in $\mathbf{A}$ with $ae = be$, hence $UaUe = UbUe$. Since Ue is an $\mathbf{X}$-epi (see 1.2), $Ua = Ub$. Thus, for $g = Ua$, $gUe = U(ae)$, and since e is U-final, there exists exactly one $\overline{g} : B \to C$ in $\mathbf{A}$ with $U\overline{g} = g$ and $\overline{g}e = ae$. Hence $\overline{g} = a$ and $\overline{g} = b$. □

The following Criterion makes use of the notion of a *semifinal solution* (Tholen [14], [1: 25.7]). A U-structured morphism $r : X \to UB$ is called a *semifinal solution* of a U-structured morphism $e : UA \to X$ provided that re is an $\mathbf{A}$-morphism $a : A \to B$ and whenever $f : X \to UC$ is a U-structured morphism such that fe is an $\mathbf{A}$-morphism $b : A \to C$ there exists exactly one $\overline{f} : B \to C$ in $\mathbf{A}$ with $U\overline{f} = f$ and $b = \overline{f}a$.

Proposition 5.2. *Let $U : \mathbf{A} \to \mathbf{X}$ be a right adjoint, $\mathbf{X}$ an $(\mathbf{E}, \mathbf{M})$-category with enough $\mathbf{E}$-projectives and let the universal $(\mathbf{E}, \mathbf{M})$-algebraic completion $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ be a full embedding. Then the following conditions are equivalent:*

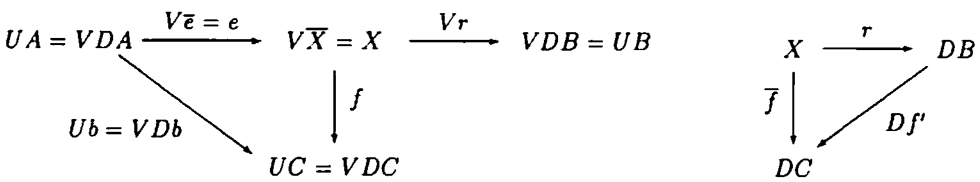
(a) $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ is reflective

(b) there exists a reflective full embedding $D : (\mathbf{A}, U) \to (\mathbf{B}, V)$ *over* $\mathbf{X}$ *into an* $(\mathbf{E}, \mathbf{M})$*-algebraic* $(\mathbf{B}, V)$

(c) every $(\mathbf{E}, \mathbf{M})$*-structure quotient has a semifinal solution.*

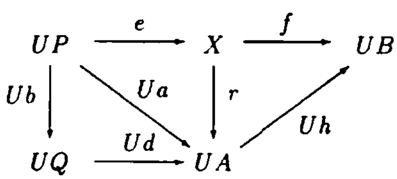
PROOF. (a) $\Rightarrow$ (b) trivial.

(b) $\Rightarrow$ (c) Let $e : UA \to X$ be any $(\mathbf{E}, \mathbf{M})$-structure quotient. Then $e : UA = VDA \to X$ is an $(\mathbf{E}, \mathbf{M})$-structure quotient with respect to V. So, e has a V-lift $\bar{e} : DA \to \overline{X}$. Let $r : \overline{X} \to DB$ be the D-reflection. Since D is full, there exists $a : A \to B$ in $\mathbf{A}$ with $Da = r\bar{e}$. We show that $Vr : X \to UB$ is a semifinal solution of e with respect to U: Let $f : X \to UC$ be a U-structured morphism such that fe is an $\mathbf{A}$-morphism $b : A \to C$.



Since $\bar{e}$ is V-final by 2.4 and $fV\bar{e} = Ub = VDb$, there exists $\bar{f} : \overline{X} \to DC$ in $\mathbf{B}$ with $V\bar{f} = f$. Since r is the D-reflection, there exists $f' : B \to C$ in $\mathbf{A}$ with $\bar{f} = Df'r$. Hence $f = V\bar{f} = Uf'Vr$. Since U is faithful, Vr is a semifinal solution of e.

(c) $\Rightarrow$ (a) Let $[P, e]$ be in $\hat{\mathbf{A}}$. Then $e : UP \to X$ has a semifinal solution $r : X \to UA$. Hence, there exists $a : P \to A$ in $\mathbf{A}$ with $Ua = re$. Let $EA = [Q, Ud]$. Since P is $U^{-1}[\mathbf{E}]$-projective and $d \in U^{-1}[\mathbf{E}]$, there exists $b : P \to Q$ in $\mathbf{A}$ with $db = a$, i.e., $r : [P, e] \to EA$ is in $\hat{\mathbf{A}}$.



$r : [P, e] \to EA$ is the E-reflection in $\hat{\mathbf{A}}$, since for any $f : [P, e] \to EB$ in $\hat{\mathbf{A}}$ there exists $g : P \to B$ in $\mathbf{A}$ with $fe = Ug$, hence, since r is a semifinal solution, there exists $h : A \to B$ in $\mathbf{A}$ with $f = (Uh)r$. Hence $f = (Eh)r$ holds in $\hat{\mathbf{A}}$. Since $r : X \to UA$ is a U-epi, $r : [P, e] \to EA$ is an E-epi. □

The regular, i.e., (regular epi, monosource)-algebraic functors (see 2.2) are one of the most important types of algebraic functors (see [1: 23.]). Complementing a well-known *topological* result (see the Introduction), the two preceding Propositions allow an internal characterization of those $(\mathbf{A}, U)$ which are fully and reflectively embeddable into a regular $(\mathbf{B}, V)$ over $\mathbf{X}$:

Theorem 5.3. *Let $U : \mathbf{A} \to \mathbf{X}$ be a right adjoint functor, $\mathbf{X}$ a (regular epi, monosource)-category with enough (regular epi)-projectives. Then the following conditions are equivalent:*

(a) the universal regular completion $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$ is a reflective full embedding

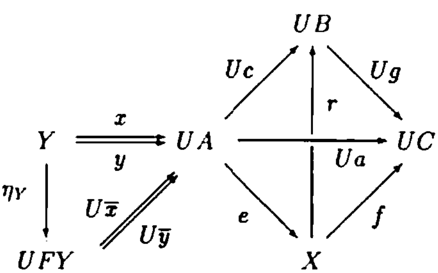
(b) there exists a reflective full embedding $D : (\mathbf{A}, U) \to (\mathbf{B}, V)$ over $\mathbf{X}$ into a regular $(\mathbf{B}, V)$

(c) $\mathbf{A}$ has coequalizers, U is amnestic and reflects regular epimorphisms.

PROOF. (a) $\Rightarrow$ (b) trivial.

(b) $\Rightarrow$ (c) By 2.6, the regular functor V reflects regular epis. Since D is a reflective full embedding, D reflects colimits, in particular regular epis (see, e.g., [1: 13.30]). Hence $U = VD$ reflects regular epis, and since $\mathbf{B}$ has coequalizers by 2.5 and 1.2, $\mathbf{A}$ has coequalizers, too.

(c) $\Rightarrow$ (a) We show that U reflects $\mathbf{E}$-morphisms into U-final morphisms (then, by 5.1, E is a full embedding): For the counit ε belonging to U, each ε_A, A in $\mathbf{A}$, is a (regular) epi in $\mathbf{A}$, since $U\varepsilon_A$ is a retraction, hence a regular epi in $\mathbf{X}$ and U reflects regular epis. Thus, U is faithful (see [1: 19.14]). Now, a faithful functor which reflects regular epis, reflects regular epis into U-final morphisms (see [1: 80] or [12: II.1.22 (a)]). Now we prove that each (regular epi, monosource)-structure quotient has a semifinal solution (then, by 5.2, E is reflective): Let $e : UA \to X$ be a (regular epi, monosource)-structure quotient. e is the coequalizer of a pair $(x, y) : Y \rightrightarrows UA$ in $\mathbf{X}$. There exist $\overline{x}, \overline{y} : FY \rightrightarrows A$ in $\mathbf{A}$ with $x = U\overline{x}\eta_Y$ and $y = U\overline{y}\eta_Y$. By assumption, the coequalizer $c : A \to B$ of $(\overline{x}, \overline{y})$ exists in $\mathbf{A}$. Since Uc equalizes (p, q), there exists $r : X \to UB$ with $Uc = re$. We prove that r is a semifinal solution of e:



Let $f : X \to UC$ be a U-structured morphism such that fe is an $\mathbf{A}$-morphism $a : A \to C$. The U-universality of η_Y implies the identity $a\overline{x} = a\overline{y}$. Hence there exists $g : B \to C$ in $\mathbf{A}$ with $a = gc$. Since U is faithful, r is a semifinal solution. $\square$

6 Universal Algebraic Completions: Examples

Since we consider here categories $(\mathbf{A}, U)$ over **Set** and **Set** has only one (proper) factorization structure, namely (surjective, point separating) and every set is surjective-projective, we call universal (surjective, point separating)-algebraic completions of $(\mathbf{A}, U)$ simply *universal algebraic completions* of $(\mathbf{A}, U)$. The following criterion can be used to verify our examples:

Proposition 6.1. *Let* $E : (\mathbf{A}, U) \to (\mathbf{B}, V)$ *be a functor over* **Set** *and let* F *be a left adjoint of* U *and* G *be a left adjoint of* V. *Then the following conditions are equivalent:*

(a) E *is a universal algebraic completion*

(b) $(\mathbf{B}, V)$ *is algebraic,* E *is dense with respect to monosources and* $EFX \cong GX$ *for all sets* X.

PROOF. (a) $\Rightarrow$ (b) We may assume that E is of the form $E : (\mathbf{A}, U) \to (\hat{\mathbf{A}}, \hat{U})$. We show that E is dense with respect to monosources: Let $[P, e]$ be in $\hat{\mathbf{A}}$. Then there exists a monosource (= point separating source) $(m_i : X \to UA_i)_I$ such that all $m_i e$ are $\mathbf{A}$-morphisms $a_i : P \to A_i$. By definition of E, $EA_i = [UA_i, Ud_i]$ with $d_i : P_i \to A_i$. Since P is $U^{-1}[\mathrm{E}]$-projective and $d_i \in U^{-1}[\mathrm{E}]$, there exists $b_i : P \to P_i$ in $\mathbf{A}$ with $a_i = d_i b_i$. Hence, $(m_i : [P, e] \to EA_i)_I$ is an $\hat{\mathbf{A}}$-source. As a faithful functor, $\hat{U}$ reflects monosources. Thus, $(m_i : [P, e] \to EA_i)_I$ is an $\hat{\mathbf{A}}$-monosource. The rest follows from 3.5.

(b) $\Rightarrow$ (a) Let $D : (\mathbf{A}, U) \to (\mathbf{C}, W)$ be over **Set** into an algebraic $(\mathbf{C}, W)$. Let B be any $\mathbf{B}$-object. By 1.3, there exists a surjective-projective cover $e : GX \to B$ in $\mathbf{B}$. Since $GX \cong EFX$, we may write $e : EFX \to B$. There exists a monosource $(m_i : B \to EA_i)_I$ in $\mathbf{B}$. As a right adjoint, V preserves monosources. Hence, $Ve : VEFX = UFX \to VB$ is a structure quotient with respect to U and also W because $WD = U$. Thus, there exists a unique W-lift $\bar{e} : DFX \to \overline{B}$ of Ve. Thus, $B \mapsto \overline{B}$ defines a (well-defined) object assignment $\overline{D} : Ob\,\mathbf{B} \to Ob\,\mathbf{C}$. Now, proceed as in the Proof of 4.2 (with the obvious change of denotations) to define a unique functor $\overline{D} : (\mathbf{B}, V) \to (\mathbf{C}, W)$ over $\mathbf{X}$ with $D = \overline{D}E$. ☐

Example 6.2. $U : \mathbf{Cat} \to \mathbf{Set}$

Let **Cat** be the category of all small categories and functors between them, and let U denote the functor which assigns the morphism set to each small category. $\widehat{\mathbf{Cat}}$ is (up to isomorphism over **Set**) the category of small directed graphs with identity, i.e., the category of all algebras (X, d, c) where d, c are unary operations on the set $\mathbf{X}$ satisfying $cd = d^2 = d$ and $dc = c^2 = c$. $E : \mathbf{Cat} \to \widehat{\mathbf{Cat}}$ is properly forgetful (E forgets the partial operation "composition of morphisms").

Example 6.3. O : $\mathbf{Ban}_1 \to \mathbf{Set}$

Let $\mathbf{Ban}_1$ be the category of all Banach spaces over $\mathbf{R}$ (or $\mathbf{C}$) and all non-expanding linear maps, and let O be the closed unit ball functor defined by $O(B) = \{x \in B | \ \| x \| \leq 1\}$ for Banach spaces B. From 6.1 and results of Pumplün/Röhrl [11: §11], $\widehat{\mathbf{Ban}_1}$ is the category of the separated totally convex spaces.

Example 6.4. Hom (–, **S**): $\mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$

Hom (–, **S**) denotes the contravariant Hom-functor on **Top** determined by the Sierpinski space **S**. $\widehat{\mathbf{Top}}^{\mathrm{op}}$ is the well-known category of all spatial frames.

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