Category Theory at Work
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Automorphisms and Full Embeddings of Categories in Algebra and Topology

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Abstract. We review some results on describing all automorphisms of certain categories, and on describing all full embeddings between certain pairs of categories, for certain well-known categories in algebra and topology. We reproduce in a simple case the proof, for one case in algebra, and one in topology, which are however already somewhat characteristic for the proof methods. For other cases we give references, although we do not claim to completeness. We also include a simple proof of a result not yet explicitly proved, namely that each isomorphism of the category of commutative semigroups onto itself is naturally isomorphic to the identity functor. This proof also leads to a new proof of the semigroup case. We give a modified (hopefully simpler) proof for the Abelian group case, too. We also prove a new result, about full embeddings F of the category of proximity spaces, or some of its subcategories, into the category of "sets with systems of pairs of non- empty subsets", with morphisms the functions preserving these pairs (like at near pairs), showing that each such F is given by a "base" of the proximity, for near pairs. Further we give analogues of this result for full embeddings F of the category of topological spaces, or some of its subcategories, into the category of "sets with systems of pairs consisting of a non-empty subset and a point of the set", with morphisms the functions preserving these pairs (like at pairs (A,x) with $\overline{A}\ni x$), showing that each such F is given by a "base" for all pairs (A,x) with $\overline{A}\ni x$. Also we describe the full embeddings of the category of (normed) linear spaces (with contractive, or bounded linear maps), into itself, and, for commutative field of scalars, we extend these results to full embeddings of full subcategories of these categories into these categories.

§1

1.1. We will deal with two questions.

1) The first one is, intuitively, the following. Suppose we are given a category of some structures. Let us consider this category as an abstract category only. The question is: does this abstract category contain as much information, as if it were given actually? E.g., given the category of semigroups, as an abstract category, can one identify (up to isomorphism) each object of the category, what semigroup it actually is, and each morphism of the category, what semigroup homomorphism it actually is. Categorically: given a category **C**, we ask about the isomorphisms $\mathbf{C} \to \mathbf{C}$ (i.e. automorphisms of **C**), whether they are naturally isomorphic to $1_{\mathbf{C}}$, or if not, how can we describe all automorphisms of **C**.

For the case of the category of semigroups the answer is evidently no. Namely for each semigroup we can introduce a new multiplication $*$, the opposite multiplication, setting $x * y = yx$. Then homomorphims w.r.t. $*$ are the same as homomorphisms w.r.t. the

This work is an original contribution and will not appear elsewhere.

original multiplication. Thus the functor carrying each semigroup to the same semigroup equipped with the opposite multiplication, and each homomorphism to itself (as a set map), is an automorphism of the category. However it is not naturally isomorphic to the identity functor. Namely a semigroup S for which $\forall x\, \forall y \quad xy = x$ is carried to a semigroup S^{op} for which $\forall x\, \forall y \quad xy = y$, which are not isomorphic for $|S| > 1$.

2) The second question, intuitively, is the following. Describe all possible axiomatizations of a category of some structures by means of some other (more general) structures. E.g. the category of topological spaces has two different axiomatizations by set systems on the same underlying set (a set system is a pair (X,τ), X a set, $\tau \subset \mathcal{P}(X)$, while $f : (X,\tau) \to (Y,\sigma)$ is a set system morphism if $f \in Y^X$ and $f^{-1}(\sigma) \subset \tau$), namely the one given by the open sets, and the one given by the closed sets of the topological spaces. Categorically, given two categories **C** and **D**, we ask about all full embeddings **C** $\to$ **D**, whether they are the ones usually given, or not, and in the second case if we can describe all of them.

1.2. We will use the notations **Set, Pos, Ab, Grp, Sgr, CommRng, Rng** for the categories of sets, partially ordered sets, Abelian groups, groups, semigroups, commutative rings, rings, respectively. Further **Top** denotes the category of topological spaces, while **Top**$_1$, **Haus**, **Reg**$_1$, **Tych**$_1$, **Met**$_c$ its full subcategories consisting of the $T_1, T_2, T_3, T_{3\frac{1}{2}}$, resp. metric spaces. Let **SetSyst**^{0} denote the category with objects (X,τ), where X is a set and $\{\emptyset, X\} \subset \tau \subset \mathcal{P}(X)$, and with morphisms $f : (X,\tau) \to (Y,\sigma)$ characterized by $f \in Y^X$ and $f^{-1}(\sigma) \subset \tau$. **Unif** resp. **Prox** denotes the category of uniform spaces, resp. that of proximity spaces (in the sense of Efremovič); the T_0 axiom is not assumed at the definition of uniform, resp. proximity spaces. For concrete categories (thus all categories in this paper) U, or sometimes U', will denote the forgetful functor to **Set**. (For **SetSyst**^{0} $U(X,\tau) = X$, $Uf = f$.) Undefined categorical notions cf. e.g. in [15], [1].

§2

2.1. We begin with

THEOREM 1. *The automorphisms of the categories* **Set** *[9] p.29,* **Pos** *[9] p.30,* **Ab** *[9] p.30,* **Grp** *[9] p.31,* **Sgr** *[3],* **CommRng** *[6],* **Rng** *[6] are naturally isomorphic to the respective identity functor, or, for the cases* **Sgr**, **Rng** *to the opposite semigroup, or ring functor (carrying a semigroup, or ring to the same semigroup, or ring, but with the opposite multiplication, and same addition, in case of rings, and the homomorphisms, as set maps, to themselves) resp. for the case* **Pos**, *to the functor reversing the partial order (and carrying the morphisms, as set maps, to themselves).*

We note that [6] actually considered the category of (commutative) R- algebras, for R a commutative integral domain with 1, and obtained a bit more complicated answer (taking in account the automorphisms of R). Categories of modules are treated in [23] and [9] p.106, [15] p.325, actually giving categorical characterizations of these categories (a more general situation is treated in [9] p.120); these characterizations are rather similar to the categorical characterization of varieties, resp. quasivarieties of finitary algebras in [23], resp. [17], cf. also [27], [15] §32, 38, [1] §23, 24. For the category of groups the opposite group functor was not listed, since it is naturally isomorphic to the identity; for a group G and its opposite group G^{op} there is an isomorphism $\varphi: G \to G^{op}$ defined by $\varphi(x) = x^{-1}$, and this is a natural transformation from identity to the opposite group functor. [54] has shown that all concrete full embeddings **Set** $\to$ **Sgr**, resp. [category of totally ordered sets, with isotone maps] $\to$ **Sgr** are given by defining xy as x, or as y, resp. as $\min\{x,y\}$, or as $\max\{x,y\}$, and there are no concrete full embeddings **Pos** $\to$ [any category of universal algebras]. Similar results are contained in [19]. [7] contains the proof of the statement, that any automorphism of the category of locally compact Abelian groups, with continuous homomorphisms, is naturally isomorphic to the identity (given there in the equivalent form that any isomorphism with the dual category is naturally isomorphic to the usual duality given by hom $(-,\mathsf{R}/\mathsf{Z})$). For more details cf. the above cited works, and also [55] (where "polygon" means monoid action), [48] p.539.

2.2. The method of proof of most of these statements goes back to P. Freyd [9], and consists of three steps.

1) Identification of the free algebra $F(1)$ on one generator. (E.g. it is regular-projective, has some minimality property among these, etc.)

2) The underlying set functor U is then identified as (being naturally isomorphic to) $\hom(1, U(-)) \cong \hom(F(1), -)$, where 1 is a one-element set.

3) There remained to identify the algebraic structure on $UC \cong \hom(F(1), C)$, for each object C of the category in question. Thus we have to define an algebra structure on a hom-set hom (A, B), which would be evident if B had an algebra structure, but as yet it is not known. However hom (A, B) is covariant in B, and contravariant in A, thus it suffices to define on A a "co-algebra" structure. (Remind that e.g. at the definition of the homotopy groups of topological spaces a group structure on the set of homotopy classes of base-point preserving continuous maps $S^n \to B$ is not obtained by any group structure on B, but by maps on S^n.)

2.3. In more detail, let us consider the simplest case, the category of Abelian groups.

Although in this case not the full power of the co-algebra concept is needed for the proof, we will give a complete definition of them in this case, since in other cases this concept is necessary for the existing proofs.

An Abelian group is a set X equipped with a multiplication $m: X \times X \to X$ satisfying among others the associative law, expressed by the commutativity of the diagram

$$\begin{array}{ccc} & (X \times X) \times X \cong X \times X \times X \cong X \times (X \times X) & \\ {\scriptstyle m \times 1_X} \swarrow & & \searrow {\scriptstyle 1_X \times m} \\ X \times X & & X \times X \\ {\scriptstyle m} \searrow & & \swarrow {\scriptstyle m} \\ & X & \end{array}$$

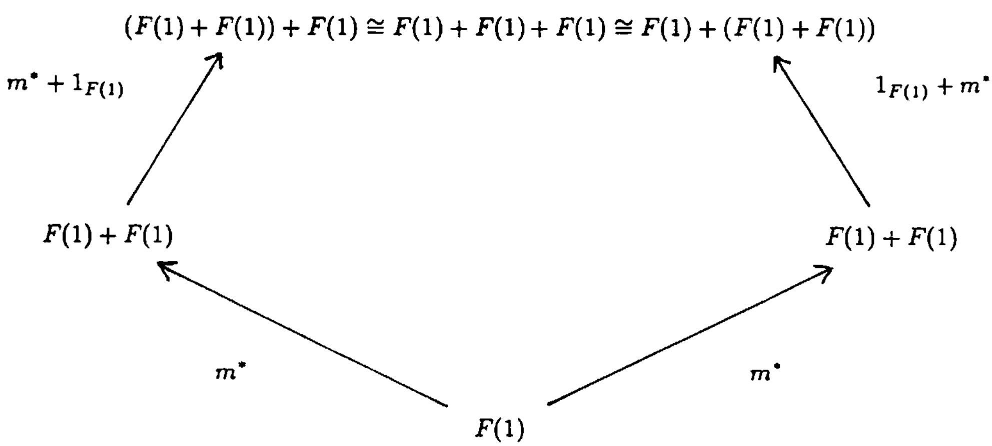
(where in the upper row the isomorphisms are the canonical ones). Considering hom $(F(1), C))$, by contravariance, this will correspond to a "co-multiplication" $m^*: F(1) \to F(1) + (F(1)$, satisfying a "co-associative" law, obtained by dualizing the above diagram.

More concretely, for an Abelian group C and two elements x_1, x_2 of UC $x_1 x_2$ can be identified as follows. Let x_i correspond at the isomorphism $UC \cong \hom(F(1), C)$ to $\varphi_i \in \hom(F(1), C)$, i.e. φ_i is determined by carrying the free generator $\bar{1}$ (coming from the adjunction) of $F(1)$ into x_i. Then we have a unique map $[\varphi_1, \varphi_2]: F(1) + F(1) \to C$, which composed with the injections $\mu_i: F(1) \to F(1) + F(1)$ to the coproduct gives φ_i. Then $x_1 x_2 = [\varphi_1, \varphi_2](\mu_1(\bar{1})\mu_2(\bar{1}))$. In other words, letting m^* be the morphism $F(1) \to F(1)+F(1)$ defined by $m^*(1) = \mu_1(\bar{1})\mu_2(\bar{1})$, we have $x_1 x_2 = [\varphi_1, \varphi_2]m^*(\bar{1})$, that is, $x_1 x_2$ corresponds to the composite homomorphism

$$F(1) \xrightarrow{m^*} F(1) + F(1) \xrightarrow{[\varphi_1, \varphi_2]} C.$$

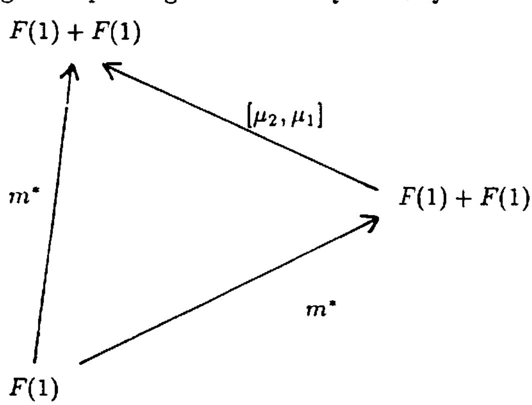
The associativity of multiplication is expressed via m^* in the following way. Let $x_i \in UC$, $i = 1, 2, 3$, and let φ_i be the corresponding morphisms $F(1) \to C$. Then $(x_1, x_2)x_3$, resp. $x_1(x_2 x_3)$ correspond to the homomorphisms $[[\varphi_1, \varphi_2]m^*, \varphi_3]m^* = [[\varphi_1, \varphi_2], \varphi_3](m^* + 1_{F(1)})m^* = [\varphi_1, \varphi_2, \varphi_3]\iota'(m^* + 1_{F(1)})m^*$, resp. to $[\varphi_1, [\varphi_2, \varphi_3]m^*]m^* = [\varphi_1, [\varphi_2, \varphi_3]](1_{F(1)} + m^*)m^* = [\varphi_1, \varphi_2, \varphi_3]\iota''(1_{F(1)} + m^*)m^*$, where $\iota': (F(1) + F(1)) + F(1) \to F(1) + F(1) +$

$F(1), \iota'': F(1)+(F(1)+F(1)) \to F(1)+F(1)+F(1)$ are the canonical isomorphisms. Equality of these, for each Abelian group C and each $\varphi_1, \varphi_2, \varphi_3 \in \hom(F(1), C)$ means that $\iota'(m^* + 1_{F(1)})m^* = \iota''(1_{F(1)} + m^*)m^*$ (put $C = F(1) + F(1) + F(1), \varphi_i =$ the i-th injection $F(1) \to F(1) + F(1) + F(1)$). However this means exactly that the diagram, obtained from the diagram expressing associativity of multiplication, by dualizing, and replacing X with $F(1)$, commutes:



This property of m^* is called co-associativity of m^*.

Similarly, commutativity of m can be expressed in the following way. We have, for $x_1, x_2, \varphi_1, \varphi_2, \mu_1, \mu_2$ as above, $[\varphi_1, \varphi_2]m^*(\bar{1}) = x_1x_2 = x_2x_1 = [\varphi_2, \varphi_1]m^*(\bar{1}) = [\varphi_1, \varphi_2]$ $[\mu_2, \mu_1]m^*(\bar{1})$, which is, like above, equivalent to the commutativity of the diagram, obtained from the diagram expressing commutativity of m, by dualization, and replacing X by $F(1)$:

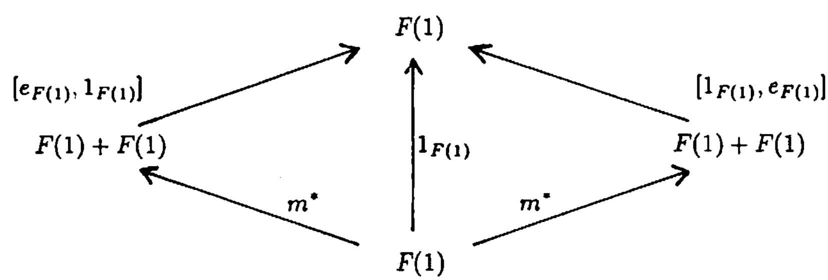


Commutativity of this diagram will be called co-commutativity of m^*.

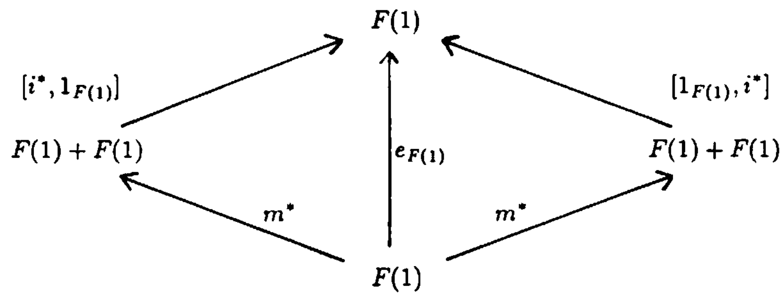
Using the commutativity of these diagrams, for $\bar{1} \in F(1)$, we see that, denoting $m^*(\bar{1}) = \mu_1(\bar{1})^k \mu_2(\bar{1})^\ell \in F(1) + F(1)$, co-associativity, resp. co-commutativity of m^* is equivalent to $k^2 = k, \ell^2 = \ell$, resp. to $k = \ell$.

We have beside m a unary operation i (inverse) and a 0-ary operation e (identity). Like above, we have for x in UC and φ like above (i.e., $\varphi(\bar{1}) = x$), $x^{-1} = \varphi((\bar{1})^{-1})$. Let i^* be the morphism $F(1) \to F(1)$ defined by $i^*(\bar{1}) = (\bar{1})^{-1}$. Then $x^{-1} = \varphi i^*(\bar{1})$, that is, x^{-1} corresponds to the composite homomorphism $F(1) \xrightarrow{i^*} F(1) \xrightarrow{\varphi} C$. Similarly, let e^* be the unique morphism $F(1) \to$ sum of 0 copies of $F(1)$ = initial object = one-element Abelian group. Then e corresponds to the composite homomorphism $F(1) \xrightarrow{e^*}$ initial object $\to C$, where the second arrow is the unique one.

We still rewrite the remaining group axioms $ex = x = xe$, $xx^{-1} = e = x^{-1}x$. We have, with φ corresponding to x, φ_e to e ($\varphi_e(\bar{1}) = e$), and $e_{F(1)}: F(1) \to F(1)$ denoting the homomorphism with $e_{F(1)}(\bar{1})$ = the identity element of $F(1)$, that $\varphi[e_{F(1)}, 1_{F(1)}]m^*(\bar{1}) = [\varphi_e, \varphi]m^*(\bar{1}) = ex = x = \varphi(\bar{1}) = xe = [\varphi, \varphi_e]m^*(\bar{1}) = \varphi[1_{F(1)}, e_{F(1)}]m^*(\bar{1})$. Also, $\varphi[i^*, 1_{F(1)}]$ $m^*(\bar{1}) = [\varphi i^*, \varphi]m^*(\bar{1}) = x^{-1}x = e = \varphi\, e_{F(1)}(\bar{1}) = xx^{-1} = [\varphi, \varphi\, i^*]m^*(\bar{1}) = \varphi[1_{F(1)}, i^*]$ $m^*(\bar{1})$. Therefore, the group axioms considered are equivalent to the commutativity of the following diagrams, which arise from the diagrams expressing the respective group axioms, by dualization, and replacing X with $F(1)$:



resp.



With $m^*(\bar{1}) = \mu_1(\bar{1})^k\,\mu_2(\bar{1})^\ell$, $i^*(\bar{1}) = (\bar{1})^j$ the commutativity of these diagrams is equivalent to $k = \ell = 1$, resp. to $(j = -1,\ k = \ell) \vee (j = 1,\ k + \ell = 0) \vee (k = \ell = 0)$.

(We note that co-algebras are defined in [6] actually for any variety, i.e. equationally definable class of algebras. Using the same correspondence $UC \cong \hom(F(1), C)$, and denoting for $x_i \in UC$ the corresponding homomorphism by φ_i – i.e. $\varphi_i(\bar{1}) = x_i$, where $\bar{1}$ is the free generator –, the value of an n-ary (possibly composite) operation f at $x_1, \ldots, x_n$ can be determined as follows. Let us consider the n-fold coproduct $F(1) + \cdots + F(1)$, with injections $\mu_1, \ldots, \mu_n$. Consider the morphism $\varphi = [\varphi_1, \ldots, \varphi_n]: F(1) + \cdots + F(1) \to C$. Then $f(x_1, \ldots, x_n) = f(\varphi_1(\bar{1}), \ldots, \varphi_n(\bar{1})) = f(\varphi\mu_1(\bar{1}), \ldots, \varphi\mu_n(\bar{1})) = \varphi(f(\mu_1(\bar{1}), \ldots, \mu_n(\bar{1})))$. That is, letting $f^*: F(1) \to F(1) + \cdots + F(1)$ be defined by $f^*(\bar{1}) = f(\mu_1(\bar{1}), \ldots, \mu_n(\bar{1}))$, $f(x_1, \ldots, x_n) = \varphi f^*(\bar{1})$ corresponds to the homomorphism φf^*. This specifies the operations f, with the help of f^*. It remains to rewrite the equations satisfied by the operations. For notational convenience we make this only for an equation of the form $f(g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m)) = f'(g'_1(y_1, \ldots, y_m), \ldots, g'_{n'}(y_1, \ldots, y_m))$, where any of the operations g_i, g'_i can be equal to an operation of smaller arity, with variables a subset of $y_1, \ldots, y_m$, resp. to some of its variables. (The general case is done the same way, but with notational complications.) Let $y_j = \psi_j(\bar{1})$, $\psi_j: F(1) \to C$. Then $[\psi_1, \ldots, \psi_m](f(g_1, \ldots, g_n))^*(\bar{1}) = f(g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m))$. Further this equals $[[\psi_{11}, \ldots, \psi_{1m}], \ldots, [\psi_{n1}, \ldots, \psi_{nm}]]\,(\coprod_i g_i^*)\,f^*(\bar{1})$, where $\psi_{ij} = \psi_j$, $i = 1, \ldots, n$, (where $\coprod_i g_i^*: \coprod_i F(1) \to \coprod_i \left(\coprod_j F(1)\right)$ is the morphism between the two n-fold coproducts equal to the coproduct of the g_i^*-s, i.e. $[\psi_1, \ldots, \psi_m][1_{F(1)+\cdots+F(1)}, \ldots, 1_{F(1)+\cdots+F(1)}]$ $(\coprod_i g_i^*)\,f^*(\bar{1})$ (where the identity map of the m-fold coproduct, say $F(m)$, appears n times). Therefore the equation $f(g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m)) = f'(g'_1(y_1, \ldots, y_m), \ldots, g'_{n'}(y_1, \ldots, y_m))$ holds for each object C iff $[1_{F(m)}, \ldots, 1_{F(m)}](\coprod_i g_i^*)f^* = [1_{F(m)}, \ldots, 1_{F(m)}]$ $(\coprod_i (g'_i)^*)(f')^*$ (where $1_{F(m)}$ appears on the left hand side n times, on the right hand side n' times).)

2.4. Now we complete the proof in case of Abelian groups, by an elementary, partly ad hoc argument. We have to identify the object $F(1) \in \mathcal{O}b\mathbf{Ab}$, and the co-multiplication $m^*: F(1) \to F(1) + F(1)$. (We note that in the paragraphs **2.4** - **2.6** the described proofs follow the original proofs, i.e. the pattern described in **2.2**, but we hope that the proofs given here are in the details either more elementary, or simpler, than the original ones.)

First we note that in **Ab** the monomorphisms are, up to equivalence at mono subobjects, the embeddings of subgroups (and the same holds for **Sgr** and the category of commutative semigroups as well). Further, the only Abelian groups, whose only subgroups are the one-element group (the terminal object of **Ab**) and themselves (the embedding being the identity), and are not one-element groups, are generated by any element other then identity, thus are the cyclic groups of prime order, which constitute the class $\mathbf{C}_1$, say. Thus we can characterize categorically the class $\mathbf{C}_2$ of Abelian groups, without elements of finite order and different from identity; namely these are the Abelian groups having no cyclic group of prime order as a subgroup, i.e. are not the codomains of monomorphisms with domains in $\mathbf{C}_1$. Then the infinite cyclic group $F(1)$ is characterized as the only object of $\mathbf{C}_2$, which is non-terminal in **Ab**, and admits a monomorphism to any object in $\mathbf{C}_2$, non-terminal in **Ab**.

We yet have to identify the co-multiplication $m^*: F(1) \to F(1) + F(1)$. However **Ab** is an Abelian category, thus has canonically identified finite products and coproducts, namely the biproducts. Now m^* can be given as $\langle 1_{F(1)}, 1_{F(1)} \rangle: F(1) \to F(1) \times F(1) \cong F(1) + F(1)$.

2.5. Now we point out some modifications, by which one can prove that also for the category of commutative semigroups each automorphism is naturally isomorphic to the identity. However we note that this result can be proved also with some evident modification of the proof of [3].

In the category of commutative semigroups the initial object is the empty semigroup, the terminal object is the one-element semigroup. Note that a (commutative) semigroup generated by an element x is either free, or there is a smallest n such that $x^n = x^{n-k}$ for some $k > 0$; then $x, \ldots, x^{n-1}$ are all elements of the semigroup generated by x, they are distinct, and multiplication is performed in the obvious way. Also $\{x^{n-k}, \ldots, x^{n-1}\}$ is a cyclic group, hence has a one-element subgroup (namely $\{x^\ell\}$, with $n - k \leq \ell < n$, ℓ a multiple of k).

Let us denote $\mathbf{C}_1$ the class consisting of the terminal commutative semigroup. Again letting $\mathbf{C}_2$ be the class of all those commutative semigroups which do not have subsemigroups (i.e., do not have mono subobjects) belonging to the class $\mathbf{C}_1$, we see $\mathbf{C}_2$ consists of all those commutative semigroups, in which each element generates a free subsemigroup. $F(1)$ is a subsemigroup of each non-initial element of $\mathbf{C}_2$, and the same holds for all subsemigroups of $F(1)$ as well, and (up to isomorphism) only for them. Let them constitute the class $\mathbf{C}_3$, say. I.e., using further additive notation, $\mathbf{C}_3$ is the class of all subsemigroups of the additive semigroup $\mathbb{N}$.

Let $C, D \in \mathbf{C}_3$, $\varphi: C \to D$. Then for $m, n \in C \subset \mathbb{N}$ we have $n\varphi(m) = \varphi(nm) = m\varphi(n)$, i.e. $\varphi(n) = \text{const} \cdot n$, const > 0. In particular, φ is a monomorphism. If $\varphi: C \to D$, then the mono subobject $\varphi(C) \hookrightarrow D$ is defined as the monomorphism factor in the regular epi-mono (i.e., surjective-injective) factorisation of φ. If $C \cong F(1) \cong \mathbb{N}$, then, assuming for simplicity $C = \mathbb{N}$, we have $\forall D \in \mathbf{C}_3 \quad \forall n \in D \quad \exists \varphi: C \to D, \quad \varphi(1) = n$, thus the union of the mono subobjects $\varphi(C), \varphi: C \to D$, in the complete lattice of all mono subobjects of D, is D. Let now $C \not\cong F(1)$. Let us choose $D = \mathbb{N} \in \mathbf{C}_3$. For $\varphi: C \to \mathbb{N}$ let us consider the subobject $\varphi(C) \hookrightarrow \mathbb{N}$. If, supposing for simplicity $\varphi(C) \subset \mathbb{N}$, we have $1 \in \varphi(C)$, then $\varphi(C) = \mathbb{N}$, thus, φ being a monomorphism, $C \cong F(1)$, a contradiction. Therefore $\varphi(C)$ is contained in the subsemigroup $\{2, 3, \dots\}$ of $\mathbb{N}$, thus the union of all mono subobjects $\varphi(C), \varphi: C \to D = \mathbb{N}$, in the complete lattice of all mono subobjects of D, is not $D = \mathbb{N}$. Thus we have shown that $C \in \mathbf{C}_3$ is isomorphic to $F(1)$ iff $\forall D \in \mathbf{C}_3$ the union of the subobjects $\varphi(C), \varphi \in \hom(C, D)$, in the complete latice of all mono subobjects of D, is D. This identifies $F(1)$ categorically.

It remains to identify $m^*: F(1) \to F(1) + F(1)$, whose domain and codomain have already been identified categorically, which commutative semigroups they actually are. $\bar{1} \in F(1)$ is the only element of $F(1)$, generating $F(1)$. $m^*(\bar{1})$ is characterized as the only element of $F(1) + F(1)$, which can be written as a product of two elements, and of at most two elements only, and if it is written as a product of two elements, then these elements are distinct. This ends the proof that for the category of commutative semigroups each isomorphism is naturally isomorphic to the identity.

2.6. Note that any semigroup, generated by one element, is commutative. Thus for the category of all semigroups we can define the classes $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ word for word as for the category of commutative semigroups. Thus like above we characterize categorically the free semigroup $F(1)$ on one generator, and hence also $F(1) + F(1)$. The property characterizing $m^*(\bar{1})$ above is now satisfied by two elements: $z' = (\mu_1\bar{1})(\mu_2\bar{1})$, and $z'' = (\mu_2\bar{1})(\mu_1\bar{1})$, while actually we have $m^*(\bar{1}) = (\mu_1\bar{1})(\mu_2\bar{1})$, cf. **2.3.** Recalling from **2.3** that for a semigroup C and its two elements x_1, x_2, with corresponding homomorphisms $\varphi_i \in \hom(F(1), C)$ we have $x_1x_2 = [\varphi_1, \varphi_2]((\mu_1\bar{1})(\mu_2\bar{1})) = [\varphi_1, \varphi_2](m^*(\bar{1}))$, we see that z' gives the original multiplication on C, while z'' defines on UC the opposite multiplication $x_1 * x_2 = x_2x_1$, since $[\varphi_1, \varphi_2]((\mu_2\bar{1})(\mu_1\bar{1})) = [\varphi_1, \varphi_2][\mu_2, \mu_1]((\mu_1\bar{1})(\mu_2\bar{1})) = [\varphi_2, \varphi_1]((\mu_1\bar{1})(\mu_2\bar{1}))$.

2.7. The characterization of categories of modules in Freyd [9], p. 106 specializes to categories $\mathbf{Vec_K}$ of vector spaces over some field **K**. Namely a category **C** is equivalent to a category $\mathbf{Vec_K}$ iff it is a cocomplete Abelian category, having a small projective generator C_0 (in the terminology of [9]) which additionally satisfies that $\hom(C_0, C_0)$, which is a ring by Abelianness, is actually a field (choose C_0 a one dimensional vector space). The proof of [9] also yields that, for fields K_1, K_2, $\mathbf{Vec_{K_1}}$ and $\mathbf{Vec_{K_2}}$ are isomorphic iff $\mathbf{K_1} \cong \mathbf{K_2}$, and that any automorphism $\mathbf{Vec_K} \to \mathbf{Vec_K}$ is naturally isomorphic to a concrete functor $F_0(\iota)$ preserving the additive structure of the vector spaces, but satisfying that for $C \in \mathcal{O}b\mathbf{Vec_K}$, $x \in UC$, $\lambda \in F$ λx in C equals $\iota(\lambda)x$ in $F_0(\iota)C$, where $\iota\colon \mathbf{K} \to \mathbf{K}$ is an isomorphism.

A small modification of Freyd's proof yields the description of all full embeddings $\mathbf{Vec_K} \to \mathbf{Vec_K}$. Let **NormVec**, resp. $\mathbf{NormVec_b}$ denote the category of normed linear spaces (real or complex), with all contractive linear maps, resp. all bounded linear maps, as morphisms. Let further $\mathbf{T_2LocConv_w}$ denote the category of $\mathbf{T_2}$ locally convex spaces (real or complex) having weak topology, with all continuous linear maps, as morphisms. For **NormVec**, resp. $\mathbf{NormVec_b}$ or $\mathbf{T_2LocConv_w}$ $F_0(\iota)$ is defined like above, but only for continuous automorphisms ι, and additionally requiring that $F_0(\iota)$ also preserves the norm, resp. the topology.

PROPOSITION 1. *Let* **C** *be* $\mathbf{Vec_K}$, *for a field* K, *or* **NormVec** *(real or complex) or* $\mathbf{NormVec_b}$ *(real) or* $\mathbf{T_2LocConv_w}$ *(real). Then each full embedding* $F\colon \mathbf{C} \to \mathbf{C}$ *is naturally isomorphic to a functor* $F_0(\iota)$, *where* $\iota = 1_\mathbf{R}$ *for the real case, and* $\iota = 1_\mathbf{C}$ *or* $\iota =$ *complex conjugation for the complex case.*

PROOF: We will identify for each object $C \in \mathcal{O}b\mathbf{C}$ and each morphism $f \in \mathcal{M}or\,\mathbf{C}$ the object C and the morphism f (up to isomorphy). This identification will be such that it can be applied to each FC and each Ff as well, yielding that $F_0(\iota)C$ and FC coincide, up to isomorphy, and similarly for $F_0(\iota)f$ and Ff. Namely, we only will use objects from the image subcategory $F(\mathbf{C})$ (thus, e.g. we will not use universal properties in **C**, which can be different in **C** and $F\mathbf{C}$, hence preservation of them by F is not immediate).

A) For any object C F carries the zero morphism $0_C\colon C \to C$ to the zero-morphism $0_{FC}\colon FC \to FC$. Then a one-dimensional vector space C_0 is characterized by the property that $\hom(C_0, C_0)$ has more than one element and does not have non-trivial divisors of 0. Hence $\dim FC_0 = 1$ as well, thus $FC_0 \cong C_0$.

B) Now we turn to the case $\mathbf{C} = \mathbf{NormVec}$. Let $U_1\colon \mathbf{NormVec} \to \mathbf{Set}$ be the closed unit ball functor. (For an object C U_1C is its closed unit ball, for a morphism $f\colon C_1 \to C_2$ $U_1f\colon U_1C_1 \to U_1C_2$ is the two-sided restriction of the linear operator f.) Then $\hom(C_0, -) \cong U_1$; for convenience we suppose $\hom(C_0, -) = U_1$.

$\Lambda = \hom(C_0, C_0)\setminus\{0_{C_0}\}$ acts on each $\hom(C_0, C)$ by composition. We can identify U: **NormVec** $\to$ **Set**, and the usual natural monotransformation $U_1 \to U$ (in a way preserved by F) as follows. Let $\lambda \in \Lambda$, $f \in \hom(C_0, C)$. We define f/λ as the equivalence class of pairs (f, λ), under the equivalence $(f_1, \lambda_1) \sim (f_2, \lambda_2) \Leftrightarrow f_1\lambda_2 = f_2\lambda_1$. The set of all these (f/λ)'s is UC. Then for $g\colon C_1 \to C_2$ in **NormVec** $U_1 g\colon U_1C_1 \to U_1C_2$ is extended to a map $Ug\colon UC_1 \to UC_2$ by $(Ug)(f/\lambda) = (gf)/\lambda$. The usual monotransformation $U_1 \to U$ is the set inclusion $(f \mapsto f/1_{C_0})$.

Now we identify **NormVec**$_\mathbf{b}$ (in a way preserved by F) as the category with the same objects as **NormVec**, and with hom-sets $\hom(C_1, C_2)$ consisting of the elements $(Ug/\nu \in (UC_2)^{UC_1}$, $g\colon C_1 \to C_2$ in **NormVec**, $\nu \in \Lambda$, where $((Ug)/\nu)(f/\lambda) = (gf)/(\nu\lambda)$, and $((Ug_1)/\nu_1)((Ug_2/\nu_2) = (U(g_1g_2))/(\nu_1\nu_2)$. (This amounts to extend F to a full embedding F_b: **NormVec**$_\mathbf{b}$ $\to$ **NormVec**$_\mathbf{b}$.

C) For **Vec**$_\mathbf{K}$, **NormVec**$_\mathbf{b}$, **T**$_2$**LocConv**$_\mathbf{w}$ we have $U \cong \hom(C_0, -)$.

D) Further we proceed like in [9], p. 106, [15], p. 313. **Vec**$_\mathbf{K}$, **NormVec**$_\mathbf{b}$, resp. **T**$_2$**LocConv**$_\mathbf{w}$ are pointed categories with finite biproducts, which makes possible to define categorically an addition on each $\hom(C_1, C_2)$ in the respective category, coinciding with the usual addition of linear operators.

We have that this addition is preserved by F, provided F preserves biproduct diagrams. Evidently F preserves zero-objects, zero-morphisms $0_C\colon C \to C$ for any $C \in \mathcal{O}b\mathbf{C}$, hence also zero-morphisms in any $\hom(C_1, C_2)$, as multiples of zero-morphisms in $\hom(C_i, C_i)$, $i = 1, 2$. Therefore biproduct diagrams for objects C_1, C_2, with C_1 or C_2 0-dimensional, are preserved by F. Let now $\dim C_1, \dim C_2 \geq 1$. Then $\mu_i\colon C_i \to C$, $\pi_i\colon C \to C_i$ $(i = 1, 2)$ form a biproduct diagram iff $\pi_i\mu_j$ is a zero-morphism for $i \neq j$, $\pi_i\mu_i = 1_{C_i}$, and $\sum \mu_i\pi_i = 1_C$ ([15], p. 308). We have then $\dim C = \dim(C_1 \oplus C_2) \geq 2$. However in this case the monoid $\hom(C, C)$ can be made to a ring with a unique additive structure only (i.e., the additive structure of the ring $\hom(C, C)$ is determined by its multiplicative structure), cf. [10], p. 864. This implies that the condition $\sum \mu_i\pi_i = 1_C$ is preserved by F, hence all biproduct diagrams are preserved by F, thus the additive structure on each $\hom(C_1, C_2)$ is preserved by F as well. (An elementary alternative proof is the following. For $\dim C_i \geq 1$, $\mu_i\colon C_i \to C$, $\pi_i\colon C \to C_i$ $(i = 1, 2)$, $\pi_i\mu_j$ a zero-map for $i \neq j$, $\pi_i\mu_i = 1_{C_i}$, the condition $\sum \mu_i\pi_i = 1_C$ can be expressed, by the multiplicative structure of $\hom(C, C)$ only, as follows. Denoting $p_i = \mu_i\pi_i$ (thus we have $0 \neq p_i = p_i^2$, $p_ip_j = 0$ for $i \neq j$, $i, j \in \{1, 2\}$), there does not exist p_3, such that $0 \neq p_i = p_i^2$, $p_ip_j = 0$ for $i \neq j$, $i, j \in \{1, 2, 3\}$. Proof is straightforward, choosing $p_3 = 1 - p_1 - p_2$.)

E) Thus, in particular, $\hom(C_0, C_0)$ has a ring structure, preserved by F, and this is the respective field of scalars, K, R or C (for K this holds if we consider right vector spaces — for left vector spaces this is K with opposite multiplication, but this also identifies K). Thus we have identified K, R or C, up to an automorphism ι.

R only has the identical automorphism.

For C with $\mathbf{C}$ = **NormVec**, observe that we also know the subset of $\hom(C_0, C_0)$, in the category $\mathbf{NormVec_b}$, which is the hom-set $\hom(C_0, C_0)$, in the category **NormVec**, i.e. the subset consisting of the contractive linear maps (in other words, the extension F_b of F preserves and reflects this subset). Thus ι is an automorphism, mapping $\{z \in \mathrm{C} \mid |z| \leq 1\}$ onto itself. However thus ι maps each $\{z \in \mathrm{C} \mid |z| \leq r\}$, r some rational number, onto itself. Therefore $|\iota(z)| = |z|$, hence ι is identical on R, and thus either $\iota = 1_{\mathrm{C}}$ or ι = conjugation.

F) Since $\hom(C_0, C_0)$ acts on each $\hom(C_0, C)$ by composition, thus we have identified the multiplication by scalars on each UC, up to an above isomorphism ι.

Addition on $UC \cong \hom(C_0, C)$ (in $\mathbf{Vec_K}$, $\mathbf{NormVec_b}$, resp. $\mathbf{T_2LocConv_w}$) is identified, since $\hom(C_0, C)$ has an already identified additive group structure (i.e. one preserved by F) in the respective category.

For **NormVec**, norm on UC can be identified, as usual, by $\|x\| = \inf\{\lambda \in \mathrm{R} \mid \lambda \geq 0, x \in \lambda \cdot U_1 C\}$.

Since for each $g \in \hom(C_1, C_2)$ Ug has already been identified (using $\hom(C_0, -)$), the proof for $\mathbf{Vec_K}$, **NormVec** is finished.

It remained to show that F also preserves the topology in case of $\mathbf{NormVec_b}$ and $\mathbf{T_2LocConv_w}$. This follows from [43], [44], whose considerations we repeat. For $\mathbf{T_2LocConv_w}$ each C has the weak topology w.r.t. $\hom(C, C_0)$. For $\mathbf{NormVec_b}$ a subset A of UC is bounded iff $\forall f \in \hom(C, C_0)$ $f(A)$ is bounded (Banach-Steinhaus); then a subset $V \ni 0$ is a neighbourhood of 0 iff $\forall A \subset UC$ bounded $\exists \varepsilon > 0$ $\varepsilon A \subset V$. □

2.8. Using some more involved results of [21], [2], [10], [43], [44] we can prove on the lines of these papers, for commutative field of scalars, a statement stronger than Proposition 1 (cf. also [62]).

We note that if $\mathbf{C}_1$ is a full subcategory of $\mathbf{Vec_K}$, and $F\colon \mathbf{C}_1 \to \mathbf{Vec_K}$ is a full embedding (or similarly with **NormVec**, $\mathbf{NormVec_b}$, $\mathbf{T_2LocConv_w}$ rather than $\mathbf{Vec_K}$), then zero objects are carried by F to zero-objects. Further, for any $C_0 \in Ob\,\mathbf{C}_1$, $\dim C_0 = 1$ we have by the proof of Proposition 1 $\dim FC_0 = 1$; and F establishes a bijection $K \cong \hom(C_0, C_0) \to \hom(FC_0, FC_0) \cong K$ (for right vector spaces), which is a semigroup isomorphism. Thus for $Ob\,\mathbf{C}_1 \subset \{C \in Ob\,\mathbf{Vec_K} \mid \dim C \leq 1\}$ a full embedding $\mathbf{C}_1 \to \mathbf{Vec_K}$ is, up to natural isomorphism, completely determined by a multiplicative isomorphism $\iota\colon K \to K$ (i.e. an isomorphism of the multiplicative structure of K), which in turn can be arbitrary. Similar statement holds for **NormVec**, **NormVec$_b$**, **T$_2$LocConv$_w$** too, but in case of **NormVec** we can make a reduction to **NormVec$_b$**, like at Proposition 1, and thus ι must preserve and reflect $\{x \mid |x| \le 1\} \subset \mathbf{R}$, resp. $\mathbf{C}$. Then for $\mathbf{R}$ $|\iota|$ is a monotonically increasing semigroup homomorphism, for $x > 0$ $\iota(x) = \iota(\sqrt{x})^2 > 0$, thus $\iota(x) = \operatorname{sgn} x \cdot |x|^c$, $c > 0$, while for $\mathbf{C}$ ι preserves and reflects $\{x \mid |x| = 1\}$, and $|\iota|$ restricted to $\{x \in \mathbf{R} \mid x \ge 0\}$ equals x^c, $c > 0$, thus $|\iota(x)| = |x|^c$, $c > 0$ (cf. also [8]).

We note yet that for **NormVec$_b$**, if a full subcategory $\mathbf{C}_1$ consists of finite dimensional spaces only, it can be considered as a full subcategory of $\mathbf{Vec_K}$, $K = \mathbf{R}$, $\mathbf{C}$, and its image by a full embedding into **NormVec$_b$** also contains finite dimensional spaces only (cf. e.g. C) of the following proof). Thus the full embeddings $F\colon \mathbf{C}_1 \to$ **NormVec$_b$** are given by those from $\mathbf{C}_1$ to $\mathbf{Vec_K}$, $K = \mathbf{R}, \mathbf{C}$.

With the above notation $F_0(\iota)$ (c.f. **2.7.**) we have

PROPOSITION 2. *Let $\mathbf{C}_1$ be a full subcategory of $\mathbf{C}$, $\mathbf{C} = \mathbf{Vec_K}$ (for a commutative field K), or $\mathbf{C} =$ **NormVec** (real or complex), or $\mathbf{C} =$ **NormVec$_b$** (real or complex, but in the complex case supposing $\mathcal{O}b\,\mathbf{C}_1 \not\subset$ {finite dimensional spaces in $\mathbf{C}$}), or **T$_2$LocConv$_w$** (real). Let $\mathcal{O}b\,\mathbf{C}_1 \not\subset \{C \in \mathcal{O}b\,\mathbf{C} \mid \dim C \le 1\}$. Then each full embedding $F\colon \mathbf{C}_1 \to \mathbf{C}$ is naturally isomorphic to the restriction of a functor $F_0(\iota)$ to $\mathbf{C}_1$, where $\iota = 1_\mathbf{R}$ for the real case, and $\iota = 1_\mathbf{C}$ or $\iota =$ complex conjugation for the complex case.*

PROOF: Like at Proposition 1, we will identify each object $C \in \mathcal{O}b\,\mathbf{C}_1$ and each morphism $f \in \mathcal{M}or\,\mathbf{C}_1$ (up to isomorphy), in a way as to be applicable to FC and Ff as well, which will yield that $F_0(\iota)C$ and FC coincide, up to isomorphy, and similarly for f.

A) First we identify the underlying set of the base field (resp. in case of **NormVec** that of the subsemigroup of the base field, consisting of the elements of absolute value ≤ 1). We have for each $C \in \mathcal{O}b\,\mathbf{C}_1$ that the centre of $\hom(C, C)$ consists just of the scalar multiples of 1_C (for **NormVec** it is $\{\lambda 1_C \mid \lambda \in \mathbf{R}$, resp. $\mathbf{C}, |\lambda| \le 1\}$). For $C', C'' \in \mathcal{O}b\,\mathbf{C}_1$, $\dim C'.\dim C'' \ge 1$, $\lambda' 1_{C'}$, $\lambda'' 1_{C''}$ (λ', λ'' scalars) satisfy $\lambda' = \lambda''$ iff for each $f\colon C' \to C''$ we have $f(\lambda' 1_{C'}) = (\lambda'' 1_{C''})f$ (for **NormVec** λ', λ'' are scalars of absolute value ≤ 1), and this is an equivalence relation among the $(\lambda 1_C)$'s. Thus the elements of the base field (for **NormVec** $\{\lambda \in$ base field $\mid |\lambda| \le 1\}$) bijectively correspond to the equivalence classes of the above equivalence relation, and these are preserved by F. Thus $F(\lambda 1_C) = \iota(\lambda) 1_{FC}$, for some multiplicative isomorphism $\iota\colon K \to K$ (resp. $\mathbf{R} \to \mathbf{R}$, $\mathbf{C} \to \mathbf{C}$; for **NormVec** $\iota\colon \{\lambda \in$ base field $\mid |\lambda| \le 1\} \to \{\lambda \in$ base field $\mid |\lambda| \le 1\}$).

B) Now we reduce the case of **NormVec** to that of **NormVec$_b$**. The objects of these two categories coincide, thus $Ob\,\mathbf{C}_1$ can be considered as a subclass of $Ob\,\mathbf{NormVec_b}$. For $C', C'' \in Ob\,\mathbf{C}_1$, we can identify the hom-set $\hom(C', C'')$ in $\mathbf{NormVec_b}$ as follows. Let Λ denote the set of the above equivalence classes of elements of the centres of $\hom(C, C)$'s. $C \in Ob\,\mathbf{C}_1$, $\dim C \geq 1$ (in the category $\mathbf{NormVec}$), minus the class consisting of the zero maps $0_C \colon C \to C$, $C \in Ob\,\mathbf{C}_1$, $\dim C \geq 1$. Λ acts on each $\hom(C', C'')$, where $\dim C'$, $\dim C'' \geq 1$, by composition, namely if $\lambda 1_{C'}, \lambda 1_{C''}$ belong to an equivalence class, the action is $f(\lambda 1_{C'}) = (\lambda 1_{C''})f = \lambda f$. Then for $f \in \hom(C', C'')$, $\dim C', \dim C'' \geq 1$, $\lambda \in \Lambda$ f/λ is defined as the equivalence class of pairs (f, λ), under the equivalence $(f_1, \lambda_1) \sim (f_2, \lambda_2) \Leftrightarrow f_1\lambda_2 = f_2\lambda_1$. The set of these (f/λ)'s, resp. the zero-morphism in $\hom(C', C'')$, if $\min(\dim C', \dim C'') = 0$, will form the hom-set $\hom(C', C'')$ in $\mathbf{NormVec_b}$.

The composition law is $(f_1/\lambda)(f_2/\lambda_2) = (f_1 f_2)/(\lambda_1\lambda_2)$, resp. (the mentioned) zero-morphisms in $\mathbf{NormVec}$ act as zero-morphisms in $\mathbf{NormVec_b}$. This identifies the class $Ob\,\mathbf{C}_1$ as a full subcategory of $\mathbf{NormVec_b}$, which will then be denoted $(\mathbf{C}_1)_\mathbf{b}$ (and this identification is preserved by F). We extend F to the hom-sets $\hom(C', C'')$ in $\mathbf{NormVec_b}$, to a map F_b, by $F_b(f/\lambda) = (Ff)/\lambda$ for $\dim C', \dim C'' \geq 1$, resp. for $\min(\dim C', \dim C'') = 0$ — which implies $\min(\dim FC', \dim FC'') = 0$ — in the unique way. F_b becomes a full embedding $(\mathbf{C}_1)_\mathbf{b} \to \mathbf{NormVec_b}$, if we define for $C \in Ob\,\mathbf{C}_1$ $F_bC = FC$.

C) In the following paragraphs C) to I) we will deal with $\mathbf{Vec_K}$, $\mathbf{NormVec_b}$ and $\mathbf{T_2LocConv_w}$.

Now we identify the base field, showing that F induces a field isomorphism $\iota \colon K \to K$ (resp. $\mathsf{R} \to \mathsf{R}$, $\mathsf{C} \to \mathsf{C}$), namely the mapping ι introduced in A). By hypothesis $\exists C \in Ob\,\mathbf{C}_1$, $\dim C \geq 2$. For $\mathbf{Vec_K}$ by [10], p. 856, for $\mathbf{NormVec_b}$, real, by [43], for $\mathbf{T_2LocConv_w}$, real, by [44], we have that for each $C \in Ob\,\mathbf{C}_1$, $\dim C \geq 2$ each semigroup isomorphism $\hom(C, C) \to \hom(FC, FC)$ is given by lifting $Uf \mapsto i(Uf)i^{-1}$, where i is a bijection of UC to UFC which is additive, is continuous together with i^{-1} for $\mathbf{NormVec_b}$, real, $\mathbf{T_2LocConv_w}$, real, and $i(\lambda x) = \alpha(\lambda)i(x)$, $\lambda \in K$ (R, C), $x \in UC$, α an automorphism of the base field. In particular, each semigroup isomorphism $\hom(C, C) \to \hom(FC, FC)$ is additive, thus is a ring isomorphism. For the semigroup isomorphism $\hom(C, C) \to \hom(FC, FC)$ induced by F we will denote i, α by i_C, α_C, resp., thus for $\dim C \geq 2$ we have $U(Ff) = i_C(Uf)i_C^{-1}$, where $i_C(\lambda x) = \alpha_C(\lambda)i_C(x)$.

For $\mathbf{NormVec_b}$, complex, by [10] p. 864, for each $C \in Ob\,\mathbf{C}_1$ satisfying $\dim C \geq 2$, each semigroup isomorphism $\hom(C, C) \to \hom(FC, FC)$ is also a ring isomorphism. By [2] p. 34, for each not finite dimensional object C, each ring isomorphism $\hom(C, C) \to \hom(FC, FC)$ is given by lifting $Uf \mapsto i(Uf)i^{-1}$, i a bijection $UC \to UFC$, which is additive, is continuous together with i^{-1}, and $i(\lambda x) = \alpha(\lambda)i(x)$, $\alpha = 1_{\mathbb{C}}$ or α = complex conjugation. For C not finite dimensional, for the ring isomorphism $\hom(C,C) \to \hom(FC, FC)$ induced by F we denote i, α by i_C, α_C, resp., thus in this case $U(Ff) = i_C(Uf)i_C^{-1}$, where $i_C(\lambda x) = \alpha_C(\lambda)i_C(x)$.

For $\mathbf{NormVec_b}$, complex, for $2 \leq \dim C$ finite, $C \in \mathcal{O}b\,\mathbf{C}_1$ can be considered as an object of $\mathbf{Vec_K}$ with $K = \mathbb{C}$, thus by [10] p. 856 we have, like above, $U(Ff) = i_C(Uf)i_C^{-1}$, i_C, α_C satisfying the properties of i, α listed at the case $\mathbf{Vec_K}$.

We have, using A), for $C \in \mathcal{O}b\,\mathbf{C}_1$, $\dim C \geq 2$, in any of the above cases, $U(\iota(\lambda)1_{FC}) = UF(\lambda 1_C) = i_C(\lambda 1_C)i_C^{-1}$, hence $U(\iota(\lambda)1_{FC})i_C = i_C U(\lambda 1_C)$, thus $\forall x \in UC \quad \iota(\lambda)i_C(x) = i_C(\lambda x)$, i.e. $\alpha_C = \iota$. Since the semigroup isomorphism $\hom(C,C) \to \hom(FC, FC)$ induced by F is a ring isomorphism (cf. above), its restriction to the centre of $\hom(C,C)$, i.e. the map $\lambda 1_C \mapsto \iota(\lambda)1_{FC}$, yields the field isomorphism $\iota\colon K \to K$ (resp. $\mathbb{R} \to \mathbb{R}$, $\mathbb{C} \to \mathbb{C}$), which shows we have identified the base field.

For $\mathbb{R}$ we have $\iota = 1_{\mathbb{R}}$. For $\mathbb{C}$ with $\mathbf{C} = \mathbf{NormVec_b}$ we have either $\iota = \alpha_C = 1_{\mathbb{C}}$ or $\iota = \alpha_C$ = complex conjugation, where $C \in \mathcal{O}b\,\mathbf{C}_1$ is not finite dimensional. (This is the only statement where we use the existence of such an object C.)

D) For $C \in \mathcal{O}b\,\mathbf{C}_1$, $\dim C \leq 1$ we have $FC \cong C$, by the remarks preceding Proposition 2. For $\dim C \geq 2$ we already know there is an i_C such that F on $\hom(C,C)$ is given by $U(Ff) = i_C(Uf)i_C^{-1}$, i_C an additive bijection $UC \to UFC$, $i_C(\lambda x) = \iota(\lambda)i_C(x)$, i together with i^{-1} continuous in case of $\mathbf{NormVec_b}$, $\mathbf{T_2LocConv_w}$. (For $\mathbf{NormVec_b}$, complex, $\dim C$ finite, continuity of i, i^{-1} holds by $\iota = \alpha_C = 1_{\mathbb{C}}$ or $\iota = \alpha_C$ = conjugation, for C not finite dimensional.) Thus F has been identified on the objects, and on hom-sets $\hom(C,C)$, $\dim C \geq 2$.

E) Supposing the one-dimensional vector space C_0 belongs to $\mathcal{O}b\,\mathbf{C}_1$, we will identify F on $\hom(C_0, C_0)$. By hypothesis $\exists C \in \mathcal{O}b\,\mathbf{C}_1$, $\dim C \geq 2$. Then $\exists \mu\colon C_0 \to C$, $\exists \pi\colon C \to C_0$, $\pi\mu = 1_{C_0}$ (Hahn–Banach). From above we have an i_C, and we define $i_{C_0} = U(F\pi)i_C(U\mu)$. This is additive, and $i_{C_0}(\lambda x) = \iota(\lambda)i_{C_0}(x)$ for $x \in UC_0$, λ scalar. We have by $\mu\pi \in \hom(C,C)$, $UF(\mu\pi) = i_C(U(\mu\pi))i_C^{-1}$ that $(U\pi)i_C^{-1}U(F\mu)\cdot U(F\pi)i_C(U\mu) = (U\pi)[i_C^{-1}U(F(\mu\pi))i_C](U\mu) = (U\pi)U(\mu\pi)(U\mu) = 1_{UC_0}$, and $U(F\pi)i_C(U\mu)\cdot(U\pi)i_C^{-1}U(F\mu) = U(F\pi)[i_C U(\mu\pi)i_C^{-1}]U(F\mu) = U(F\pi)(UF(\mu\pi))U(F\mu) = 1_{UFC_0}$, hence i_{C_0} is a bijection with inverse $i_{C_0}^{-1} = (U\pi)i_C^{-1}U(F\mu)$.

We will show that for $\varphi \in \hom(C_0, C_0)$ $UF\varphi = i_{C_0}U(\varphi)i_{C_0}^{-1}$. It suffices to show $U(F\mu)i_{C_0}U(\varphi)i_{C_0}^{-1} = U(F\mu)U(F\varphi)$, i.e. $U(F\mu)U(F\pi)i_C U(\mu)U(\varphi)U(\pi)i_C^{-1}U(F\mu) = U(F\mu)U(F\varphi)$. We have $U(F\mu)U(F\pi)[i_C U(\mu\varphi\pi)i_C^{-1}]U(F\mu) = U(F\mu)U(F\pi)U(F(\mu\varphi\pi))\cdot U(F\mu) = U(F\mu)U(F\varphi)$, as asserted. Thus F has been identified on each $\hom(C,C)$, $C \in \mathcal{O}b\,\mathbf{C}_1$.

We note that of course i_{C_0} may also depend on C, μ and π (and the choice of i_C), which is not shown by notation. However, if an additive bijection i_{C_0} satisfies $i_{C_0}(\lambda x) = \iota(\lambda) i_{C_0}(x)$ and $\forall \varphi \in \hom(C_0, C_0)$ $UF\varphi = i_{C_0} U(\varphi) i_{C_0}^{-1}$, then any non-zero scalar multiple of i_{C_0} satisfies the analogous relations (note that $(\nu i_{C_0})^{-1} = (\iota^{-1}(\nu))^{-1} i_{C_0}^{-1}$, for $\nu \neq 0$ scalar). Further any two $i'_{C_0}, i''_{C_0}\colon UC_0 \to UC_0$ with the above properties are non-zero scalar multiples of each other, thus i_{C_0} is unique up to a non-zero scalar multiple.

F) It remains to identify F on hom-sets $\hom(C', C'')$, where $C', C'' \in Ob\,\mathrm{C}_1$. Zero-morphisms $0_C \in \hom(C, C)$ are carried to zero-morphisms 0_{FC} (are preserved by F), hence zero-morphisms in any $\hom(C', C'')$ are carried to zero-morphisms as well, as right, resp. left multiples of $0_{C''}$, resp. $0_{C'}$. Thus for zero-morphisms the commutativity of the diagram expressing natural isomorphism is evident (for any choice of the components of the natural isomorphism). Henceforward we deal with non-zero morphisms f (for these Ff is a non-zero morphism as well). In particular, for $f\colon C' \to C''$, $C', C'' \in Ob\,\mathrm{C}_1$, we will thus suppose $\dim C', \dim C'' \geq 1$.

First we deal with the case $\min(\dim C', \dim C'') = 1$.

If $\dim C'_0 = \dim C''_0 = 1$, $C'_0, C''_0 \in Ob\,\mathrm{C}_1$, $f\colon C'_0 \to C''_0$, f not a zero-morphism, then $f = ig$, where $g\colon C'_0 \to C'_0$, and $i\colon C'_0 \to C''_0$ is a fixed isomorphism. Then both UFi and $i_{C''_0} U(i) i_{C'_0}^{-1}$ are non-zero morphisms $FC'_0 \to FC''_0$, where $\dim FC'_0 = \dim FC''_0 = 1$, thus they are non-zero scalar multiples of each other. Therefore $UFf = UF(i)UF(g) = \mathrm{const} \cdot i_{C''_0} U(i) i_{C'_0}^{-1} \cdot i_{C'_0} U(g) i_{C'_0}^{-1} = \mathrm{const} \cdot i_{C''_0} U(f) i_{C'_0}^{-1}$, the constant different from 0 and only depending on C'_0, C''_0 (and the choice of $i_{C'_0}, i_{C''_0}$ and i).

Now we turn to $\hom(C_0, C), \hom(C, C_0)$, where $\dim C_0 = 1 < \dim C$, $C_0, C \in Ob\,\mathrm{C}_1$. Let $f\colon C_0 \to C$, $g\colon C \to C_0$. We will show $UFf = i_C(Uf) i_{C_0}^{-1}$, resp. $UFg = i_{C_0} U(g) i_C^{-1}$, provided i_{C_0} has been defined just with this C (and some μ, π). For f we have, using $f\pi \in \hom(C, C)$, $i_C U(f) i_{C_0}^{-1} = [i_C(Uf)(U\pi) i_C^{-1}] U(F\mu) = UF(f\pi) U(F\mu) = UF(f)$, as asserted. Similarly for g we have, using $\mu g \in \hom(C, C)$, $i_{C_0} U(g) i_C^{-1} = U(F\pi)[i_C(U\mu)(Ug) i_C^{-1}] = U(F\pi) UF(\mu g) = UFg$, as asserted. Therefore, for i_{C_0} defined via an arbitrary C (and μ, π) — then we have by the end of E) that thus i_{C_0} changes by a non-zero constant factor only — we have $UFf = \mathrm{const} \cdot i_C U(f) i_{C_0}^{-1}$, $UFg = \mathrm{const} \cdot i_{C_0} U(g) i_C^{-1}$, the constants different from 0 and only depending on i_{C_0}, C (and the choice of μ, π for C).

G) There remained the case $C', C'' \in Ob\,\mathrm{C}_1$, $f \in \hom(C', C'')$, $\min(\dim C', \dim C'') \geq 2$. We are going to show that, like above, also in this case $U(Ff) = \mathrm{const} \cdot i_{C''}(Uf) i_{C'}^{-1}$, where the constant is different from 0 and only depends on C', C'' (but not on f).

For this we recall some parts of the proofs of [10], [43], [44] about the determination of C by the semigroup $\hom(C,C)$, for $\dim C \geq 2$. (The statements about C, $\hom(C,C)$ will equally apply to FC, $\hom(FC,FC)$.) Let us denote $\hom_1(C,C) = \{f \in \hom(C,C) \mid \dim f(C) \leq 1\} = \{f \in \hom(C,C) \mid f = \langle -, x^*\rangle x,\ x \in UC,\ x^* \in UC^*\}$, where C^* is the algebraic, resp. topological dual of C (for $C = \mathbf{Vec_K}$, resp. for $\mathbf{NormVec_b}$ or $\mathbf{T_2LocConv_w}$). This is the minimal non-zero (semigroup) ideal of $\hom(C,C)$ (thus is preserved by F). Then the minimal non-zero right ideals of $\hom_1(C,C)$ are of the form $I(C') = \{f \in \hom_1(C,C) \mid f(C) \subset C'\}$, where C' is some one-dimensional subspace of C. Evidently these also are preserved by F.

Let now $f\colon C' \to C''$, and C_1', C_1'' be one-dimensional subspaces of C', C'' (with corresponding subspaces $i_{C'}C_1' \subset FC'$, $i_{C''}C_1'' \subset FC''$). Let $0 \neq x' \in UC_1'$, $0 \neq x'' \in UC_1''$. Then $f(x') = 0 \Leftrightarrow [\forall g \in I(C_1')\quad fg = 0 \in \hom(C', C'')] \Leftrightarrow [\forall Fg \in FI(C_1') = \{h \in \hom(FC', FC') \mid i_{C'}^{-1}(Uh)i_{C'}(C') \subset C_1'\} = \{h \in \hom(FC', FC') \mid h(FC') \subset i_{C'}(C_1')\} = I(i_{C'}(C_1'))\quad (Ff)(Fg) = 0 \in \hom(FC', FC'')] \Leftrightarrow (Ff)(i_{C'}(x')) = 0$. Further $f(x') = \mathrm{const}\cdot x'' \neq 0 \Leftrightarrow [\exists g \in I(C_1'), fg \neq 0 \in \hom(C', C''), \exists h \in I(C_1''), fg = hfg] \Leftrightarrow [\exists Fg \in FI(C_1') = I(i_{C'}(C_1')\quad (Ff)(Fg) \neq 0 \in \hom(FC', FC''), \exists Fh \in FI(C_1'') = I(i_{C''}(C_1''))\quad (Ff)(Fg) = (Fh)(Ff)(Fg)] \Leftrightarrow (Ff)(i_{C'}(x')) = \mathrm{const}\cdot i_{C''}(x'') \neq 0$. Hence, choosing $x'' = f(x')$, we see $\forall x' \in UC'\quad (Ff)(i_{C'}(x')) = \mathrm{const}\cdot i_{C''}(f(x'))$, with a constant depending on x', and always different from 0.

Suppose now that there are $x_1', x_2' \in UC'$, with $f(x_1'), f(x_2') \neq 0$, such that the respective constants, say λ_1, λ_2 $(\neq 0)$ are different. Then x_1', x_2' are linearly independent (this follows from $i_{C'}(\lambda x') = \iota(\lambda)i_{C'}(x')$, $i_{C''}(\lambda x'') = \iota(\lambda)i_{C''}(x'')$). Suppose first $f(x_2') = \lambda f(x_1')$. Then $f(x_2' - \lambda x_1') = 0$, hence $0 = (Ff)(i_{C'}(x_2' - \lambda x_1')) = (Ff)(i_{C'}(x_2')) - \iota(\lambda)(Ff)(i_{C'}(x_1')) = \lambda_2 i_{C''}f(x_2') - \iota(\lambda)\lambda_1 i_{C''}(f(x_1')) = i_{C''}[\iota^{-1}(\lambda_2)f(x_2') - \lambda\iota^{-1}(\lambda_1)f(x_1')]$, therefore $\iota^{-1}(\lambda_2)f(x_2') = \lambda\iota^{-1}(\lambda_1)f(x_1')$. Since also $f(x_2') = \lambda f(x_1') \neq 0$, $\iota^{-1}(\lambda_2) = \iota^{-1}(\lambda_1)$, thus $\lambda_2 = \lambda_1$, a contradiction.

Second suppose $f(x_1'), f(x_2')$ are linearly independent. Then for some λ we have $\lambda_1 i_{C''}(f(x_1')) + \lambda_2 i_{C''}(f(x_2')) = (Ff)(i_{C'}(x_1' + x_2')) = \lambda i_{C''}(f(x_1' + x_2')) = \lambda i_{C''}f(x_1') + \lambda i_{C''}f(x_2')$. By our hypothesis also $i_{C''}f(x_1')$, $i_{C''}f(x_2')$ are linearly independent (since $i_{C''}$ is a bijection, we have $f(x_2') = \nu f(x_1') \Leftrightarrow i_{C''}f(x_2') = \iota(\nu)i_{C''}f(x_1')$), thus $\lambda_1 = \lambda = \lambda_2$, a contradiction. Therefore $(Ff)(i_{C'}(x')) = \lambda i_{C''}(f(x'))$, where λ only depends on f, and $\lambda \neq 0$.

H) Now we show that, in case $\min(\dim C', \dim C'') \geq 2$, for all $f\colon C' \to C''$ we have $(U(Ff))i_{C'} = \lambda i_{C''}(Uf)$, with $\lambda \neq 0$ only depending on C', C'' (multiplication by a scalar meant pointwise, in the respective vector space). Suppose therefore that, for $j = 1,2$, $0 \neq f_j \in \hom(C', C'')$ we have $(U(Ff_j))i_{C'} = \lambda_j i_{C''}(Uf_j)$, where $\lambda_j \neq 0$. Let $0 \neq x' \in UC'$,

$0 \neq x'^* \in U(C'^*)$, $0 \neq x'' \in UC''$, $0 \neq x''^* \in U(C''^*)$. Letting $g' = \langle -, x'^* \rangle x'$, $g'' = \langle -, x''^* \rangle x''$, we have $g'' f_j g' = \langle -, x'^* \rangle \langle f_j x', x''^* \rangle x''$. Letting $h = \langle -, x'^* \rangle x''$, this implies $(Fg'')(Ff_j)(Fg') = F(\langle f_j x', x''^* \rangle h) = \iota(\langle f_j x', x''^* \rangle) F(h)$ (since for λ scalar, $f\colon C' \to C''$ we have $F(\lambda f) = F(\lambda 1_{C''} f) = F(\lambda 1_{C''}) F(f) = \iota(\lambda) 1_{FC''} F(f) = \iota(\lambda) F(f)$). Here $U(Fg') = i_{C'}(Ug') i_{C'}^{-1}$, $U(Fg'') = i_{C''}(Ug'') i_{C''}^{-1}$, $U(Fh) = \lambda i_{C''} U(h) i_{C'}^{-1}$ for some $\lambda \neq 0$, hence we have $\lambda_j i_{C''}(Ug'')(Uf_j)(Ug') i_{C'}^{-1} = U(Fg'') U(Ff_j) U(Fg') = \iota(\langle f_j x', x''^* \rangle) \lambda i_{C''} U(h) i_{C'}^{-1}$. That means (since $g'' f_j g' = \langle f_j x', x''^* \rangle h$), $\iota(\langle f_j x', x''^* \rangle) \lambda_j i_{C''} U(h) i_{C'}^{-1} = \iota(\langle f_j x', x''^* \rangle) \lambda i_{C''} U(h) i_{C'}^{-1}$. This implies by $h \neq 0$ that $\iota(\langle f_j x', x''^* \rangle) \lambda_j = \iota(\langle f_j x', x''^* \rangle) \lambda$. Now fix x'^*, x'', thus λ is also fixed. We have $f_1, f_2 \neq 0$, hence $\exists x'$, $x' \notin f_1^{-1}(0) \cup f_2^{-1}(0)$, thus $f_j x' \neq 0$, $j = 1, 2$. Then $\exists x''^*$, $\langle f_j x', x''^* \rangle \neq 0$, $j = 1, 2$. Then both above equations can be shortened, yielding $\lambda_1 = \lambda = \lambda_2$.

This shows that for all $f\colon C' \to C''$ we have $(U(Ff)) i_{C'} = \lambda(C', C'') i_{C''}(Uf)$, with $0 \neq \lambda(C', C'')$ only depending on C', C'' ($f = 0$ included), for $\min(\dim C', \dim C'') \geq 2$. By F) the same holds for $\min(\dim C', \dim C'') \leq 1$, thus we have this relation always (for $\min(\dim C', \dim C'') = 0$ trivially).

I) Now we show that F is naturally isomorphic to $F_0(\iota) \mid \mathbf{C}_1$, for $\mathbf{Vec_K}$, $\mathbf{NormVec_b}$, $\mathbf{T_2LocConv_w}$. Let $C', C'', C''' \in \mathcal{O}b\, \mathbf{C}_1$, each of dimension ≥ 1. Then we have $\lambda(C', C'') \lambda(C'', C''') = \lambda(C', C''')$, since choosing $f\colon C' \to C''$, $g\colon C'' \to C'''$, $gf \neq 0$, we have $\lambda(C', C''') i_{C'''} U(gf) i_{C'}^{-1} = UF(gf) = UF(g) UF(f) = \lambda(C'', C''') i_{C'''}(Ug) i_{C''}^{-1} \lambda(C', C'') i_{C''}(Uf) i_{C'}^{-1}$. Therefore for $C', C'' \in \mathcal{O}b\, \mathbf{C}_1$, $\dim C', \dim C'' \geq 1$ we have $\lambda(C', C'') = \lambda(C'')/\lambda(C')$, for some non-zero scalars $\lambda(C)$, $C \in \mathcal{O}b\, \mathbf{C}_1$. Thus we have for $f\colon C' \to C''$, $C', C'' \in \mathcal{O}b\, \mathbf{C}_1$, $\dim C', \dim C'' \geq 1$ $U(Ff) i_{C'} = [\lambda(C'')/\lambda(C')] i_{C''}(Uf)$, or $U(Ff) \lambda(C') i_{C'} = \lambda(C'') i_{C''}(Uf)$, i.e., the functions $\lambda(C) i_C$, for $C \in \mathcal{O}b\, \mathbf{C}_1$, $\dim C \geq 1$, resp. the unique functions for $C \in \mathcal{O}b\, \mathbf{C}_1$, $\dim C = 0$, can be lifted to the components of a natural transformation $(F_0(\iota) \mid \mathbf{C}_1) \to F$. (These will be morphisms in the respective category by the very definition of $F_0(\iota)$, since $\lambda(C) i_C(\lambda x) = \iota(\lambda) \lambda(C) i_C(x)$.) By C), D) each $\lambda(C) i_C$ is an isomorphism in the respective category $\mathbf{Vec_K}$, $\mathbf{NormVec_b}$ or $\mathbf{T_2LocConv_w}$.

J) In B) we have reduced the case of $\mathbf{NormVec}$ to that of $\mathbf{NormVec_b}$. However, for $\mathbf{NormVec_b}$, complex, we have not supposed $\exists C \in \mathcal{O}b\, \mathbf{C}_1$, C is not finite dimensional, which hypothesis was however only used for establishing $\iota = 1_{\mathbf{C}}$ or $\iota =$ complex conjugation. Now, for $\mathbf{NormVec}$ we can determine ι, like in Proposition 1. Namely, for any $C \in \mathcal{O}b\, \mathbf{C}_1$, $\dim C \geq 1$, F_b preserves and reflects the contractive elements of $\hom(C, C)$ ($F_b\colon (\mathbf{C}_1)_\mathbf{b} \to \mathbf{NormVec_b}$ is the full embedding constructed in B), which is the extension of $F\colon \mathbf{C}_1 \to \mathbf{NormVec}$). Therefore F_b preserves and reflects the contractive elements of the centre of $\hom(C, C)$, thus

ι maps $\{z \in \mathbf{C} \mid |z| = 1\}$ onto itself, implying like above $\iota = 1_{\mathbf{C}}$ or $\iota =$ conjugation.

By what has been proved above we have that F_b is naturally isomorphic to $F_0(\iota) \mid (\mathbf{C}_1)_{\mathbf{b}}$. However the components of the natural isomorphism are isomorphisms in $\mathbf{NormVec_b}$, not in $\mathbf{NormVec}$. Let us take some $C \in \mathcal{O}b\,\mathbf{C}_1$, $\dim C \geq 1$, and some $x \in UC$, $||x|| = 1$. Then $\forall C' \in \mathcal{O}b\,\mathbf{C}_1$ $\{x' \in UC' \mid \exists f\colon C \to C'$ (in NormVec), $f(x) = x'\} = U_1C' = \{x' \in UC' \mid ||x'|| \leq 1\}$. Namely on the one hand each considered f is a contraction; on the other hand for any $x' \in U_1C'$ we can choose an f of the form $\langle -, x^*\rangle x'$, where $||x^*|| = 1 = \langle x, x^*\rangle$. Therefore, by naturality, we have $\lambda(C')i_{C'}U_1C' = \{\lambda(C')i_{C'}f(x) \mid f\colon C \to C'$ (in NormVec)$\} = \{(Ff)(\lambda(C)i_Cx) \mid f\colon C \to C'$ (in NormVec)$\} = \{g(\lambda(C)i_Cx) \mid g\colon FC \to FC'$ (in NormVec)$\} = ||\lambda(C)i_C(x)||\{g(\lambda(C)i_Cx/||\lambda(C)i_Cx||) \mid g\colon FC \to FC'$ (in NormVec)$\} = ||\lambda(C)i_Cx||U_1FC'$. Hence $\lambda(C')i_{C'}$ lifts to an isomorphism $F_0(\iota)C' \to FC'$ (in $\mathbf{NormVec_b}$) which is $||\lambda(C)i_Cx||$ times an isometry. Therefore $\lambda(C')i_{C'}/||\lambda(C)i_Cx||$ is an isometry, which lifts to a natural isomorphism $(F_0(\iota) \mid \mathbf{C}_1) \to F$ in NormVec. □

§3

3.1. Till now we have considered automorphisms of categories, mainly from algebra. Now we turn to results about full embeddings of categories, in topology. Clearly, the description of all full embeddings $F\colon \mathbf{C} \to \mathbf{D}$ includes the description of all automorphisms $F_1\colon \mathbf{C} \to \mathbf{C}$ (provided there exists such a functor F), since if F is fixed, F_1 is arbitrary, then all composite functors $FF_1\colon \mathbf{C} \to \mathbf{D}$ yield full embeddings. Also one can compose with automorphisms $F_2\colon \mathbf{D} \to \mathbf{D}$, but these may act like identity on the image of F.

The reason that for categories in topology one can also describe full embeddings, is that we have all constant mappings, and thus full embeddings are concrete. In contrast, for categories in algebra, one has in general full embeddings, which can be constructed with a rather great degree of arbitrariness, cf. the book [47].

THEOREM 2. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a full embedding. Then for* a) $\mathbf{C} = \mathbf{Top}$, $\mathbf{D} = \mathbf{SetSyst^0}$, *b)* $\mathbf{C} = \mathbf{Top_1}$, $\mathbf{D} = \mathbf{SetSyst^0}$, *c)* $\mathbf{C} = \mathbf{Haus}$, $\mathbf{D} = \mathbf{SetSyst^0}$, *d)* $\mathbf{C} = \mathbf{Reg_1}$, $\mathbf{D} = \mathbf{SetSyst^0}$, *e)* $\mathbf{C} = \mathbf{Met_c}$, $\mathbf{D} = \mathbf{SetSyst^0}$, *f)* $\mathbf{C} = \mathbf{Pos}$, $\mathbf{D} = \mathbf{Top}$, *g)* $\mathbf{C} = \mathbf{Prox}$, $\mathbf{D} = \mathbf{Unif}$, *h)* $\mathbf{C} = \mathbf{D} = \mathbf{Unif}$ *F is naturally isomorphic to the concrete functor, given on the objects $C \in \mathcal{O}b\mathbf{C}$ by* a) *either $FC = (UC, \{$ open sets of $C\})$ or $FC = (UC, \{$ closed sets of $C\})$ [9] p.32, [16] p.563, [52], [5], [51] Proposition 6, b) either $FC = (UC, \{$ open sets of $C\})$ or $FC = (UC, \{$ closed sets of $C\})$ [51] Proposition 5, c) either $FC = (UC,$ some open subbase of $C)$ or $FC = (UC,$ some closed subbase of $C)$ [51] Corollary 9, d) (provided there is no mea-*

surable cardinal; which is consistent with ZFC, provided ZFC is consistent) either $FC =$ (UC, some open subbase of C) or $FC =$ (UC, some closed subbase of C) [51] Proposition 10, Lemma 6, e) either $FC = (UC, \{$open sets of $C\})$ or $FC = (UC, \{$closed sets of $C\})$ [51] Proposition 13, f) either $FC =$ topology on UC, defined by $A \subset UC \Longrightarrow \overline{A} = \{x \in UC | \exists a \in A,\ x \leq a\}$ or $FC =$ topology on UC, defined by $A \subset UC \Longrightarrow \overline{A} = \{x \in UC | \exists a \in A,\ x \geq a\}$ [45], [53], g) $FC =$ the unique compatible precompact uniformity (proved by M. Hušek - J. Pelant, their proof appearing in [42] §7), h) $F = 1_{\mathbf{Unif}}$ (proved by J. Pelant - J. Reiterman, their proof appearing in [42] §7). i) For **C** = **Tych**, **D** = **SetSyst**^{0} *there are as many not naturally isomorphic concrete full embeddings* **C** $\to$ **D**, *as there are subclasses of a proper class [41] Corollary 4.*

We note that both [16] and [52] actually characterized the category **Top** among all (abstractly given) categories, the characterization of [52] being based on a similar characterization of **Set** by [24]. A simpler characterization of **Set** has been given by [55], and a characterization of **Pos** by [58]. For i) and some similar situations the existence of a proper class of not naturally isomorphic concrete full embeddings was shown earlier by [16] p. 561, [51] pp. 545, 547; a statement similar to i) cf. also in [14].

There are some results on non-existence of full embeddings **C** $\to$ **D**, where **C** is a category of structures, whose usual definition uses more complicated structures than the usual definition of **D**. E.g. there is no full embedding of the category of proximity spaces (in the sense of Efremovič) to **SetSyst**^{0} [42] §5, 6. Similar results cf. in [5], [48] p.537, [4].

We also note that for i) and similar situations the above mentioned proofs used full embeddings, which were differing from each other on "large" spaces. Therefore the result of [50] is of interest: even the full subcategory of **Tych**, consisting of finite simplicial complexes (actually the one consisting of all locally connected $T_{3\frac{1}{2}}$ spaces) has infinitely many not naturally isomorphic concrete full embeddings into **SetSyst**^{0}, which are different on each non-discrete space C. One of these full embeddings has object part $(UC, \{A \subset UC | A$ is the intersection of an open and a closed set in C, and the components of A are closed in $C\})$.

Some further results of the type of the results in Theorem 2 cf. [9] p.32, [16] (especially pp. 560-561), [59A], [51], [59B], [4], [39], [14], [41], [42]. ([59B] was unavailable to the author, but [59A] by the same author means by convergence structures the neighbourhood spaces = Čech closure spaces.)

3.2. The method of proof of most of the statements in Theorem 2 goes back in a special case to P. Freyd [9] p. 32, and in the general case to M. Hušek [16], and consists of three steps.

1) First one shows $F: \mathbf{C} \to \mathbf{D}$ is naturally isomorphic to a concrete functor. (This is an easy step, cf. later.) From now on for notational convenience we will suppose F itself is concrete.

2) Choose some class $\mathbf{C}_1 \subset Ob\,\mathbf{C}$, which either projectively, or inductively generates $\mathbf{C}$. Then prove for each $C_1 \in Ob\mathbf{C}_1$ that the fact that U hom (FC_1, FC_1) equals U hom(C_1, C_1) $(\subset (UC_1)^{UC_1})$, implies that the structure (D-object) FC_1 on UC_1 is uniquely determined, or there are at most two possibilities, for the restriction of F to $\mathbf{C}_1$. (In applications, inductively generating classes usually consist of nearly discrete structures, while projectively generating classes are more diverse, but often the space $[0, 1]$ plays a role there. We mention that a D-object D in a concrete category $\mathbf{D}$ is called special, it for any other D-object D' with $UD' = UD$ $\;U$ hom$(D', D') = U$hom(D, D) implies $D' = D$. Results about speciality of certain classes of objects in some categories were proved e.g. in [56], [29], [30], [57], [31], [32], [61], [49], [40], [37], [38], [42]; cf. also the surveys [33], [11], [34], [35], [36], and also more recent papers of K. D. Magill Jr. and his collaborators. Cf. also the works of the Baku school of topology, e.g. E. A. Babaev, M. I. Burtman, R. B. Feĭzullaev, F. A. Ismailov, V. Š. Jusufov (Yusufov), A. A. Mahmudov (Makhmudov), F. H. (Kh.) Muradov, L. G. Mustafaev, mainly in Izv. Akad. Nauk Azerbaĭdžan SSR Ser. Fiz. - Tehn. Mat. Nauk, from about 1965, who also deal with structures other than semigroups of self-maps, in categories of topological spaces, linear spaces (here without preassigning the underlying set), etc. For (more or less unique) determination of linear spaces, or normed linear spaces by their endomorphism rings, endomorphism semigroups, linear homeomorphism groups, etc., cf. e.g. [8], [28], [10], [43], and also many more special papers, e.g. in more recent volumes of Linear Algebra Appl., that operations on certain sets of matrices, preserving some properties, are of certain special form. A more special problem is that of determining e.g. a topological space by its autohomeomorphism group, cf. e.g. [25], [26].)

3) Now we turn to an arbitrary object $C \in Ob\mathbf{C}$. If, say, $\mathbf{C}_1$ projectively generates $\mathbf{C}$, then C has a C-initial structure w.r.t. some morphisms $\{\varphi_\alpha\}$ to objects of $\mathbf{C}_1$ (more exactly, w.r.t. $\{U\varphi_\alpha\}$). Then FC is finer than the D-initial structure w.r.t. the morphisms $\{F\varphi_\alpha\}$.

A) In many cases there is some other class $\mathbf{C}_2 \subset Ob\mathbf{C}$ inductively generating $\mathbf{C}$, and then one similarly obtains that FC is coarser than the D-final structure w.r.t. morphisms $\{F\psi_\beta\}$, ψ_β's some C-morphisms from objects of $\mathbf{C}_2$ to C. If the above D-initial and D-final structures on the set UC coincide, then FC is identified, as being equal to both of them.

B) In some other cases, choosing some concrete full embedding $F_0: \mathbf{C} \to \mathbf{D}$, one can consider $\mathbf{C}$ as a subcategory of $\mathbf{D}$. It may happen that, beside that $\mathbf{C}_1 \subset Ob\mathbf{C}$ projectively

(C-) generates **C**, also $\mathbf{C}_1$, as a subclass of $\mathcal{O}b\mathbf{D}$, projectively (D-) generates **D**. (In particular, this holds for $\mathbf{C} = \mathbf{D}$.) Then any C-object C has the C-initial structure w.r.t. all C-morphisms from C to $\mathbf{C}_1$-objects, and also $FC \in \mathcal{O}b\mathbf{D}$ has the D-initial structure w.r.t. all D-morphisms from FC to $\mathbf{C}_1$-objects. Supposing the restriction of F to $\mathbf{C}_1$ (considered as a full subcategory of **C**) is the concrete embedding $\mathbf{C}_1 \hookrightarrow \mathbf{D}$, we have that FC is projectively (D-) generated by all morphisms to objects FC_1, $C_1 \in \mathbf{C}_1$, i.e. by the class of morphisms $\cup\{\hom(FC, FC_1) | C_1 \in \mathbf{C}_1\}$. However, since F is a concrete full embedding, $U \hom(FC, FC_1) = U \hom(C, C_1)$. Thus C is projectively C-generated by the source $\cup\{U \hom(C, C_1) | C_1 \in \mathbf{C}_1\}$ in **Set**, and FC is projectively D-generated by the same source in **Set**. This may suffice to show the equality $FC = C$ (actually $FC = F_0C$). (This second approach is mainly applicable for the case $\mathbf{C} = \mathbf{D}$, but below it will be used for $\mathbf{C} = \mathbf{Top}$, $\mathbf{D} = \mathbf{SetSyst}^0$. Of course, there is a "dual" of this approach, using inductive generation.)

3.3. As the probably simplest case, we will reproduce the proof from the papers listed at Theorem 2, a), and [53], for the case of full embeddings F: $\mathbf{Top} \to \mathbf{SetSyst}^0$, showing that F is naturally isomorphic either to the concrete functor, given on objects C as $(UC, \{$ open sets of $C\})$, or to the concrete functor, given on objects as $(UC, \{$ closed sets of $C\})$. We follow the steps as described in **3.2.**

1) Observe that in $\mathbf{SetSyst}^0$, like in **Top**, all constant functions underlie morphisms. Denote by 1, resp. $1'$, the up to isomorphism unique object on a one-point set in **Top**, resp. in $\mathbf{SetSyst}^0$. 1 is identified categorically, up to isomorphism, by the fact that it is a terminal object a **Top**. In particular, $|\hom(1,1)| = 1$, and, since F is a full embedding, $|\hom(F1, F1)| = 1$ as well. Note that for any $D \in \mathcal{O}b\ \mathbf{SetSyst}^0$ with $|UD| \geq 2$ we have $|\hom(D, D)| \geq 3$, since $U \hom(D, D)$ contains beside 1_{UD} all constant functions $UD \to UD$. Therefore $|U(F1)| \leq 1$. Therefore either, up to isomorphism, we have $F1 = 1'$, or $F1 = 0' =$ object of $\mathbf{SetSyst}^0$ on the empty set. However in the second case for any $C \in \mathcal{O}b$ **Top**, $C \not\cong 1$ we have $FC \not\cong 0'$, and $1 = |\hom(C, 1)| = |\hom(FC, 0')| = 0$, a contradiction. Hence the first case holds. We have for the forgetful functors U: **Top** $\to$ **Set**, U': $\mathbf{SetSyst}^0 \to$ **Set** $U \cong \hom(1, -)$, $U' \cong \hom(1', -) \cong \hom(F1, -)$. Thus $U'F \cong \hom(F1, F(-))$. However, since F is a full embedding, $\hom(F1, F(-)) \cong \hom(1, -)$ (the restriction of F to $\hom(1, C)$ giving the component of the natural isomorphism $\hom(1, -) \to \hom(F1, F(-))$ at C), showing $U'F \cong U$.

From now on we will suppose for notational convenience that actually $U'F = U$, i.e. F is concrete.

2) Let us consider the Sierpiński space C_1 on the set $\{0, 1\}$, with $\{1\}$ as the only non-

trivial open set. Then $\{C_1\}$ projectively generates **Top**, since $\forall C \in \mathcal{O}b\,\mathbf{Top}$ we have $\{\varphi^{-1}(1)|\varphi \in \hom(C, C_1)\} = \{$ open sets of $C\}$, thus in particular is an open subbase of C. We are going to determine FC_1. We have $|\hom(C_1, C_1)| = 3$, thus $|\hom(FC_1, FC_1)| = 3$, and $U'(FC_1) = \{0,1\}$. There are four $\mathbf{SetSyst}^0$ structures $D = (\{0,1\}, \tau)$ on $\{0,1\}$. The ones with $\tau = \{\emptyset, \{0,1\}\}$, $\tau = \mathcal{P}(\{0,1\})$ satisfy $|\hom(D,D)| = 4$, thus FC_1 must be one of $(\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$ and $(\{0,1\}, \{\emptyset, \{0\}, \{0,1\}\})$. Note that $D_1 = (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$ $\in \mathcal{O}b\ \mathbf{SetSyst}^0$ projectively generates $\mathbf{SetSyst}^0$, since $\forall D \in \mathcal{O}b\ \mathbf{SetSyst}^0$ we have $D = (U'D, \{\psi^{-1}(1)|\psi \in \hom(D, D_1)\})$ (actually this is equivalent to projective generation).

3) In the first of the above cases, i.e. when $FC_1 = D_1$ we have $\forall C \in \mathcal{O}b$ **Top** $FC = (UC, \{\psi^{-1}(1)|\psi \in \hom(FC, FC_1)\}) = (UC, \{\varphi^{-1}(1)|\varphi \in \hom(C, C_1)\}) = (UC, \{$ open sets of $C\})$. Now suppose we have the second case, i.e. $FC_1 = (\{0,1\}, \{\emptyset, \{0\}, \{0,1\}\})$. Then let us define the concrete automorphism G: $\mathbf{SetSyst}^0 \to \mathbf{SetSyst}^0$ by setting for $(X, \tau) \in \mathcal{O}b$ $\mathbf{SetSyst}^0$ $G(X, \tau) = (X, \{A \subset X | X \setminus A \in \tau\})$. Then GF:**Top** $\to \mathbf{SetSyst}^0$ is a concrete full embedding, satisfying $GFC_1 = D_1$. Hence, by what has been shown above, $\forall C \in \mathcal{O}b$ **Top** $GFC = (UC, \{$ open sets of $C\})$, thus $FC = (UC, \{$ closed sets of $C\})$, which finishes the proof of the characterisation of full embeddings F:**Top**$\to\mathbf{SetSyst}^0$.

3.4. As we have seen above, there are several results about full embeddings of some interesting categories in topology into $\mathbf{SetSyst}^0$ (cf. also the papers cited in **3.1** after Theorem 2), so this case can be considered as rather well understood. However there are a lot of structures, considered in topology, which are defined in a more involved way, than by set systems. E.g. proximity spaces (we mean these in the sense of Efremovič, cf. [18]) are defined by systems of pairs of subsets, uniformities by systems of subsets of the square of the underlying set (if defined by entourages), or by systems of systems of subsets (if defined by uniform covers, like in [18]).

E.g. we may ask the following. Let F be a concrete full embedding of the category **Prox** of proximity spaces into the category $\mathbf{SetPairSyst}_0$ with objects $(X, \mathcal{A})$, X a set, $\mathcal{A}$ some set of unordered pairs of subsets of X, where for technical reasons we suppose $(A', A''| \in \mathcal{A} \Rightarrow A', A'' \neq \emptyset$, and with morphisms $f: (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$ characterized by $f \in X_2^{X_1}$ and $(A', A''| \in \mathcal{A}_1 \Rightarrow (f(A'), f(A'')) \in \mathcal{A}_2$. (For $\mathbf{SetPairSyst}_0$ the forgetful functor is defined by $U(X, \mathcal{A}) = X$, $Uf = f$, and similarly for analogous categories.) We will show in **3.5** that in this case for each proximity space C $FC = (UC, \mathcal{A})$, with $\mathcal{A}$ in a natural sense a base (for near pairs) of C. (A reason for supposing $A', A'' \neq \emptyset$ is the following: if it were not supposed, then a concrete full embedding could be obtained by setting for $C \in \mathcal{O}b\,\mathbf{Prox}$ $FC = (UC, \{(A', A'')|A', A'' \subset UC,\ A'\delta A''\} \cup \{(A', \emptyset)|A',$ as a subspace of C, belongs to any class of proximity spaces, closed under proximally continuous images }), hence in this case $\mathcal{A} \not\subset$ { near pairs of subsets of C }. Also, all information about δ in its usual definition is carried by the pairs (A', A'') with $A', A'' \neq \emptyset$. Since such pairs, as well as pairs of the form $(A', \emptyset)$ are preserved by any function, the question would fall into two parts, and it is not clear if $(UC, \{(A', A'') \in \mathcal{A} | A', A'' \neq \emptyset\})$ gives a concrete full embedding **Prox** $\to$ **SetPairSyst$_0$**, where $FC = (UC, \mathcal{A})$; although we conjecture this holds.)

The case of uniform spaces seems to be more complicated. Let **D** denote the category with objects $(X, \mathcal{B})$, X a set, $\mathcal{B} \subset \mathcal{P}(\mathcal{P}(X))$, where for similar technical reasons we assume $\{B_\gamma | \gamma \in \Gamma\} \in \mathcal{B} \Rightarrow \forall \gamma \in \Gamma \quad B_\gamma \neq \emptyset$, and with morphisms $f : (X_1, \mathcal{B}_1) \to (X_2, \mathcal{B}_2)$ characterized by $f \in X_2^{X_1}$ and $\{B_\gamma | \gamma \in \Gamma\} \in \mathcal{B}_1 \Rightarrow \{f(B_\gamma) | \gamma \in \Gamma\} \in \mathcal{B}_2$. Then there are two very different concrete full embeddings F:**Unif**$\to$ **D**. One is given by all near systems, i.e. by all systems $\{B_\gamma | \gamma \in \Gamma\} \subset \mathcal{P}(X)$, such that for each entourage V we have $\cap\{VB_\gamma | \gamma \in \Gamma\} \neq \emptyset$ (where $VB_\gamma = \{x_2 \in X | \exists x_1 \in B_\gamma, (x_1, x_2) \in V\}$), or, equivalently, for each uniform cover $\mathcal{V}$ we have $\cap\{st(B_\gamma, \mathcal{V}) | \gamma \in \Gamma\} \neq \emptyset$. The other concrete full embedding is given by all micromeric systems of non-empty sets (cf. e.g. [20]), i.e. by all systems $\{B_\gamma | \gamma \in \Gamma\} \subset \mathcal{P}(X)$, such that $\forall \gamma \in \Gamma \quad B_\gamma \neq \emptyset$, and for each entourage V $\exists \gamma \in \Gamma$ $B_\gamma \times B_\gamma \subset V$, or, equivalently, for each uniform cover $\mathcal{V}$ $\exists \gamma \in \Gamma$, $\exists W \in \mathcal{V}$, $B_\gamma \subset W$. Is it true that each concrete full embedding F:**Unif**$\to$ **D** satisfies for each $C \in Ob$ **Unif** $FC = (UC, \mathcal{B})$, with $\mathcal{B}$ in some sense a (sub)base for all near systems, or satisfies for each $C \in Ob$ **Unif** $FC = (UC, \mathcal{B})$, with $\mathcal{B}$ in some sense a (sub)base for all micromeric systems of non-empty sets, or are there many more concrete full embeddings?

Of course, since by now a lot of categories have been investigated in topology (cf. e.g. [13]), there arise many more questions on automorphisms and full embeddings of these.

3.5. Besides **Prox** we also deal with a full subcategory of **Prox**. Let **Prox$_{mc}$** denote the category of those proximity spaces whose completion is a compact pseudometric space. (For $C \in Ob$ **Prox** we mean by the completion γC of C the compact proximity space that contains C as a dense proximity subspace, and for which two of its points have the same closure only if both belong to UC. We mean by a pseudometric space a space obeying the same axioms as a metric space, but that different points may have distance 0.)

LEMMA 1. *Let $C \in Ob$ **Prox**. Then hom $(C, C) = (UC)^{UC}$ iff C is either indiscrete or discrete.*

PROOF (cf. [40], Lemma 1). If C is not indiscrete, then for some $x_1, x_2 \in UC \quad \{x_1\}\bar{\delta}\{x_2\}$. Then for any $A \subset UC$ define $\varphi: UC \to UC$, by $\varphi(A) \subset \{x_1\}$, $\varphi((UC) \setminus A) \subset \{x_2\}$. Since φ

underlines a proximally continuous map, $A\bar{\delta}((UC)\setminus A)$. □

LEMMA 2. *Let* $(X_1, \mathcal{A}_1), (X_2, \mathcal{A}_2) \in Ob\,\mathbf{SetPairSyst}_0$. *Then each constant function* $\varphi: X_1 \to X_2$ *belongs to* $U\,hom\,((X_1, \mathcal{A}_1), (X_2, \mathcal{A}_2))$ *iff* $\mathcal{A}_1 \neq \emptyset \Rightarrow \{(\{x\}, \{x\}) | x \in X_2\} \subset \mathcal{A}_2$.

PROOF is obvious. □

COROLLARY 1. *Each concrete full embedding* F: **Prox** $\to$ $\mathbf{SetPairSyst}_0$ *satisfies* $\forall C \in Ob$ **Prox** $\{(\{x\}, \{x\}) | x \in UC\} \subset \mathcal{A}$, *where* $FC = (UC, \mathcal{A})$. *Conversely, if* F: **Prox** $\to$ $\mathbf{SetPairSyst}_0$ *is a full embedding, for which* $\forall C \in Ob$ **Prox** $\{(\{x\}, \{x\}) | x \in UFC\} \subset \mathcal{A}$, *where* $FC = (UFC, \mathcal{A})$, *then* F *is naturally isomorphic to a concrete functor. The same holds also for* $\mathbf{Prox}_{mc}$ *rather than* **Prox**.

PROOF: Let $C_1 \in Ob$ **Prox** be non-discrete, non-indiscrete. Then by Lemma 1 U hom$(C_1, C_1) \neq (UC_1)^{UC_1}$, thus U hom$(FC_1, FC_1) \neq (UC_1)^{UC_1}$ either, hence $\mathcal{A}_1 \neq \emptyset$, where $FC_1 = (UC_1, \mathcal{A}_1)$. Choosing $(X_1, \mathcal{A}_1) = (UC_1, \mathcal{A}_1)$, the "only if" part of Lemma 2 gives for each $C_2 \in Ob$ **Prox**, with $FC_2 = (UC_2, \mathcal{A}_2) = (X_2, \mathcal{A}_2)$, that $\{(\{x\}, \{x\}) | x \in UC_2\} \subset \mathcal{A}_2$.

Conversely, by the "if" part of Lemma 2 each constant function $\varphi: X_1 \to X_2$ belongs to $U\,$hom$((X_1, \mathcal{A}_1), (X_2, \mathcal{A}_2))$, if $\{(\{x\}, \{x\}) | x \in X_2\} \subset \mathcal{A}_2$. Then proceeding like in **3.3.** 1), and noting that each set X with $|X| \leq 1$ carries a unique structure $(X, \mathcal{A})$ satisfying $\{(\{x\}, \{x\}) | x \in X\} \subset \mathcal{A}$, further that between two such objects of $\mathbf{SetPairSyst}_0$ each constant function gives a morphism, we obtain the statement of Corollary 1. For $\mathbf{Prox}_{mc}$ the some proof works. □

We let $\mathbf{SetPairSyst}_0^0$ be the full subcategory of $\mathbf{SetPairSyst}_0$, consisting of the objects $(X, \mathcal{A})$, satisfying $\{(\{x\}, \{x\}) | x \in X\} \subset \mathcal{A}$. We will investigate full embeddings F: **Prox** $\to$ $\mathbf{SetPairSyst}_0^0$, resp. $\mathbf{Prox}_{mc} \to \mathbf{SetPairSyst}_0^0$. For F fixed we will denote FC as $(UFC, \mathcal{A}(C))$ (or, if F is concrete, which can be supposed, we will write $(UC, \mathcal{A}(C))$); the same notation will be used later with analogous categories.

Before turning to the propositions we recall projectively and inductively generating classes for $\mathbf{Prox}_{mc}$, resp. **Prox**. For both of them $\{[0,1]\}$ (with the usual proximity) is projectively generating, since $C \in Ob\,\mathbf{Prox}_{mc}$ $(Ob\,\mathbf{Prox})$, $A', A'' \subset UC$, $A'\bar{\delta}A'' \Rightarrow \exists f: C \to [0,1]$, $f(A') \subset \{0\}$, $f(A'') \subset \{1\}$.

For $\mathbf{Prox}_{mc}$ the "Cauchy sequence" space, i.e. $N^* = \{n^{-1} | n \in \mathbb{N}\}$ with its usual proximity forms a one-element inductively generating class. Namely for $C \in Ob\,\mathbf{Prox}_{mc}$, $A', A'' \subset UC$, $A'\delta A''$ the closures of A', A'' in the completion γC of C have a common point, and A', A'' contain sequences $(x_{2k-1} | k \in \mathbb{N})$, resp. $(x_{2k} | k \in \mathbb{N})$ in UC converging in γC to this point.

Then define a map $f: \{n^{-1} | n \in \mathbb{N}\} \to C$ by $f(n^{-1}) = x_n$.

For **Prox** an inductively generating class can be given as follows [46]. Let us call an ultraproximity a proximity space obtained as follows. Let X be a set, $|X| \geq 2$, and let $\mathcal{U}_1, \mathcal{U}_2$ be two different ultrafilters on X (both can be, independently, fixed or free). Let $A', A'' \subset X$ be near iff either $A' \cap A'' \neq \emptyset$ or $A' \in \mathcal{U}_1, A'' \in \mathcal{U}_2$ or $A' \in \mathcal{U}_2$, $A'' \in \mathcal{U}_1$. (This corresponds to the "compactification", meant here as a dense map of X to a compact T_2 space, obtained from the Stone-Čech compactification of X with discrete topology, as a quotient by identifying the points corresponding to $\mathcal{U}_1, \mathcal{U}_2$, the map being the canonical one.) Then the class of all ultraproximities is an inductively generating class for **Prox**. Namely let $C \in \mathcal{O}b\mathbf{Prox}$, $B', B'' \subset UC$, $B'\delta B''$, where we may suppose $B' \cap B'' = \emptyset$. There is a point y in γC belonging to the closures of B', B'' in γC. The trace of the neighbourhood filter of y on UC is a filter $\mathcal{F}$ on UC, each of whose elements intersects both B' and B''. Thus $\mathcal{F} \cup \{B'\} \subset \mathcal{U}_1$, $\mathcal{F} \cup \{B''\} \subset \mathcal{U}_2$ for some ultrafilters $\mathcal{U}_1, \mathcal{U}_2$ on UC, which are different. From the ultraproximity on UC, constructed with $\mathcal{U}_1, \mathcal{U}_2$ the map identical on UC is proximally continuous to C. In fact, if A', A'' are two disjoint near sets in the ultraproximity, say $A' \in \mathcal{U}_1$, $A'' \in \mathcal{U}_2$, then their images in C, i.e. A', A'' are near, since their closures in γC both contain y. Further $B' \in \mathcal{U}_1$, $B'' \in \mathcal{U}_2$ are near in the ultraproximity, and their images are the subsets B', B'' of UC. This shows that C is inductively generated by all ultraproximities on UC.

PROPOSITION 1. *Each full embedding* $F: \mathbf{Prox}_{\mathrm{mc}} \to \mathbf{SetPairSyst}_0^0$ *is naturally isomorphic to a concrete functor* F_0, *satisfying for* $C \in \mathcal{O}b\,\mathbf{Prox}_{\mathrm{mc}}$, $F_0C = (UC, \mathcal{A})$ *that* $\{(A', A'') | A', A'' \subset UC, A'\delta A''$ *in* $C\} = \mathcal{A}_\delta(C) \supset \mathcal{A} \supset \mathcal{A}_\delta^*(C) = \{(A', A'') | A', A'' \subset UC, |A'| = |A''| = \aleph_0$, $A' \cup A''$ *yields a Cauchy sequence in (the compatible precompact uniformity of)* $C\} \cup \{(A', A'') | A', A'' \subset UC$, $1 \leq |A'| < \aleph_0$, $|A''| = \aleph_0$, A'' *yields a sequence converging to some* $a' \in A'$ *in* $C\} \cup \{(A', A'') | (A', A'' \subset UC$, $1 \leq |A'|, |A''| < \aleph_0$, $\exists a' \in A', \exists a'' \in A''$, $\overline{\{a'\}} = \overline{\{a''\}}$ *in* $C\}$. *(Thus* $\mathcal{A}$ *is in an evident sense a base, for near pairs, of* δ.*)*

Before the proof we note that the map F^* defined on objects by $F^*C = (UC, \mathcal{A}_\delta^*(C))$ is actually the object part of a concrete full embedding $\mathbf{Prox}_{\mathrm{mc}} \to \mathbf{SetPairSyst}_0^0$. In fact for $f: C_1 \to C_2$ in $\mathbf{Prox}_{\mathrm{mc}}$ we have that Uf, being uniformly continuous between the respective compatible precompact uniformities, lifts to a morphism $(UC_1, \mathcal{A}_\delta^*(C_1)) \to (UC_2, \mathcal{A}_\delta^*(C_2))$ in $\mathbf{SetPairSyst}_0^0$. Conversely, if $\varphi: UC_1 \to UC_2$ does not lift to a proximally continuous map, then it does not lift to a uniformly continuous map between the compatible precompact uniformities, thus there are $c_n' \in UC_1$, $c_n'' \in UC_1$, $d_1(c_n', c_n'') \to 0$, such that $d_2(\varphi(c_n'), \varphi(c_n'')) \geq \text{const} > 0$ (d_i denotes a compatible pseudometric on the completion of C_i). By precompactness we may assume that c_n' form a Cauchy sequence, in which case the sequence $c'_1, c''_1, \ldots, c'_n, c''_n, \ldots$ also is a Cauchy sequence. Also we may suppose c'_n are either all distinct or all coincide, and the same for c''_n. Similarly we may assume that $\varphi(c'_n)$ form a Cauchy sequence, with all terms distinct or all terms coinciding, and the same for $\varphi(c''_n)$. Then letting $A' = \{c'_n\}$, $A'' = \{c''_n\}$ we see φ does not lift to a morphism $(UC_1, \mathcal{A}_\delta^*(C_1)) \to (UC_2, \mathcal{A}_\delta^*(C_2))$.

Proof of Proposition 1. By Corollary 1 we may and will suppose F is concrete. The proof will consist of three steps. First we investigate the above introduced space $N^* = \{n^{-1} | n \in \mathbb{N}\}$ and $\mathcal{A}(N^*)$. Second we show by using projective generation that $\forall C \in \mathcal{O}b$ $\mathbf{Prox_{mc}}$ $\mathcal{A}(C) \subset \mathcal{A}_\delta(C) = \{(A', A'') | A', A'' \subset UC,\ A'\delta A'' \text{ in } C\}$. Lastly, once more investigating $\mathcal{A}(N^*)$ and using inductive generation, we show the remaining statement of the proposition. We note that the first and second steps work equally well in case of a full embedding F: $\mathbf{Prox} \to \mathbf{SetPairSyst}_0^0$.

1) We have U hom $(N^*, N^*) = \{\varphi \in (UN^*)^{UN^*} | \lim \varphi(n^{-1}) = 0\} \cup \{\varphi \in (UN^*)^{UN^*} | \exists n_0 \in \mathbb{N},\ n \geq n_0 \Longrightarrow \varphi(n^{-1}) = \varphi(n_0^{-1})\}$ (consider for $f: N^* \to N^*$ its extension $g: \gamma N^* \to \gamma N^* (= N^* \cup \{0\})$ and distinguish the cases that $g(0) = 0$ or $g(0) \in UN^*$).

Suppose $\mathcal{A}(N^*) \subset \{(A, A) | A \subset UN^*\} \cup \{(A', A'') | A', A'' \subset UN^*, \min(|A'|, |A''|) = 1\}$. If $(A', A'') \in \mathcal{A}(N^*)$ then, since $\forall f: N^* \to N^*$ Uf underlies a morphism $(UN^*, \mathcal{A}(N^*)) \to (UN^*, \mathcal{A}(N^*))$, we have $\forall f: N^* \to N^*$ $(f(A'), f(A'')) \in \mathcal{A}(N^*)$. If $(A, A) \in \mathcal{A}(N^*)$, then $B \subset UN^*$, $1 \leq |B| \leq |A| \Longrightarrow (B, B) \in \mathcal{A}(N^*)$. In fact, for $|A| < \aleph_0$ obviously $B = f(A)$, thus $(B, B) = f(A), f(A))$, for some $f: N^* \to N^*$; for $|A| = \aleph_0 > |B|$ $B = f(A)$ with an $f: N^* \to N^*$ satisfying $\exists n_0 [n \geq n_0 \Longrightarrow f(n^{-1}) = f(n_0^{-1})]$; for $|A| = |B| = \aleph_0$ $B = f(A)$ with an $f: N^* \to N^*$ satisfying $\lim f(n^{-1}) = 0$. Hence $\{A \subset UN^* | (A, A) \in \mathcal{A}(N^*)\} = \{A \subset UN^* | 1 \leq |A| < \alpha\}$, for some α, $1 < \alpha \leq \aleph_1$. If $(A', A'') \in \mathcal{A}(N^*)$, $\min(|A'|, |A''|) = 1$ — say, $|A'| = 1$ — then we can have two cases. Either for this (A', A'') we have $A' \subset A''$, or $A' \cap A'' = \emptyset$. For $A' \subset A''$ we see similarly like above that $B' \subset B'' \subset UN^*$, $|B'| = 1 \leq |B''| \leq |A''| \Longrightarrow (B', B'') \in \mathcal{A}(N^*)$. For $A' \cap A'' = \emptyset$ we see similarly that $B', B'' \subset UN^*$, $|B'| = 1 \leq |B''| \leq |A''| \Longrightarrow (B', B'') \in \mathcal{A}(N^*)$. Hence $\{(A', A'') \in \mathcal{A}(N^*) \,|\, |A'| = 1\} = \{(A', A'') \,|\, A' \subset A'' \subset UN^*,\ |A'| = 1 \leq |A''| < \beta\} \cup \{(A', A'') | A', A'' \subset UN^*,\ |A'| = 1 \leq |A''| < \gamma\}$ for some β, γ, $1 \leq \gamma \leq \beta \leq \aleph_1$, $1 < \beta$. Hence under our hypothesis we have $\mathcal{A}(N^*) = \{(A, A) \,|\, A \subset UN^*,\ 1 \leq |A| < \alpha\} \cup \{(A', A'') \,|\, A' \subset A'' \subset UN^*, |A'| = 1 \leq |A''| < \beta\} \cup \{(A', A'') \,|\, A', A'' \subset UN^*, |A'| = 1 \leq |A''| < \gamma\}$. Thus U hom $((UN^*, \mathcal{A}(N^*)), (UN^*, \mathcal{A}(N^*)) = (UN^*)^{UN^*}$, while U hom $(N^*, N^*) \neq (UN^*)^{UN^*}$ by Lemma 1, a contradiction.

Thus we have shown $\exists (A', A'') \in \mathcal{A}(N^*)$, $A' \neq A''$, $\min(|A'|, |A''|) \geq 2$.

2) Let us now suppose that for some $C \in \mathcal{O}b\,\mathbf{Prox_{mc}}$ (resp. $C \in \mathcal{O}b\,\mathbf{Prox}$) we have

$\exists(B', B'') \in \mathcal{A}(C)$, $B'\delta B''$. Then, since $B', B'' \neq \emptyset$ (by the definition of $\mathbf{SetPairSyst}_0^0$), $\exists f: C \to [0,1]$, $f(B') = \{0\}$, $f(B'') = \{1\}$. Thus $(\{0\}, \{1\}) \in \mathcal{A}([0,1])$.

We also have $(\{0\}, \{0\})$, $(\{1\}, \{1\}) \in \mathcal{A}([0,1])$, by the definition of $\mathbf{SetPairSyst}_0^0$. We are going to show that also $(\{0,1\}, \{0,1\})$, $(\{0\}, \{0,1\})$, $(\{1\}, \{0,1\}) \in \mathcal{A}([0,1])$.

Actually we will show that for any $D \in \mathcal{O}b\,\mathbf{Prox}_{\mathrm{mc}}$ (resp. $D \in \mathcal{O}b\,\mathbf{Prox}$), and any $a, b \in UD$, $a \neq b$ we have $(\{a,b\}, \{a,b\})$, $(\{a\}, \{a,b\})$, $(\{b\}, \{a,b\}) \in \mathcal{A}(D)$. For this observe that U hom $(N^*, D) \supset \{\varphi: UN^* \to UD \mid \exists N_0 \subset \mathbb{N}$, N_0 is finite, $[n \in N_0 \implies \varphi(n^{-1}) = b]$, $[n \in \mathbb{N} \setminus N_0 \implies \varphi(n^{-1}) = a]\}$.

We have an $(A', A'') \in \mathcal{A}(N^*)$, $A' \neq A''$, $|A'|, |A''| \geq 2$. If $A' \cap A'' = \emptyset$, then $\exists f: N^* \to D$, $f(UN^*) \subset \{a,b\}$, such that $f(A') = f(A'') = \{a,b\}$, and $f(n^{-1}) = a$ except for finitely many n. The same holds for $|A' \cap A''| \geq 2$, and also for $|A' \cap A''| = 1$. This implies $(\{a,b\}, \{a,b\}) \in \mathcal{A}(D)$.

Once more consider $(A', A'') \in \mathcal{A}(N^*)$, $A' \neq A''$, $|A'|, |A''| \geq 2$. Let us suppose e.g. $A' \setminus A'' \neq \emptyset$, and let $A' \setminus A'' \ni n_0^{-1}$. Let $f: N^* \to D$ be defined by $f(n_0^{-1}) = b$, $f((UN^*) \setminus \{n_0^{-1}\}) = \{a\}$. Then $f(A') = \{a,b\}$, $f(A'') = \{a\}$. Hence also $(\{a\}, \{a,b\}) \in \mathcal{A}(D)$, and similarly $(\{b\}, \{a,b\}) \in \mathcal{A}(D)$.

Recapitulating, we have shown that each pair of non-empty subsets of $\{0,1\}$ belongs to $\mathcal{A}([0,1])$. Then however each $\varphi \in (U[0,1])^{U[0,1]}$ with $\varphi([0,1]) \subset \{0,1\}$ underlies a morphism in hom $((U[0,1], \mathcal{A}([0,1])), (U[0,1], \mathcal{A}([0,1])))$. However the same is not true for morphisms in hom $([0,1],[0,1])$, a condradiction. This shows that for every $C \in \mathcal{O}b\,\mathbf{Prox}_{\mathrm{mc}}$ (resp. $C \in \mathcal{O}b\,\mathbf{Prox}$) $\mathcal{A}(C) \subset \{(B', B'') \mid B', B'' \subset UC,\ B'\delta B'' \text{ in } C\} = A_\delta(C)$.

3) Once more we consider N^* and $\mathcal{A}(N^*)$. Suppose $\mathcal{A}(N^*) \subset \{(A', A'') \mid A', A'' \subset UN^*,\ A' \cap A'' \neq \emptyset\}$. Let $a, b \in UN^*$, $a \neq b$. Let $\varphi \in (UN^*)^{UN^*}$ be any function, such that $\varphi(UN^*) \subset \{a,b\}$. φ underlies a morphism in hom $((UN^*, \mathcal{A}(N^*)), (UN^*, \mathcal{A}(N^*)))$, provided $\forall (A', A'') \in \mathcal{A}(N^*)$ $(\varphi(A'), \varphi(A'')) \in \mathcal{A}(N^*)$. However we have $\varphi(A') \cap \varphi(A'') \neq \emptyset$ by hypothesis, which means $(\varphi(A'), \varphi(A''))$ can only be one of $(\{a,b\}, \{a,b\})$, $(\{a\}, \{a,b\})$, $(\{b\}, \{a,b\})$, $(\{a\}, \{a\})$, $(\{b\}, \{b\})$. However, as shown in step 2) (choosing $D = N^*$), we have $(\{a,b\}, \{a,b\})$, $(\{a\}, \{a,b\})$, $(\{b\}, \{a,b\}) \in \mathcal{A}(N^*)$, as well as $(\{a\}, \{a\})$, $(\{b\}, \{b\}) \in \mathcal{A}(N^*)$. Therefore any function $\varphi \in (UN^*)^{UN^*}$, such that $\varphi(UN^*) \subset \{a,b\}$, underlies a morphism in hom $((UN^*, \mathcal{A}(N^*)), (UN^*, \mathcal{A}(N^*)))$. However, the same is not true for hom (N^*, N^*), a contradiction. This contradiction shows that our hypothesis $\mathcal{A}(N^*) \subset \{(A', A'') \mid A', A'' \subset UN^*,\ A' \cap A'' \neq \emptyset\}$ is false, i.e. $\exists (B', B'') \in \mathcal{A}(N^*)$, $B' \cap B'' = \emptyset$. Also $B'\delta B''$, since $(B', B'') \in \mathcal{A}(N^*) \subset A_\delta(N^*)$, as shown in step 2). Therefore B' and B'' are disjoint infinite subsets of UN^*.

Now let $C \in Ob\,\mathbf{Prox_{mc}}$, $(A', A'') \in \mathcal{A}_\delta^*(C)$. We will show $(A', A'') \in \mathcal{A}(C)$. Clearly in any of the three cases, in the definition of $\mathcal{A}_\delta^*(C)$, there exists an $f\colon N^* \to C$, satisfying $f(B') = A'$, $f(B'') = A''$. Since $(B', B'') \in \mathcal{A}(N^*)$, we have $(A', A'') \in \mathcal{A}(C)$. Therefore $\mathcal{A}_\delta^*(C) \subset \mathcal{A}(C)$. □

Now we turn to investigate how many not naturally isomorphic full embeddings $F\colon \mathbf{Prox_{mc}} \to \mathbf{SetPairSyst}_0^0$ there are. We again may and will suppose F concrete. If T_0 is not supposed, we have at least a proper class many such functors F, namely we can take $\mathcal{A}(C)$ as $\{(A', A'') \mid A', A'' \subset UC,\ A'\delta A'' \text{ in } C,\ |A'|, |A''| < \alpha\}$, for any cardinal $\alpha > \aleph_0$. (Remind that for any $C \in Ob\,\mathbf{Prox_{mc}}$ any two near sets A', A'' in C have near subsets of cardinalities $\leq \aleph_0$.) Now we turn to the T_0 case.

PROPOSITION 2. *The cardinality of the set of non-naturally isomorphic (concrete) full embeddings F from the full subcategory of* $\mathbf{Prox_{mc}}$, *consisting of the T_0 spaces contained in it, to* $\mathbf{SetPairSyst}_0^0$ *is* $\exp\exp\exp\aleph_0$.

PROOF: We may suppose concreteness by Corollary 1.

In the considered subcategory of $\mathbf{Prox_{mc}}$ each space C is a proximity subspace of the Hilbert cube, thus there are at most $\exp\exp\aleph_0$ non-isomorphic such spaces, and each of them has a cardinality $\leq \exp\aleph_0$. For each infinite set X the family of sets $\mathcal{A}$ of unordered pairs of subsets of X has a cardinality $\exp\exp|X|$. Thus there are at most $\exp\exp\exp\aleph_0$ ways of choosing an $\mathcal{A}$ for each isomorphism class of C's, thus also at most this many (concrete) full embeddings between our considered categories.

Conversely, we have to construct $\exp\exp\exp\aleph_0$ not naturally isomorphic (concrete) full embeddings. We use the well-known technics of strongly rigid classes of spaces cf. e.g. [14]. By [12] $\mathbf{R}^2$ contains $\exp\exp\aleph_0$ connected subspaces D_α, $|UD_\alpha| > 1$ (thus $|UD_\alpha| = \exp\aleph_0$), such that any continuous $g\colon D_{\alpha_1} \to D_{\alpha_2}$ is either constant, or is an identity (with $\alpha_1 = \alpha_2$). Each of these D_α's, as a separable metric space, has a metric compactification, which induces on it a T_0 proximity space $C_\alpha \in Ob\,\mathbf{Prox_{mc}}$. We also have that any $f\colon C_{\alpha_1} \to C_{\alpha_2}$ is either constant, or is an identity (with $\alpha_1 = \alpha_2$), and $|UC_\alpha| = \exp\aleph_0$. Let $\{C_\alpha\} = \mathbf{C}$, where $|\mathbf{C}| = \exp\exp\aleph_0$.

We have two different concrete full embeddings F_0, F_1 between the considered categories, namely the ones given on objects by $F_0C = (UC, \mathcal{A}_\delta^*(C))$, $F_1C = (UC, \mathcal{A}_\delta(C))$, satisfying $\mathcal{A}_\delta^*(C) \subset \mathcal{A}_\delta(C)$. These are different, and, in fact, not isomorphic, for any $C_\alpha \in \mathbf{C}$, since $(A', A'') \in \mathcal{A}_\delta^*(C_\alpha) \Longrightarrow |A'|, |A''| \leq \aleph_0$, while for $\mathcal{A}_\delta(C_\alpha)$ this is not true, by $|C_\alpha| = \exp\aleph_0$.

Let $\mathbf{C}' \subset \mathbf{C}$ be an arbitrary subclass of $\mathbf{C}$. We define for each T_0-space $C \in Ob\,\mathbf{Prox_{mc}}$ $\mathcal{A}_{\mathbf{C}'}(C) = \mathcal{A}_\delta^*(C) \cup \bigcup\{f(\mathcal{A}_\delta(C_\alpha)) | C_\alpha \in \mathbf{C}',\ f\colon C_\alpha \to C\}$. One easily verifies that $\mathcal{A}_{\mathbf{C}'}(C)$

consists of near pairs, and is in the evident sense a base for all near pairs, for each C (i.e., beside $\mathcal{A}_{C'}(C) \subset \mathcal{A}_\delta(C)$, we have $[A', A'' \subset UC,\ A'\delta A''$ in $C \Longrightarrow \exists(B', B'') \in \mathcal{A}_{C'}(C),\ B' \subset A',\ B'' \subset A'']$). Thus if $\varphi: UC_1 \to UC_2$ underlies a morphism $(UC_1, \mathcal{A}_{C'}(C_1)) \to (UC_2, \mathcal{A}_{C'}(C_2))$, then also it underlies a morphism $C_1 \to C_2$. One easily sees, by the definition of $\mathcal{A}_{C'}(C)$, that also the converse implication holds. Therefore $C \mapsto (UC, \mathcal{A}_{C'}(C))$ is the object part of a concrete full embedding $F_{C'}$ between our categories. Further, since any $f: C_{\alpha_1} \to C_{\alpha_2}$ is either constant or is an identity, we have for $C_\alpha \in \mathbf{C}'$ $\mathcal{A}_{C'}(C_\alpha) = \mathcal{A}_\delta(C_\alpha)$, while for $C_\alpha \in \mathbf{C} \setminus \mathbf{C}'$ $\mathcal{A}_{C'}(C_\alpha) = \mathcal{A}_\delta^*(C_\alpha)$. This means that for different $\mathbf{C}'$-s the respective concrete full embeddings $F_{C'}$ are not naturally isomorphic. Since $\mathbf{C}'$ can be chosen in $\exp\exp\exp\aleph_0$ ways, we have this many not naturally isomorphic concrete full embeddings between our considered categories. □

PROPOSITION 3. *Each full embedding F: **Prox** $\to$ **SetPairSyst**$_0^0$ is naturally isomorphic to a concrete functor F_0 satisfying, for $C \in Ob$ **Prox**, $F_0C = (UC, \mathcal{A})$ that $\{(A', A'') | A', A'' \subset UC,\ A'\delta A''$ in $C\} = \mathcal{A}_\delta(C) \supset \mathcal{A}$, and $[(A', A'') \in \mathcal{A}_\delta(C) \Longrightarrow \exists(B', B'') \in \mathcal{A},\ B' \subset A',\ B'' \subset A'']$. (Thus $\mathcal{A}$ is, in an evident sense a base, for near pairs, of δ.)*

PROOF: As noted in the proof of Proposition 1, steps 1) and 2) carry over for the case F: **Prox** $\to$ **SetPairSyst**$_0^0$, which we will suppose concrete. Thus we have $\forall C \in Ob$ **Prox** $\mathcal{A}(C) \subset \mathcal{A}_\delta(C)$.

It remained to show that $\forall C \in Ob$ **Prox** $\mathcal{A}(C)$ is in the above sense a base, for near pairs, of δ. For $|UC| \le 1$ this is trivial.

Let $|UC| = 2$, e.g. $UC = \{a, b\}$. Then by step 2) of the proof of Proposition 1, we have $(\{a, b\}, \{a, b\})$, $(\{a\}, \{a, b\})$, $(\{b\}, \{a, b\}) \in \mathcal{A}(C)$, and of course, $(\{a\}, \{a\})$, $(\{b\}, \{b\}) \in \mathcal{A}(C)$. For $C = C_d$ discrete, this amounts to $\mathcal{A}_\delta(C_d) \subset \mathcal{A}(C_d)$. Since we already know $\mathcal{A}(C_d) \subset \mathcal{A}_\delta(C_d)$, we have $\mathcal{A}(C_d) = \mathcal{A}_\delta(C_d)$. For $C = C_i$ indiscrete, on $UC = \{a, b\}$ we have that $1_{\{a,b\}}$ underlies a morphism $C_d \to C_i$, but not conversely. Thus the same holds for $(\{a, b\}, \mathcal{A}(C_d))$ and $(\{a, b\}, \mathcal{A}(C_i))$, hence $\mathcal{A}(C_i) = \{(A', A'') | \emptyset \ne A', A'' \subset UC\} = \mathcal{A}_\delta(C_i)$.

Let now $|UC| > 2$, and let $A_0', A_0'' \subset UC$, $A_0'\delta A_0''$ in C. If $A_0' \cap A_0''$ contains some $b \in UC$, then $B' = B'' = \{b\}$ can be chosen. Let now $A_0' \cap A_0'' = \emptyset$. Then, as mentioned before Proposition 1, there is on UC an ultraproximity C_0 finer than C, in which A_0' and A_0'' are near. Let $f: C_0 \to C$ be the morphism with $Uf = 1_{UC}$. Then it is enough to show that for some $B' \subset A_0'$, $B'' \subset A_0''$ $(B', B'') \in \mathcal{A}(C_0)$, since this implies $(B', B'') = (f(B'), f(B'')) \in \mathcal{A}(C)$.

Therefore it is enough to investigate C_0, and to prove the statement for C_0, rather than C, and for A_0', A_0''. Let $a, b \in UC_0$, $\{a\}\overline{\delta}\{b\}$ in C_0. These exist because of $|UC_0| > 2$. As shown in step 2) of the proof of Proposition 1, we have $(\{a, b\}, \{a, b\})$, $(\{a\}, \{a, b\})$, $(\{b\}, \{a, b\}) \in$

$\mathcal{A}(C_0)$, and, of course, $(\{a\},\{a\})$, $(\{b\},\{b\}) \in \mathcal{A}(C_0)$. Analogously as in Proposition 1, if we suppose $\mathcal{A}(C_0) \subset \{(A',A'')|A',A'' \subset UC_0,\ A' \cap A'' \neq \emptyset\}$, then $\forall\varphi \in (UC_0)^{UC_0}$, such that $\varphi(UC_0) \subset \{a,b\}$, underlies a morphism in hom $((UC_0,\mathcal{A}(C_0)),(UC_0,\mathcal{A}(C_0)))$, since by our hypothesis $\forall(A',A'') \in \mathcal{A}(C_0)\quad \varphi(A') \cap \varphi(A'') \neq \emptyset$, thus $(\varphi(A'),\varphi(A'')) \in \mathcal{A}(C_0)$. However if for each $\varphi \in (UC_0)^{UC_0}$, such that $\varphi(UC_0) \subset \{a,b\}$, $\{a\}\bar{\delta}\{b\}$ in C_0, we have that φ underlies a morphism in hom $((UC_0,\mathcal{A}(C_0)),(UC_0,\mathcal{A}(C_0)))$, thus a one in hom (C_0,C_0) too, then C_0 is discrete, a contradiction. This contradiction shows our hypothesis $\mathcal{A}(C_0) \subset \{(A',A'')|A',A'' \subset UC_0,\ A' \cap A'' \neq \emptyset\}$ is false, i.e. $\exists(A_1',A_1'') \in \mathcal{A}(C_0)$, $A_1' \cap A_1'' = \emptyset$. Also, $(A_1',A_1'') \in \mathcal{A}(C_0) \subset \mathcal{A}_\delta(C_0)$ implies $A_1'\delta A_1''$ in C_0.

Let $\mathcal{U}_1,\mathcal{U}_2$ be the ultrafilters used for defining C_0. Then, since also it has already been supposed that our investigated pair (A_0',A_0''), with $A_0', A_0'' \subset UC$, $A_0'\delta A_0''$ (in C_0), satisfies $A_0' \cap A_0'' = \emptyset$, we have that one of A_0', A_0'' belongs to $\mathcal{U}_1$, the other to $\mathcal{U}_2$, and similarly for A_1', A_1'' (by the definition of the ultraproximity). We may suppose $A_0', A_1' \in \mathcal{U}_1$, $A_0'', A_1'' \in \mathcal{U}_2$. Then we will choose $\emptyset \neq B' = A_0' \cap A_1' \subset A_0'$, $\emptyset \neq B'' = A_0'' \cap A_1'' \subset A_0''$.

We only have to show $(B',B'') \in \mathcal{A}(C_0)$, where we know $(A_1',A_1'') \in \mathcal{A}(C_0)$). For this we define $g\colon C_0 \to C_0$ as follows: g is identical on $B' \cup B''$, $g(A_1' \setminus B') \subset B'$, $g(A_1'' \setminus B'') \subset B''$, g on $(UC_0) \setminus (A_1' \cup A_1'')$ is arbitrary. $g\colon C_0 \to C_0$ holds since each near pair of subsets of UC_0 with empty intersection, say (B_1',B_1''), with $B_1' \in \mathcal{U}_1$, $B_1'' \in \mathcal{U}_2$, is carried by g to a near pair, because of $g(B_1') \supset B' \cap B_1' \in \mathcal{U}_1$, $g(B_1'') \supset B'' \cap B_1'' \in \mathcal{U}_2$. We have $g(A_1') = B'$, $g(A_1'') = B''$, hence $(B',B'') \in \mathcal{A}(C_0)$, as was to be shown. □

Now we turn to the analogues of Propositions 1 and 3 for 0-dimensional proximity spaces (i.e. proximity spaces C for which $A',A'' \subset UC$, $A'\bar{\delta}A'' \Longrightarrow \exists B \subset UC$, $A' \subset B$, $A'' \subset (UC) \setminus B$, $B\bar{\delta}((UC) \setminus B)$)). Let $\mathbf{Prox}_0$, resp $\mathbf{Prox}_{\mathbf{mc}0}$ denote the full subcategories of $\mathbf{Prox}$, resp. $\mathbf{Prox}_{\mathbf{mc}}$ consisting of their zero-dimensional objects. The category **SetPairSyst** is defined in the same way as $\mathbf{SetPairSyst}_0$, but without requiring $(A',A'') \in \mathcal{A} \Longrightarrow A',A'' \neq \emptyset$.

We have

LEMMA 3. *Let* $(X_1,\mathcal{A}_1)$, $(X_2,\mathcal{A}_2) \in \mathcal{O}b$ **SetPairSyst**. *Then each constant function* $\varphi\colon X_1 \to X_2$ *belongs to* U *hom* $((X_1,\mathcal{A}_1),(X_2,\mathcal{A}_2))$ *iff* $[\exists(A',A'') \in \mathcal{A}_1,\ A',A'' \neq \emptyset \Longrightarrow \forall x \in X_2\ (\{x\},\{x\}) \in \mathcal{A}_2]$, $[\exists(A,\emptyset) \in \mathcal{A}_1,\ A \neq \emptyset \Longrightarrow \forall x \in X_2\ (\{x\},\emptyset) \in \mathcal{A}_2]$, $[X_2^{X_1} \neq \emptyset,\ (\emptyset,\emptyset) \in \mathcal{A}_1 \Longrightarrow (\emptyset,\emptyset) \in \mathcal{A}_2]$.

PROOF: is obvious. □

Observe that step 1) of the proof of Proposition 1 actually gives that for any concrete full embedding $F\colon \mathbf{Prox} \to \mathbf{SetPairSyst}$ $\exists(A',A'') \in \mathcal{A}(N^*)$, $A' \neq A''$, $\min(|A'|,|A''|) \geq 2$. (In the contrary case by the same proof $\mathcal{A}(N^*)$ is the union of the system given in step

1) of Proposition 1, and possibly of $\{(A,\emptyset)|A \subset UN^*,\ 1 \leq |A| < \varepsilon\}$, where $1 < \varepsilon \leq \aleph_1$, and possibly of $\{(\emptyset,\emptyset)\}$; in each of these cases $U\hom((UN^*,\mathcal{A}(N^*)),\ (UN^*,\mathcal{A}(N^*))) = (UN^*)^{UN^*}$, a contradiction.) Also, for any concrete full embedding F: **Prox** → **SetPairSyst**, if $\exists C \in Ob\,\mathbf{Prox}$, $\exists\emptyset \neq A \subset UC$, $(A,\emptyset) \in \mathcal{A}(C)$, then by Lemma 3 $\forall C \in Ob\,\mathbf{Prox}$ $\forall x \in UC$ $(\{x\},\emptyset) \in \mathcal{A}(C)$.

COROLLARY 2. *Each concrete full embedding* F: **Prox** → **SetPairSyst** *either factorizes through the embedding* $\mathbf{SetPairSyst}_0 \hookrightarrow \mathbf{SetPairSyst}$, *and then* $\forall C \in Ob\,\mathbf{Prox}$ $\{(\{x\},\{x\})|x \in UC\} \subset \mathcal{A}(C)$, *or else* $[\exists C_0 \in Ob\,\mathbf{Prox},\ \exists\emptyset \neq A \subset UC_0,\ (A,\emptyset) \in \mathcal{A}(C_0) \Longrightarrow \forall C \in Ob\,\mathbf{Prox}\ \{(\{x\},\{x\})|x \in UC\} \cup \{(\{x\},\emptyset)|x \in UC\} \subset \mathcal{A}(C)]$. *and* $[\exists C_1 \in Ob\,\mathbf{Prox},\ (\emptyset,\emptyset) \in \mathcal{A}(C_1) \Longrightarrow \forall C \in Ob\,\mathbf{Prox}\ (UC \neq \emptyset \Longrightarrow \{(\{x\},\{x\})|x \in UC\} \cup \{(\emptyset,\emptyset)\} \subset \mathcal{A}(C))]$. *Conversely, if* F: **Prox** → **SetPairSyst** *is a full embedding, either factorizing through* **Set-Pair**

$\mathbf{Syst}_0 \hookrightarrow \mathbf{SetPairSyst}$, *and then satisfying* $\forall C \in Ob\,\mathbf{Prox}$ $\{(\{x\},\{x\})|x \in UFC\} \subset \mathcal{A}(C)$, *or else satisfying* $\forall C \in Ob\,\mathbf{Prox}$ $\{(\{x\},\{x\})|x \in UFC\} \subset \mathcal{A}(C)$, *and* $[\exists C_0 \in Ob\,\mathbf{Prox},\ \exists\emptyset \neq A \subset UFC_0,\ (A,\emptyset) \in \mathcal{A}(C_0) \Longrightarrow \forall C \in Ob\,\mathbf{Prox}\ \{(\{x\},\emptyset)|x \in UFC\} \subset \mathcal{A}(C)]$, *and* $[\exists C_1 \in Ob\,\mathbf{Prox},\ (\emptyset,\emptyset) \in \mathcal{A}(C_1) \Longrightarrow \forall C \in Ob\,\mathbf{Prox}\ (UFC \neq \emptyset \Longrightarrow (\emptyset,\emptyset) \in \mathcal{A}(C))]$, *then* F *is naturally isomorphic to a concrete functor. The same holds for* $\mathbf{Prox}_{\mathrm{mc}}$, $\mathbf{Prox}_0$, $\mathbf{Prox}_{\mathrm{mc}0}$ *as well, rather than* **Prox**.

PROOF follows from the observations before Corollary 2 and the proof of Corollary 1. The only difference is that in the converse direction, when we show that each constant function $X_1 \to X_2$ belongs to U hom $((X_1,\mathcal{A}_1),(X_2,\mathcal{A}_2)$ $((X_i,\mathcal{A}_i) = FC_i$, for some $C_i \in Ob\,\mathbf{Prox})$, we have to exclude the case $(X_1,\mathcal{A}_1) = (\emptyset,\{(\emptyset,\emptyset)\})$, $(X_2,\mathcal{A}_2) = (\emptyset,\emptyset)$. However these are two non-isomorphic objects, such that no object FC, $C \in Ob\,\mathbf{Prox}$, non-isomorphic to both of them, admits a morphism to any of them, which is a contradiction, since F is a full embedding. □

We let $\mathbf{SetPairSyst}^0$ be the full subcategory of **SetPairSyst**, consisting of the objects $(X,\mathcal{A})$, satisfying $\{(\{x\},\{x\})|x \in X\} \cup \{(\{x\},\emptyset)|x \in X\} \cup \{(\emptyset,\emptyset)\} \subset \mathcal{A}$. (We might have required $(\emptyset,\emptyset) \notin \mathcal{A}$, or $(\emptyset,\emptyset) \in \mathcal{A} \Longleftrightarrow X \neq \emptyset$, but each of these makes the full embedding F naturally isomorphic to a concrete functor, and preserving the pair $(\emptyset,\emptyset)$ is no restriction for a function $X_1 \to X_2$, thus a full embedding remains one, if, for each $(X,\mathcal{A})$ in the image, we add to $\mathcal{A}$ $(\emptyset,\emptyset)$.)

PROPOSITION 4. *Each full embedding* F *of* $\mathbf{Prox}_{\mathrm{mc}0}$ *resp.* $\mathbf{Prox}_0$ *into any of* $\mathbf{SetPairSyst}^0$ *or* $\mathbf{SetPairSyst}^0_0$ *is naturally isomorphic to a concrete functor* F_0, *satisfying for* $C \in Ob\,\mathbf{Prox}_{\mathrm{mc}0}$ $\mathcal{A}_\delta(C) \supset \mathcal{A}(C) \cap \{(A',A'')|A',A'' \subset UC,\ A',A'' \neq \emptyset\} \supset \mathcal{A}^*_\delta(C)$, *resp. for* $C \in$

$\mathcal{O}b\,\mathbf{Prox}_0\ \mathcal{A}_\delta(C) \supset \mathcal{A}(C) \cap \{(A',A'')|A',A'' \subset UC,\ A',A'' \neq \emptyset\}$ *and* $[(A',A'') \in \mathcal{A}_\delta(C) \Longrightarrow \exists (B',B'') \in \mathcal{A}(C),\ \emptyset \neq B' \subset A',\ \emptyset \neq B'' \subset A'']$.

PROOF. By Corollary 2 we suppose F concrete. We first restrict ourselves to the full subcategories $\mathbf{T_0\,Prox_{mc0}}$ resp. $\mathbf{T_0\,Prox_0}$ of $\mathbf{Prox_{mc0}}$ resp. $\mathbf{Prox_0}$, consisting of their T_0 objects, and investigate the restriction of F to them. The projectively generating class for $\mathbf{T_0\,Prox_{mc0}}$ or $\mathbf{T_0\,Prox_0}$ will be { Cantor set } (referred to further as constructed in $[0,1]$ in the usual way). The inductively generating classes for $\mathbf{T_0\,Prox_{mc0}}$, resp. $\mathbf{T_0Prox_0}$ will be the same, as for $\mathbf{Prox_{mc}}$, resp. as for $\mathbf{Prox}$, but excepting the ultraproximities constructed with two fixed ultrafilters (by the same proof, noting that for C T_0, the ultraproximities constructed at proving inductive generation cannot be ones constructed with two fixed ultrafilters).

Suppose first that for the case of $\mathbf{SetPairSyst^0}$ we have $\forall C \in \mathcal{O}b\,\mathbf{T_0\,Prox_{mc0}}$ resp. $\mathcal{O}b\,\mathbf{T_0\,Prox_0}$ $(A,\emptyset) \in \mathcal{A}(C) \Longrightarrow |A| \leq 1$. Then $\{A|(A,\emptyset) \in \mathcal{A}(C)\} = \{\{x\}|x \in UC\} \cup \{\emptyset\}$, and $\varphi \in U$ hom $(((UC_1,\mathcal{A}(C_1)),(UC_2,\mathcal{A}(C_2))) \iff \varphi \in U$ hom $((UC_1,\mathcal{A}(C_1) \cap \{(A_1',A_1'')|A_1',A_1'' \subset UC_1,\ A_1',A_1'' \neq \emptyset\}),(UC_2,\mathcal{A}(C_2) \cap \{(A_2',A_2'')|A_2',A_2'' \subset UC_2,\ A_2',A_2'' \neq \emptyset\}))$. Therefore this case is reduced to that of $\mathbf{SetPairSyst_0^0}$, and thus in the case of $\mathbf{SetPairSyst^0}$ we may and will assume $\exists C_1 \in \mathcal{O}b\,\mathbf{T_0\,Prox_{mc0}}$, resp. $\mathcal{O}b\,\mathbf{T_0Prox_0}$, $\exists (A_1,\emptyset) \in \mathcal{A}(C_1)$, $|A_1| \geq 2$.

We have $N^* \in \mathcal{O}b\,\mathbf{T_0Prox_{mc0}} \subset \mathcal{O}b\ \mathbf{T_0\,Prox_0}$ and the result of step 1) of the proof of Proposition 1 remains valid, as noted after Lemma 3.

For step 2), supposing that for some $C \in \mathcal{O}b\,\mathbf{T_0\,Prox_{mc0}}$ resp. $\mathcal{O}b\,\mathbf{T_0\,Prox_0}$ $\exists (A',A'') \in \mathcal{A}(C)$, $A',A'' \neq \emptyset$, $A'\bar{\delta}A''$ in C, we obtain for the Cantor set $(\{0\},\{1\}) \in \mathcal{A}$ (Cantor set). If we consider $\mathbf{SetPairSyst_0^0}$, we see like in Proposition 1 that each pair of non-empty subsets of $\{0,1\}$ belongs to $\mathcal{A}$ (Cantor set), which leads to a contradiction like in Proposition 1. If we consider $\mathbf{SetPairSyst^0}$, beside each pair of non-empty subsets of $\{0,1\}$ also $(\{0\},\emptyset)$, $(\{1\},\emptyset)$, $(\emptyset,\emptyset)$ belong to $\mathcal{A}$ (Cantor set). If also $(\{0,1\},\emptyset) \in \mathcal{A}$ (Cantor set), we get a contradiction like in Proposition 1. However, consider the above space C_1 and $A_1 \subset UC_1$, with $(A_1,\emptyset) \in \mathcal{A}(C_1)$, $|A_1| \geq 2$. Since $C_1 \in \mathcal{O}b\,\mathbf{T_0\,Prox_{mc0}}$, resp. $C_1 \in \mathcal{O}b\,\mathbf{T_0\,Prox_0}$, $\exists f\colon C_1 \to$ Cantor set, $f(UC_1) = f(A_1) = \{0,1\}$. By $(A_1,\emptyset) \in \mathcal{A}(C_1)$ we have $(\{0,1\},\emptyset) \in \mathcal{A}$ (Cantor set), which leads to the desired contradiction.

Thus we have shown $\forall C \in \mathcal{O}b\,\mathbf{T_0\,Prox_{mc0}}$, resp. $\mathcal{O}b\,\mathbf{T_0\,Prox_0}$ $\mathcal{A}(C) \cap \{(A',A'')|A',A'' \subset UC,\ A',A'' \neq \emptyset\} \subset \mathcal{A}_\delta(C)$.

For step 3), we first consider the case of $\mathbf{T_0\,Prox_{mc0}}$. Suppose $\mathcal{A}(N^*) \cap \{(A',A'')|A',A'' \subset UN^*, A',A'' \neq \emptyset\} \subset \{(A',A'')|A',A'' \subset UN^*,\ \ A' \cap A'' \neq \emptyset\}$. For the case of $\mathbf{SetPairSyst_0^0}$

we proceed like at Proposition 1, and obtain a contradiction. For the case of **SetPairSyst**^{0} we have with the notations of Proposition 1, step 3) $(\{a\},\emptyset),(\{b\},\emptyset),(\emptyset,\emptyset)\in\mathcal{A}(N^*)$. If also $(\{a,b\},\emptyset)\in\mathcal{A}(N^*)$, we get the contradiction like above. However, like above, $\exists f: C_1\to N^*$, $f(UC_1)=f(A_1)=\{a,b\}$. By $(A_1,\emptyset)\in\mathcal{A}(C_1)$ this implies $(\{a,b\},\emptyset)\in\mathcal{A}(N^*)$, leading to the desired contradiction. This implies $\exists(B',B'')\in\mathcal{A}(N^*)$, $B',B''\neq\emptyset$, (thus $B'\delta B''$ in N^*), $B'\cap B''=\emptyset$, which in turn implies $\forall C\in \mathcal{O}b\,\mathbf{T_0\,Prox_{mc0}}$ (actually $\forall C\in\mathcal{O}b\,\mathbf{Prox_{mc0}}$) $\mathcal{A}(C)\cap\{(A',A'')|A',A''\subset UC,\quad A',A''\neq\emptyset\}\supset\mathcal{A}^*_\delta(C)$, like in Proposition 1.

Now we deal with step 3), in case of $\mathbf{T_0\,Prox_0}$, and will show $(A'_0,A''_0)\in\mathcal{A}_\delta(C)\implies\exists(B',B'')\in\mathcal{A}(C),\emptyset\neq B'\subset A'_0,\emptyset\neq B''\subset A''_0$. Like at Proposition 3, we assume $A'_0\cap A''_0=\emptyset$. The case of spaces C with $|UC|\leq 2$ is evident, because they are now discrete. For $|UC|>2$ we turn to the ultraproximity C_0 on UC, constructed in Proposition 3 for $A'_0,A''_0\subset UC$, which, as noted above, also belongs to $\mathcal{O}b\,\mathbf{T_0\,Prox_0}$. Suppose $\mathcal{A}(C_0)\cap\{(A',A'')|A',A''\subset UC_0,A',A''\neq\emptyset\}\subset\{(A',A'')|A',A''\subset UC_0,\quad A'\cap A''\neq\emptyset\}$. For **SetPairSyst**$^0_0$, proceeding like in Proposition 3, we get a contradiction. For **SetPairSyst**^{0} we have for $a,b\in UC_0$, $\{a\}\bar\delta\{b\}$, that $(\{a\},\emptyset),(\{b\},\emptyset)$, $(\emptyset,\emptyset)\in\mathcal{A}(C_0)$ by definition, while $(\{a,b\},\emptyset)\in\mathcal{A}(C_0)$ by using C_1, as above with N^*. This gives a contradiction like at Proposition 3. Thus both for **SetPairSyst**$^0_0$ and **SetPairSyst**^{0} we gain $\exists(A'_1,A''_1)\in\mathcal{A}(C_0)$, $A'_1,A''_1\neq\emptyset$, $A'_1\cap A''_1=\emptyset$, and hence $\forall C\in\mathcal{O}b\,\mathbf{T_0\,Prox_0}$ $(A'_0,A''_0)\in\mathcal{A}_\delta(C)\implies\exists(B',B'')\in\mathcal{A}(C)$, $\emptyset\neq B'\subset A'_0$, $\emptyset\neq B''\subset A''_0$, like in Proposition 3.

Up to now we have dealt with the subcategories $\mathbf{T_0\,Prox_{mc0}}$, $\mathbf{T_0\,Prox_0}$ only, and have shown that for their objects the statement of the proposition holds. As step 4), we extend the result for $\mathbf{Prox_{mc0}}$, $\mathbf{Prox_0}$. The inclusion concerning $\mathcal{A}^*_\delta(C)$ has already been shown above. For the remaining three relations observe that the mentioned subcategories both projectively and inductively generate the respective categories. Namely, for any non-T_0 object C we can consider its T_0-reflection map $C\to rC=$ the T_0-reflection of C, for projective generation; and any right inverse of this map, and maps of N^* to two-point indiscrete subspaces of C, for inductive generation. Thus the respective relations for $\mathcal{A}(rC)$, and the fact that $\exists(B',B'')\in\mathcal{A}(N^*)$, $B',B''\neq\emptyset$, $B'\cap B''=\emptyset$ (implying that, for any given two-point indiscrete subspace $\{b',b''\}$ of C, $\exists f:N^*\to C$, $f(N^*)=\{b',b''\}$, $f(B')=\{b'\}$, $f(B'')=\{b''\}$) prove the remaining three relations for $\mathcal{A}(C)$, too. □

REMARKS. 1) By Proposition 4, for the full embeddings of $\mathbf{Prox_{mc0}}$, resp. $\mathbf{Prox_0}$ into **SetPairSyst**^{0} $\{A|(A,\emptyset)\in\mathcal{A}(C)\}$, where $C\in\mathcal{O}b\,\mathbf{Prox_{mc0}}$, resp. $\mathcal{O}b\,\mathbf{Prox_0}$, can be an arbitrary system of subspaces, containing all subspaces with ≤ 1 points, and preserved by all morphisms $f:C_1\to C_2$ in the respective category, even if $\mathcal{A}(C)\cap\{(A',A'')|A',A''\subset$

UC, $A', A'' \neq \emptyset\}$ is fixed for each C.

2) This also implies that there are at least a proper class of not naturally isomorphic concrete full embeddings of any of $\mathbf{Prox_{mc}}$, $\mathbf{Prox_{mc0}}$, $\mathbf{Prox_0}$ to $\mathbf{SetPairSyst^0}$ (take $\mathcal{A}(C) = \mathcal{A}_\delta(C) \cup \{(A, \emptyset) | A \subset UC, |A| < \alpha\}$ for some cardinal $\alpha > 1$). For **Prox** there are even as many not naturally isomorphic concrete full embeddings to $\mathbf{SetPairSyst^0}$, as there are subclasses of a proper class. Namely by [22] there is a proper class $\mathbf{D} = \{D_\alpha\}$ of $T_{3\frac{1}{2}}$ spaces, hence also a proper class $\mathbf{C} = \{C_\alpha\}$ of proximity spaces, with the property that any mapping between them is either constant or is an identity, and $|UD_\alpha|, |UC_\alpha| > 1$ (choosing for each D_α a compatible proximity C_α). Considering now any subclass $\mathbf{C}' \subset \mathbf{C}$, we let for $C \in \mathit{Ob}\,\mathbf{Prox}$ $\mathcal{A}_{\mathbf{C}'}(C) = \mathcal{A}_\delta(C) \cup \{(\{x\}, \emptyset) | x \in UC\} \cup \{(\emptyset, \emptyset)\} \cup \{(f(A_\alpha), \emptyset) | A_\alpha \subset UC_\alpha, |A_\alpha| \leq 2, C_\alpha \in \mathbf{C}', f: C_\alpha \to C\}$. Like at Proposition 2 it follows that in this way we obtain concrete full embeddings with object parts $C \mapsto (UC, \mathcal{A}_{\mathbf{C}'}(C))$, which are not naturally isomorphic for different subclasses $\mathbf{C}'$ of $\mathbf{C}$.

3) The analogue of Proposition 2 for full embeddings of **Prox**, $\mathbf{T_0\,Prox_{mc0}}$, $\mathbf{Prox_0}$ to $\mathbf{SetPairSyst_0^0}$ remains open. A lower bound for the case of $\mathbf{T_0\,Prox_{mc0}}$ would follow if one found a large set $\{C_\alpha\} \subset \mathit{Ob}\,\mathbf{T_0\,Prox_{mc0}}$, such that $|UC_\alpha| = \exp\aleph_0$ and $[\alpha_1 \neq \alpha_2 \implies \forall f: C_{\alpha_1} \to C_{\alpha_2}\ |f(UC_{\alpha_1})| \leq \aleph_0]$ (by replacing in the proof of Proposition 2 $\mathcal{A}_\delta^*(C)$ by $\{(A', A'') | A', A'' \subset UC,\ A'\delta A'',\ |A'|, |A''| \leq \aleph_0\}$).

3.6. An analogue of the question treated in **3.5** is that about the full embedings of **Top** into the category **SetPointSyst**, whose objects are of the form $(X, \mathcal{A})$, X a set, $\mathcal{A}$ a set of ordered pairs (A, x), $A \subset X \ni x$, and whose morphisms $f: (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$ are characterized by $f \in X_2^{X_1}$, $f(\mathcal{A}_1) \subset \mathcal{A}_2$. One concrete full embedding F is given of course by $FC = (UC, \{(A, x) | A \subset UC \ni x,\ \overline{A} \ni x\})$, where $C \in \mathit{Ob}\,\mathbf{Top}$. (The forgetful functor for **SetPointSyst** is defined by $U(X, \mathcal{A}) = X$, $Uf = f$.)

We note that such an $\mathcal{A}$, as above, is an element of $2^{2^X \times X} \cong (2^X)^{2^X} \cong (2^{2^X})^X$ $(2^Y = \mathcal{P}(Y))$. This shows that the category **SetPointSyst** is concretely isomorphic to the category with objects (X, ψ), X a set, $\psi: 2^X \to 2^X$, and morphisms $f: (X_1, \psi_1) \to (X_2, \psi_2)$ characterized by $f \in X_2^{X_1}$, $A \subset X_1 \implies f(\psi_1(A)) \subset \psi_2(f(A))$; and also to the category with objects (X, Ψ), X a set, $\Psi: X \to 2^{2^X}$, and morphisms $f: (X_1, \Psi_1) \to (X_2, \Psi_2)$ characterized by $f \in X_2^{X_1}, x \in X_1 \implies f(\Psi_1(x)) \subset \Psi_2(f(x))$. Thus the above concrete full embedding $F: \mathbf{Top} \to \mathbf{SetPointSyst}$ goes over to the concrete full embedding given by $\psi(A) = \overline{A}$; resp. to that given by $\Psi(x) = \{A | A \subset UC \ni x,\ \overline{A} \ni x\}$, where $C \in \mathit{Ob}\,\mathbf{Top}$, well known from the elements of topology. For convenience we will use among these three concretely isomorphic categories **SetPointSyst**.

We are going to show the analogues of Propositions 1, 2, 3, 4 for full embeddings of **Top**, and certain subcategories of **Top**, to **SetPointSyst**. The proofs are rather similar to the above proofs, and also are simpler, so some details will be left to the reader.

LEMMA 4. *([61], proof of Theorem 4.1). Let $C \in Ob\,\mathbf{Top}$. Then hom $(C,C) = (UC)^{UC}$ iff C is either indiscrete or discrete.* □

LEMMA 5. *Let $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2) \in Ob\,\mathbf{SetPointSyst}$. Then each constant function $\varphi: X_1 \to X_2$ belongs to U hom $((X_1,\mathcal{A}_1), (X_2,\mathcal{A}_2))$ iff $[\exists (A,x) \in \mathcal{A}_1,\ A \neq \emptyset \Longrightarrow \forall x \in X_2 \quad (\{x\},x) \in \mathcal{A}_2]$ and $[\exists x \in X_1,\ (\emptyset,x) \in \mathcal{A}_1 \Longrightarrow \forall x \in X_2 \quad (\emptyset,x) \in \mathcal{A}_2]$.* □

For a full embedding $F: \mathbf{Top} \to \mathbf{SetPointSyst}$ we denote $FC = (UFC, \mathcal{A}(C))$, and the same notation will be used for analogous situations.

COROLLARY 3. *A full embedding $F: \mathbf{Top} \to \mathbf{SetPointSyst}$ is naturally isomorphic to a concrete functor iff $\forall C \in Ob\,\mathbf{Top}$ $\{(\{x\},x) | x \in UFC\} \subset \mathcal{A}(C)$, and $[\exists C_0 \in Ob\,\mathbf{Top}\ \exists x \in UFC_0 \quad (\emptyset,x) \in \mathcal{A}(C_0) \Longrightarrow \forall C \in Ob\,\mathbf{Top} \quad \forall x \in UFC \quad (\emptyset,x) \in \mathcal{A}(C)]$. The same holds for the full subcategories of* **Top**, *with objects all T_1, Tychonoff, 0-dimensional, pseudometric, resp. pseudometric and 0-dimensional spaces, rather than* **Top**.

PROOF is similar to that of Corollary 1, noting for the "only if" side that C non-discrete, non-indiscrete implies by Lemmas 4 and 5 that $\exists (A,x) \in \mathcal{A}(C),\ A \neq \emptyset$. □

We let $\mathbf{SetPointSyst}_0$, $\mathbf{SetPointSyst}_0^0$, $\mathbf{SetPointSyst}^0$ be the full subcategories of **SetPointSyst** consisting of the objects $(X,\mathcal{A})$, satisfying $[(A,x) \in \mathcal{A} \Longrightarrow A \neq \emptyset]$, resp. $[((A,x) \in \mathcal{A} \Longrightarrow A \neq \emptyset)$ and $\{(\{x\},x) | x \in X\} \subset \mathcal{A}]$, resp. $[\{(\{x\},x) | x \in X\} \cup \{(\emptyset,x) | x \in X\} \subset \mathcal{A}]$. We are interested in full embeddings $\mathbf{Top} \to \mathbf{SetPointSyst}_0^0$, and $\mathbf{Top} \to \mathbf{SetPointSyst}^0$ (resp. in the same for the subcategories of **Top** listed in Corollary 3). However preservation of pairs $(\emptyset,x)$ is no restriction for a function $f: X_1 \to X_2$, thus the concrete functor I with object part $I(X,\mathcal{A}) = (X, \mathcal{A} \cup \{(\emptyset,x) | x \in X\})$ is a concrete isomorphism $\mathbf{SetPointSyst}_0^0 \to \mathbf{SetPointSyst}^0$. Hence the questions of full embeddings into these two categories are equivalent; for convenience we will treat $\mathbf{SetPointSyst}_0^0$.

The full subcategories of **Top**, listed in Corollary 3, will be denoted in that order by **Top**, $\mathbf{Top}_1$, **Tych**, **0-dim**, $\mathbf{PsMet_c}$, $\mathbf{PsMet0\text{-}dim_c}$. Following [51], we will use the following projectively and inductively generating classes for them (listed in the above order). For projective generation we use {Sierpiński space}, {infinite spaces with cofinite topology}, $\{[0,1]\}$, $\{\hat{N}\}$, $\{[0,1]\}$, $\{\hat{N}\}$. Here $\hat{N} = \{n^{-1} | n \in \mathrm{N}\} \cup \{0\}$ with its usual topology. For inductive generation we use {topological ultraspaces} for **Top**, {free topological ultraspaces} for $\mathbf{Top}_1$, {free topological ultraspaces} ∪ {topological sums of a two-point indiscrete space and an arbitrary discrete space} for **Tych**, **0-dim**, $\{\hat{N}\}$ for $\mathbf{PsMet_c}$, **PsMet0-** $\mathbf{dim_c}$. A topological ultraspace is obtained as follows (cf. e.g. [51]). Let X be a set, $|X| \geq 2$, and let $\mathcal{U}$ be an ultrafilter (free or fixed) on X, $x \in X$, $\mathcal{U} \neq$ the fixed ultrafilter corresponding to x. Each point in $X \setminus \{x\}$ forms an open set, and the neighbourhood system of x is given by $\{V \cup \{x\} | V \in \mathcal{U}\}$. We say that this topological ultraspace is free, if $\mathcal{U}$ is a free ultrafilter.

PROPOSITION 5. *Each full embedding F of* **Top**, $\mathbf{Top_1}$, **Tych**, **0-dim**, $\mathbf{PsMet_c}$ *resp.* **PsMet** $\mathbf{0\text{-}dim_c}$ *into* $\mathbf{SetPointSyst_0^0}$ *is naturally isomorphic to a concrete functor F_0, satisfying for $F_0C = (UC, \mathcal{A})$ (C any object of the respective category) $\{(A,x) | A \subset UC \ni x,\ \overline{A} \ni x\} = \mathcal{A}_{cl}(C) \supset \mathcal{A}$, and $[(A,x) \in \mathcal{A}_{cl}(C) \Longrightarrow \exists(B,x| \in \mathcal{A},\ B \subset A]$. (Thus $\mathcal{A}$ is, in an evident sense, a base for $\mathcal{A}_{cl}(C)$.) For* $\mathbf{PsMet_c}$ *and* $\mathbf{PsMet0\text{-}dim_c}$ *we have more exactly that $\mathcal{A} \supset \mathcal{A}^*_{cl}(C) = \{(A,x) | A \subset UC \ni x,\ |A| = \aleph_0,\ A$ yields a sequence converging to $x\} \cup \{(A,x) | A \subset UC \ni x,\ 1 \leq |A| < \aleph_0,\ \exists a \in A,\ \overline{\{a\}} = \overline{\{x\}}\}$. Further the map F^* defined an objects by $F^*C = (UC, \mathcal{A}^*_{cl}(C))$ is itself the object part of a concrete full embedding of* $\mathbf{PsMet_c}$ *resp.* $\mathbf{PsMet0\text{-}dim_c}$ *to* $\mathbf{SetPointSyst_0^0}$.

PROOF: By Corollary 3 we suppose F concrete. The proof consists of three steps. First we investigate $\hat{N}$ and $\mathcal{A}(\hat{N})$. Second we show by projective generation that for each object C we have $\mathcal{A}(C) \subset \mathcal{A}_{cl}(C)$. Third by inductive generation we show the remaining statements of the proposition.

1) A) $\hat{N}$ is an object of each of the six categories. Supposing $\mathcal{A}(\hat{N}) \subset \{(A,x) | A \subset U\hat{N} \ni x,\ |A| = 1\}$ we gain like at Proposition 1 that $\mathcal{A}(\hat{N}) = \{(\{x\},x) | x \in U\hat{N}\}$ or $\mathcal{A}(\hat{N}) = \{(\{y\},x) | y,\ x \in U\hat{N}\}$, implying U hom $((U\hat{N}, \mathcal{A}(\hat{N})), (U\hat{N}, \mathcal{A}(\hat{N}))) = (U\hat{N})^{U\hat{N}} \neq U$ hom $(\hat{N}, \hat{N})$, a contradiction. Therefore $\exists(A_0, x_0) \in \mathcal{A}(\hat{N}),\ |A_0| \geq 2$.

B) This implies for any D, object of the respective category, and any $a, b \in UD$, $a \neq b$ that $(\{a,b\},a) \in \mathcal{A}(D)$. In fact, $\hat{N}$ is zero-dimensional, hence for two distinct points of A_0 ((A_0, x_0) taken from $\mathcal{A}$)) there is a clopen set B of $\hat{N}$ containing exactly one of these points. Then either $x_0 \in B$ or $x_0 \in (U\hat{N}) \setminus B$, and we may suppose $x_0 \in B$. Define $f: \hat{N} \to D$, $f(B) = \{a\}$, $f((U\hat{N}) \setminus B) = \{b\}$. Then Uf lifts to a morphism $(U\hat{N}, \mathcal{A}(\hat{N})) \to (UD, \mathcal{A}(D))$, hence by $(A_0, x_0) \in \mathcal{A}(\hat{N})$ we have $(\{a,b\},a) = (f(A_0), f(x_0)) \in \mathcal{A}(D)$.

C) This further implies that for any non-discrete, non-indiscrete object D of the respective category $\exists(A,x) \in \mathcal{A}(D),\ x \notin A$. Namely otherwise for a non-indiscrete two-point subspace $\{a,b\}$ of D each $\varphi: UD \to UD$, such that $\varphi(UD) \subset \{a,b\}$, underlies a morphism $(UD, \mathcal{A}(D)) \to (UD, \mathcal{A}(D))$. In fact, we then have for each $(A,x) \in \mathcal{A}(D)$ $A \ni x$, hence $\{a,b\} \supset \varphi(A) \ni \varphi(x)$. Thus $(\varphi(A), \varphi(x))$ is one of $(\{a,b\},a)$, $(\{a,b\},b)$, $(\{a\},a)$, $(\{b\},b)$. However, by B) and by definition, each of these belongs to $\mathcal{A}(D)$, showing that in fact φ

underlies a morphism $(UD, \mathcal{A}(D)) \to (UD, \mathcal{A}(D))$. Then each such φ underlies a continuous map $D \to D$ as well, and, since $\{a, b\}$ is not indiscrete, D is discrete, a contradiction.

D) The result of B) also implies that for any non-discrete object D of the respective category, and any $a, b \in UD$, $a \neq b$, such that $\{a, b\}$ is not an indiscrete subspace of D, it cannot hold that both $(\{a\}, b), (\{b\}, a) \in \mathcal{A}(D)$. Namely otherwise each $\varphi: UD \to UD$, such that $\varphi(UD) \subset \{a, b\}$, underlies a morphism $(UD, \mathcal{A}(D)) \to (UD, \mathcal{A}(D))$. In fact, then for each $(A, x) \in \mathcal{A}(D)$ we have $\varphi(A) \subset \{a, b\} \ni \varphi(x)$, and each pair (B, y) with $B \subset \{a, b\} \ni y$ belongs to $\mathcal{A}(D)$, by B), by definition, and by our hypothesis. Therefore in fact φ underlies a morphism $(UD, \mathcal{A}(D)) \to (UD, \mathcal{A}(D))$. Then each such φ underlies a continuous map $D \to D$ as well, which gives a contradiction like at C).

E) Now we show in any of the six cases that for any 0-dimensional object C of the respective category $\forall (A, x) \in \mathcal{A}(C)$ $\overline{A} \ni x$ (i.e. $\mathcal{A}(C) \subset \mathcal{A}_{cl}(C)$). Namely otherwise there is a clopen set B of C, $x \in B$, $B \cap A = \emptyset$. Let $a, b \in U\hat{N}$, $a \neq b$. Recalling that $A \neq \emptyset$ by definition, define $f, g: C \to \hat{N}$, $f(B) = g((UC) \setminus B) = \{b\}$, $g(B) = f((UC) \setminus B) = \{a\}$. Then $(A, x) \in \mathcal{A}(C)$ implies $(\{a\}, b) = (f(A), f(x)) \in \mathcal{A}(\hat{N})$, $(\{b\}, a) = (g(A), g(x)) \in \mathcal{A}(\hat{N})$. This gives a contradiction by D).

F) By C) we have $\exists (A_1, x_1) \in \mathcal{A}(\hat{N})$, $x_1 \notin A_1$. For this (A_1, x_1) therefore by E) $x_1 \in \overline{A_1} \setminus A_1$. Hence $x_1 = 0$, and $A_1 \subset (U\hat{N}) \setminus \{0\}$, A_1 is infinite.

G) For the case of **Top** consider $f: \hat{N} \to S =$ Sierpiński space on $\{0, 1\}$ — with $\{1\}$ as the only non-trivial open set —, defined by $f(0) = 0$, $f((U\hat{N}) \setminus \{0\}) = \{1\}$. We have from F) an $(A_1, x_1) \in \mathcal{A}(\hat{N})$, for which we have $(\{1\}, 0) = (f(A_1), f(x_1)) \in \mathcal{A}(S)$. Hence, also using B), we have $\mathcal{A}(S) \supset \{(\{0\}, 0), (\{1\}, 1), (\{1\}, 0), (\{0, 1\}, 0), (\{0, 1\}, 1)\}$, and by D) here actually equality holds.

2) Now we show for each of the six categories that for any object C $\mathcal{A}(C) \subset \mathcal{A}_{cl}(C)$. For **0-dim**, **PsMet0-dim$_c$** this follows from 1E). We turn to **Top**, **Top$_1$**, **Tych**, **PsMet$_c$**. Let us suppose the contrary, i.e. $\exists (A, x) \in \mathcal{A}(C), (\emptyset \neq) \overline{A} \not\ni x$. Consider in each of these four cases continuous functions from C to the following spaces (in the above order): S; $C_{cof} =$ a space of cardinality $|UC| + \aleph_0$, with the cofinite topology; $[0, 1]$; $[0, 1]$, resp., as follows. $f(\overline{A}) = \{0\}$, $f((UC) \setminus \overline{A}) = \{1\}$; f, g satisfying $f(\overline{A}) = \{g(x)\} = \{a\}$, $g(\overline{A}) = \{f(x)\} = \{b\}$ for some $a, b \in UC_{cof}$, $a \neq b$, while the restrictions of f, g to $(UC) \setminus (\overline{A} \cup \{x\})$ yield injections into $(UC_{cof}) \setminus \{a, b\}$; in the last two cases f, g satisfying $f(\overline{A}) = \{g(x)\} = \{0\}$, $g(\overline{A}) = \{f(x)\} = \{1\}$. Then the images of (A, x) by these functions will belong to $\mathcal{A}$ (codomain of these functions). Thus we have in each of the four cases, respectively: $(\{0\}, 1) \in \mathcal{A}(S)$, contradicting 1G); $(\{a\}, b), (\{b\}, a) \in \mathcal{A}(C_{cof})$, contradicting 1D); in the last two cases

$(\{0\},1),(\{1\},0) \in \mathcal{A}([0,1])$, contradicting 1D).

3) Now we show for each of the six categories that $\mathcal{A}(C)$ is in the given sense a base for $\mathcal{A}_{cl}(C)$.

First we deal with **PsMet**$_c$, **PsMet0-dim**$_c$. By 1F) $\exists A_1 \subset (U\hat{N}) \setminus \{0\}$, $|A_1| = \aleph_0$, $(A_1,0) \in \mathcal{A}(\hat{N})$. Clearly any $(A,x) \in \mathcal{A}^{\bullet}_{cl}(C)$, $C \in \mathcal{O}b\,$**PsMet**$_c$ or $\mathcal{O}b\,$**PsMet0-dim**$_c$ is of the form $(f(A_1), f(0))$ for some $f: \hat{N} \to C$, thus $(A,x) \in \mathcal{A}(C)$. The fact that the map $F^{\bullet}$ defined on objects by $F^{\bullet}C = (UC, \mathcal{A}^{\bullet}_{cl}(C))$ is the object part of a concrete full embedding to **SetPointSyst**$_0^0$, is evident.

Second we deal with **Top**, **Top**$_1$, **Tych**, **0-dim**, where the inductively generating class was {topological ultraspaces} for **Top**, {free topological ultraspaces} for **Top**$_1$, {free topological ultraspaces} $\cup$ {topological sums of a two-point indiscrete space and an arbitrary discrete space} for **Tych** and **0-dim**. Let C be an object of one of these categories. For $|UC| \le 1$ evidently $\mathcal{A}(C) = \mathcal{A}_{cl}(C)$. Let $|UC| \ge 2$, and let $A_0 \subset UC \ni x_0$, $\overline{A}_0 \ni x_0$. We will show $\exists (B,x_0) \in \mathcal{A}(C), B \subset A_0$. We suppose $A_0 \not\ni x_0$, since otherwise we can choose $B = \{x_0\}$.

Let the neighbourhood filter of x_0 in C be $\mathcal{F}$. Then $\mathcal{F} \cup \{A_0\} \subset \mathcal{U}$ for some ultrafilter $\mathcal{U}$ on UC. If $\mathcal{U}$ is fixed, then the point $u \in UC$ corresponding to $\mathcal{U}$ satisfies $u \in A_0$, and since $x_0 \notin A_0$, $x_0 \ne u$. In case of **Top**$_1$ $\mathcal{U}$ is free, since $x \in (UC) \setminus \{x_0\} \Longrightarrow (UC) \setminus \{x\} \in \mathcal{F} \subset \mathcal{U}$, and $(UC) \setminus \{x_0\} \supset A_0 \in \mathcal{U}$. Let C_0' denote the topological ultraspace on the set $UC_0' = UC$, constructed with $\mathcal{U}$ and x_0 (where $\mathcal{U} \ne$ the fixed ultrafilter associated with x_0, since otherwise $A_0 \ni x_0$). We have that 1_{UC} lifts to a continuous map $f: C_0' \to C$, and $x_0 \in \overline{A_0}$ holds also in C_0'.

In case of **Top** and **Top**$_1$ we let $C_0 = C_0'$. Also in case of **Tych** and **0-dim**, if $\mathcal{U}$ is free, we let $C_0 = C_0'$. In case of **Tych** and **0-dim**, if $\mathcal{U}$ is fixed, say $\cap\mathcal{U} = \{u\}$, we let C_0 the topological space with $UC_0 = UC_0'$, where each point $\ne x_0, u$ forms an open set, and a neighbourhood base of x_0, as well as of u, is $\{x_0, u\}$. We then have in C_0' $\overline{\{u\}} \ni x_0$, thus the same holds in C as well. Since now the T_0-reflection of C ($\in \mathcal{O}b\,$**Tych** or $\mathcal{O}b\,$**0-dim**) is T_1, we have $\overline{\{x_0\}} \ni u$ as well, thus also here 1_{UC} lifts to a continuous map $f: C_0 \to C$, and, since C_0 is coarser than C_0', $x_0 \in \overline{A_0}$ holds also in C_0.

Like in Proposition 3, it is enough to show $\exists B \subset A_0$, $(B, x_0) \in \mathcal{A}(C_0)$, since then $(B, x_0) = (f(B), f(x_0)) \in \mathcal{A}(C)$. Thus we turn to C_0.

First we deal with the case $C_0 = C_0'$. By 1C) $\exists (A_1, x_1) \in \mathcal{A}(C_0)$, $x_1 \notin A_1$. By 2) $x_1 \in \overline{A_1}$ in C_0. Thus $x_1 \in \overline{A_1} \setminus A_1$, which implies $x_1 = x_0$. Thus we have $x_0 \in \overline{A_0}$, $x_0 = x_1 \in \overline{A_1}$ in C_0, which implies by $x_0 \notin A_0$, $x_0 = x_1 \notin A_1$ that $A_0, A_1 \in \mathcal{U}$. Then we choose $\emptyset \ne B = A_0 \cap A_1 \subset A_0$. It remains to show $(B, x_0) \in \mathcal{A}(C_0)$. For this we define $g: C_0 \to C_0$ as follows:

g is identical on $B \cup \{x_0\}$, $g(A_1 \setminus B) \subset B$, g on $(UC_0) \setminus (A_1 \cup \{x_0\})$ is arbitrary. Thus $(B, x_0) = (g(A_1), g(x_0)) = (g(A_1), g(x_1)) \in \mathcal{A}(C_0)$, as was to be shown.

Second we turn to the case $C_0 \neq C_0'$, when, as mentioned above, C_0 is the topological sum $I + D$, I is a two-point indiscrete space $\{x_0, u\}$, D is a discrete space, $x_0 \notin A_0 \ni u$. We are going to show $(\{u\}, x_0) \in \mathcal{A}(C_0)$, which is sufficient, since $\{u\} \subset A_0$. We define $h: \hat{N} \to C_0$, $h(0) = x_0$, $h((U\hat{N}) \setminus \{0\}) = \{u\}$. By 1F) we have $(\{u\}, x_0) = (h(A_1), h(0)) \in \mathcal{A}(C_0)$, as was to be shown. □

PROPOSITION 6. *There exists a proper class of not naturally isomorphic concrete full embeddings* **PsMet**$_c$ $\to$**SetPointSyst**$_0^0$, *resp.* **PsMet0-dim**$_c$ $\to$**SetPointSyst**$_0^0$. *Moreover, if there does not exists a proper class of measurable cardinals (which is consistent with ZFC, provided ZFC is consistent), then there exist as many not naturally isomorphic concrete full embeddings* **PsMet**$_c$ $\to$ **SetPointSyst**$_0^0$, *as there are subclasses of a proper class.*

PROOF: Analogously to the remarks before Proposition 2, we have concrete full embeddings F_α from **PsMet**$_c$ or **PsMet0-dim**$_c$ to **SetPointSyst**$_0^0$, defined on objects by $F_\alpha C = (UC, \{(A, x) | A \subset UC \ni x,\ \overline{A} \ni x,\ |A| < \alpha\})$ — where $\alpha > \aleph_0$ is any cardinal —, which are pairwise not naturally isomorphic.

If there does not exist a proper class of measurable cardinals, then by [60] there exists a proper class $\mathbf{C} = \{C_\alpha\}$ of metric spaces with $|UC_\alpha| > 1$, such that any $g: C_{\alpha_1} \to C_{\alpha_2}$ is either constant or is an identity (with $\alpha_1 = \alpha_2$). Evidently each C_α is connected, hence $|UC_\alpha| \geq \exp \aleph_0$. If $\mathbf{C}' \subset \mathbf{C}$ is any subclass, define for $C \in Ob\,$**PsMet**$_c$ $F_{\mathbf{C}'}(C) = \{(A, x) | A \subset UC \ni x,\ \overline{A} \ni x,\ |A| \leq \aleph_0\} \cup \bigcup\{(f(B), f(y)) | B \subset UC_\alpha \ni y,\ \overline{B} \ni y,\ C_\alpha \in \mathbf{C}',\ f: C_\alpha \to C\}$. Like at Proposition 2 we see that this is the object part of a concrete full embedding **PsMet**$_c$ $\to$ **SetPointSyst**$_0^0$, and that these are not naturally isomorphic for different subclasses $\mathbf{C}'$ of $\mathbf{C}$. □

Analogously to Remark 3 after Proposition 4, we would have as many not naturally isomorphic concrete full embeddings **PsMet0-dim**$_c$ $\to$ **SetPointSyst**$_0^0$, as there are subclasses of a proper class, provided there was a proper class $\{C_\alpha\} \subset Ob\,$**PsMet0-dim**$_c$, such that $|UC_\alpha| > \aleph_0$, and $[\alpha_1 \neq \alpha_2 \Longrightarrow \forall f: C_{\alpha_1} \to C_{\alpha_2}\quad |f(UC_{\alpha_1})| < |UC_{\alpha_2}|]$ (using the formula for $F_{\mathbf{C}'}(C)$ from Proposition 6).

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