Category Theory at Work
H. Herrlich, H.-E. Porst (eds.)

149–164

Free Spectra of Concrete Categories and Mixed Structures

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There is no doubt that the concept of an adjoint functor is a fundamental notion of Categorical Algebra. It has been particularly well expressed by reducing the nature of "monadicity" of a functor to the existence of free objects and the validity of Beck's criterion [10]. However, in the theory of topological functors the importance of free objects is incomparably smaller - recall, for instance, that in the category of topological spaces free objects are discrete spaces.

Both classes: that of monadic and that of topological categories are opposite, "extreme wings" of the large class of concrete categories having, roughly speaking, mixed "topologically-algebraic character". This can be easily seen if we restrict to finitary case over the category of sets: finitary monadic categories are varieties of finitary algebras while finitary topological categories may be identified with categories of models for Horn theories of finitary, pure relational languages without equality. A wide class of categories of algebraic systems which are naturally recognized as categories of mixed structures is out of the reach of both theories. Thus the search of a theory containing both particular theories and describing possibly large class of categories of "topological-algebraic structures" arose as a natural consequence of the development of Categorical Algebra and Topology.

The fundamental question we have to answer trying to construct such a theory is the following: what is a proper categorical setting for it? The mentioned "asymmetry" of importance of free objects in both particular theories indicates that the notion of a right adjoint functor is not the best categorical setting for such a theory. Moreover, in categories such as: local commutative rings, locally compact spaces which we would like to include in our theory, the corresponding underlying functors into sets have no left adjoints.

Analyzing properties of underlying functors of many particular

This work is an original contribution and will not appear elsewhere.

concrete categories which are generally recognized as "categories of mixed structures" we observed a common characteristic feature of these functors. This led us to the notion of a *partially adjoint functor* which we suggest to consider as a starting point for a construction of a unifying theory.

By **Pos** we denote the category of posets and order preserving maps.

For a given functor $U:\mathbf{A} \longrightarrow C$, a subcategory D of **A** is called *U-perfect* if ObD = Ob**A** and for each **A**-morphism f, if $f \cdot e \in D$ for some $e \in D$ such that Ue is a coretraction in C, then f is in D, too.

Definition 1. A functor $U:\mathbf{A} \longrightarrow C$ is called D-adjoint for a given U-perfect subcategory $D \subseteq \mathbf{A}$ if there is a triple $(S, \tilde{J}, \eta)$ such that

$S: \mathrm{Ob}C \longrightarrow \mathrm{Ob}\mathbf{Pos}$,

$\tilde{J} = (\tilde{J}_X : SX \longrightarrow \mathbf{A} : X \in \mathrm{Ob}C)$,

$\eta = (\eta_X : \Delta X \longrightarrow U\tilde{J}_X : X \in \mathrm{Ob}C)$ (Δ denotes the "constant" functor)

subject to the following condition:

for each U-morphism $(h:X \longrightarrow UA, A)$ there exists a unique $i \in SX$ together with a unique D-morphism $\tilde{h}:\tilde{J}_X(i) \longrightarrow A$ such that $U\tilde{h}\cdot\eta_i^X = h$. If, moreover, $h_1:\tilde{J}_X(j) \longrightarrow A$ is such that $Uh_1 \cdot \eta_j^X = h$ then $j \le i$ and $h_1 = \tilde{h} \cdot \tilde{J}_X(j \le i)$.

We call the triple $(S, \tilde{J}, \eta)$ a *D-spectrum* of U. For an $X \in \mathrm{Ob}C$, the poset SX as well as the ordered family $(\tilde{J}_X(i) : i \in SX)$ is called a *D-spectrum of* U *over* X. A functor $U:\mathbf{A} \longrightarrow C$ is said to be *partially adjoint* iff it is D-adjoint for some $D \subseteq \mathbf{A}$.

If U is monadic, then U is right adjoint i.e. **A**-adjoint and every spectrum is a one-point set.

If U is topological and fibre-small, then U is In-adjoint, where In denotes the subcategory of U-initial morphisms. In this case the spectrum SX is the U-fibre over X.

The following examples will hopefully clarify this definition.

1. The underlying functor $U:\mathbf{Reg} \longrightarrow \mathbf{Set}$ from the category of regular spaces into sets is adjoint with respect to the subcategory of perfect continuous maps [2]. For a set X the spectrum SX consists of all subspaces of the ultrafilter space over X containing all prime ultrafilters. This remains true if we consider the restriction of U to the subcategory $\mathbf{Tich} \subseteq \mathbf{Reg}$ of Tichonov spaces. If we restrict to

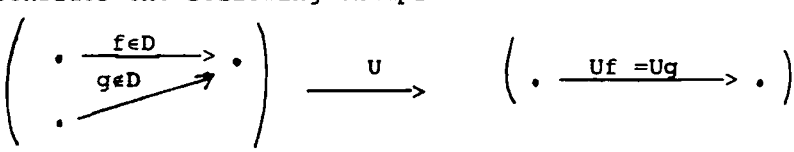
the subcategory Lcomp of locally compact spaces, then this fact remains true too, but then we have to restrict spectra SX to open sets they contain.

2. The underlying functor U:Lcrng——>Set from the category of local commutative rings is adjoint with respect to the subcategory of homomorphisms reflecting invertible elements. In this case for every set X the associated spectrum is the prime spectrum of the free commutative ring Z[X] but considered with the dual order.

3. Let U^{Ω}:PalgΩ——>Set be the underlying functor from the category of all partial algebras of a type Ω into sets. Recall that a homomorphism of partial Ω-algebras $h:(A,(q^A;q\in \Omega))$——>$(B,(q^B;q\in \Omega))$ is *strong* [4] provided for every $q\in \Omega_n$ and $a\in A^n$, $q^A(a)$ is defined iff $q^B(ha)$ is defined (and then, of course, $hq^A(a) = q^B(ha)$).

The functor U^{Ω} is adjoint with respect to the subcategory of strong homomorphisms. The spectrum of U^{Ω} over a set X is then the complete lattice of all initial segments over X (a subset X_o of the set of terms ΩX is an *initial segment* provided $X\subseteq X_o$ and for any term $t\in \Omega X$, whenever $t\in X_o$, then every subterm of t is in X_o, too).

4. A functor U:**A**——>C has a left multiadjoint [3] iff U is **A**-adjoint and then every spectrum SX is a discrete poset. More generally, if U:**A**——>C is D-adjoint, then the restriction $U_{|D}$:**D**——>C has a left multiadjoint. But the converse fails to be true. To see this, consider the following example:



The aim of this note is to present few results illustrating the usefulness of the concept of spectrum in investigation of categories of mixed structures. These results are chosen mainly from our earlier papers [6] - [8] .

Partially monadic categories. Up to now we use an intuitive notion of a "category of mixed topologically-algebraic character". Now, using the introduced concept we distinguish a wide subclass of these categories. This class will be the object of our investigation here.

Let U:**A**——>C be a D-adjoint functor with the D-spectrum $(S,\tilde{J},\eta)$.

Let $(\ )^*$ be an operator assigning to every pair $(f:Y \longrightarrow J_X(i), i \in SX)$ the pair $(f^* = U\tilde{f}:J_Y(j) \longrightarrow J_X(i),\ j \in SY)$, where $\tilde{f}:\tilde{J}_Y(j) \longrightarrow \tilde{J}_X(i)$ is the D-extension of $(f, \tilde{J}_X(i))$.

The 4-tuple $S_U = (S, J = U\tilde{J}, \eta, (\ )^*)$ is called a *spectral algebraic theory* (SAT, for short) generated by a D-adjoint functor $U:\mathbf{A} \longrightarrow C$.

Let S_U-Alg be the category such that objects (S_U-*algebras)* are triples $(A, i \in SA, a:J_A(i) \longrightarrow A)$ such that $a \cdot \eta_i^A = id$ and for any $f,g:X \longrightarrow J_A(i)$, $a \cdot f^* = a \cdot g^*$ provided $af = ag$.

Morphisms from (A,i,a) to (B,j,b) are C-morphisms $h:A \longrightarrow B$ such that $h \cdot a = b \cdot (\eta_j^B \cdot h)^* \cdot J_A(i \le k)$, where $dom(\eta_j^B \cdot h)^* = J_A(k)$.

If, moreover, $i = k$, then h is called *perfect*.

By $U_S:S_U\text{-Alg} \longrightarrow C$ we denote the obvious underlying functor.

Observe that U_S is adjoint with respect to the subcategory of perfect morphisms.

It can be easily checked that

$H:\mathbf{A} \longrightarrow S_U\text{-Alg}$ given by $H(A) = (UA,\ i \in SA, (id_{UA})^*)$, where $(id_{UA})^* = U(\widetilde{id}_{UA}):J_{UA}(i) \longrightarrow A$, is a well defined concrete functor.

Definition 2. A D-adjoint functor $U:\mathbf{A} \longrightarrow C$ is said to be D-monadic if the the comparison functor $H:\mathbf{A} \longrightarrow S_U\text{-Alg}$ is an isomorphism preserving and reflecting distinguished subcategories (i.e. Hf is perfect iff f is in D).

A functor $U:\mathbf{A} \longrightarrow C$ is said to be *partially monadic* if U is D-monadic for some U-perfect subcategory $D \subseteq \mathbf{A}$ and then $(\mathbf{A}, U)$ is called a *partially monadic category* (p.m. category, for short).

Remark. In a complete analogy with the theory of monadic categories one can formulate an abstract definition of an SAT and its (partially monadic) category of algebras (compare [6]).

The introduced notion generalizes the concept of a monadic functor in the sense that a right adjoint functor $U:\mathbf{A} \longrightarrow C$ is monadic iff U is $\mathbf{A}$-monadic. The associated SAT reduces then to the "algebraic theory in extension form" ([9]). In this particular case every S-algebra (A,i,a) is fully described by the pair (A,a) since every spectrum is a one element set.

Every topological category with small fibres is In-monadic and the associated SAT is the corresponding topological theory in the sense of Wyler. Every S-algebra (A,i,a) is then fully described by the pair (A,i) since the structure map $a:J_A(i) \longrightarrow A$ must be the identity morphism.

All the functors discussed in the examples above are D-monadic, except the underlying functor U:Tich——>Set.

The class of partially monadic functors is distinguished within the class of partially adjoint functors essentially in the same way as monadic functors have been distinguished within the class of right adjoint functors. This similarity is particularly well expressed by the following fundamental result:

Theorem 1. *(generalized Beck's criterion)* $U:\mathbf{A} \longrightarrow C$ is D-monadic iff U is D-adjoint and the following holds:
for any pair of parallel D-morphisms $f,g:A \longrightarrow B$ such that (Uf,Ug) may be completed to a contractible coequalizer (Uf,Ug,e,s,t) in C (i.e. $e\cdot Uf = e\cdot Ug$, $e\cdot s = id$, $Uf\cdot t = id$, $s\cdot e = Ug\cdot t$), there is a unique lift $\bar{e}:B \longrightarrow C$ of e in D and, moreover, $\bar{e}$ is a coequalizer of the pair (f,g) in D preserved by the embedding $D \subseteq \mathbf{A}$.

Proof. Because of space-limitation we omit the proof. We hope that the reader familiar with the classic Beck's theorem for monadic functors ([10]) will be able to prove this theorem without any serious difficulties (compare also [6]).

Example. Let $S^{\Omega} = (S^{\Omega}, J, \eta, (\ )^{*})$ be the SAT corresponding to the functor $U^{\Omega}:\mathrm{Palg}\Omega \longrightarrow \mathrm{Set}$. Three first components of S^{Ω} have been described earlier. The operator $(\ )^{*}$ is defined using inverse images as follows: for a given initial segment $X_o \subseteq \Omega X$ and $f:Y \longrightarrow X_o$ we construct the commutative diagram

$$\begin{array}{ccc} \Omega Y & \xrightarrow{\ \widetilde{mf}\ } & \Omega X \\ \uparrow & & \uparrow m \\ (\widetilde{mf})^{-1}(X_o) & \xrightarrow{\ f^{*}\ } & X_o \\ \uparrow & \nearrow f & \\ Y & & \end{array}$$

(where $\widetilde{mf}$ is the homomorphic extension of $mf:Y \longrightarrow \Omega X$).

A partial Ω-algebra $\mathcal{A}$ determines the S^{Ω}-algebra $\mathcal{A} = (A = U^{\Omega}\mathcal{A}, A_1, a)$, where A_i is an initial segment consisting of all Ω-terms over A determined in $\mathcal{A}$ while the structure map $a:A_i \longrightarrow A$ assigns to each Ω-term t in A_i its value in $\mathcal{A}$.

Thus the fundamental concepts of the theory of p.m.categories are introduced very much in the same way as it has been done in Categorical Algebra with the theory of monadic functors. The only but fundamental difference is that the concept of a right adjoint functor is replaced here by a partially adjoint functor. However, many results concerning monadic (and topological) functors can be generalized for p.m.functors. In the present note we concentrate mainly on these problems and results which have no nontrivial counterparts in both particular theories.

Completeness and completions. In contrary to monadic and topological categories, there are p.m. categories over complete categories which are not concretely complete. Recall, for instance the categories **Lcomp** and **Lcrng**.

Using the concept of spectrum we get the following criterion for concrete completeness.

Theorem 2. Let $U:\mathbf{A} \longrightarrow C$ be a D-monadic functor with the associated SAT $S = (S,J,\eta,(\ )^*)$. Assume that C is complete. Then

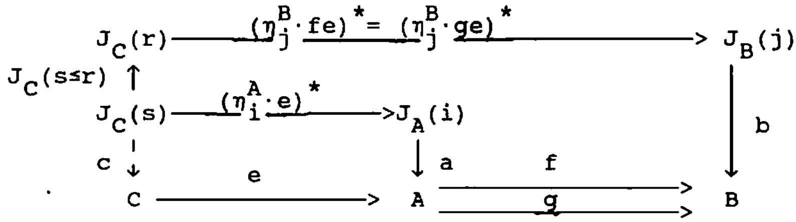
$(\mathbf{A},U)$ is concretely complete iff every SX is a complete lattice and the operator $(\ )^*$ preserves infima, i.e.

for any $f:Y \longrightarrow J_X(i)$, with $i = \inf K$ for some $K \subseteq SX$, if

$$(J_X(i \le k)\cdot f)^*:J_Y(j(k)) \longrightarrow J_X(k)) \quad \text{for every } k \text{ in } K, \text{ then}$$

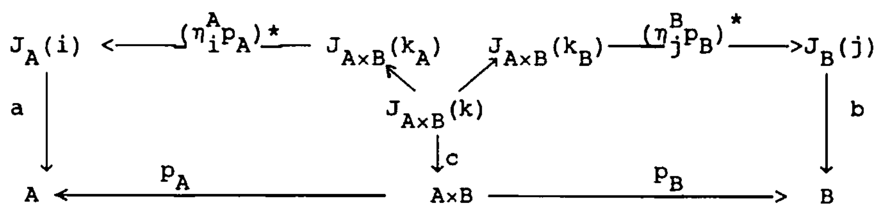
$f^*:J_Y(j) \longrightarrow J_X(i)$, where $j = \inf(j(k);k \in K)$.

Proof. We may assume $\mathbf{A}$ = S-Alg. First we show that $\mathbf{A}$ has concrete equalizers. Let $f,g:(A,i,a) \longrightarrow (B,j,b)$ in S-Alg and assume that e is an equalizer of (f,g) in C. Consider the commutative diagram:



Thus there exists $c: J_C(s) \longrightarrow C$ such that $a \cdot (\eta_i^A e)^* = e \cdot c$.
Now it is a routine to check that $(C, s \in SC, c)$ is an S-algebra and e is an equalizer in S-Alg of the pair considered.

It remains to discuss the existence of products. For notational simplicity, we restrict to finite products and finite infima. Assume (A,i,a), (B,j,b) are S-algebras and AxB is a product of A and B in C with projections p^A and p^B, resp. Consider the commutative diagram



where $k = \inf(k_A, k_B)$ in S(AxB).

It is easily checked that (AxB,k,c) is a product of the considered pair of S-algebras.

To prove necessity, for given $i, j \in SX$ consider a product of $\tilde{J}_X(i)$ and $\tilde{J}_X(j)$ and a map $\eta: X \longrightarrow J_X(i) \times J_X(j)$ determined by the pair (η_i^X, η_j^X). Let $\tilde{\eta}: J_X(k) \longrightarrow \tilde{J}_X(i) \times \tilde{J}_X(j)$ be the perfect extension of η. One can prove that $k = \inf(i,j)$.
We left the details of the proof to the reader (compare [7]).

In particular, this criterion gives methods of "completion of a partially monadic category to a complete one" based on well known methods of completion of posets to complete lattices ([7]).
We formulate here one of them:

Theorem 3 (*universal completion*). For any p.m. category $((\mathbf{A},U), D \subseteq \mathbf{A})$ over a complete base category C there is a concretely complete p.m. category $((\mathbf{A}_C, U_C), D_C)$ together with a concrete embedding $Z: \mathbf{A} \longrightarrow \mathbf{A}_C$

such that $Z(D) \subseteq D_c$ and the following holds true:

for any concrete functor $F: \mathbf{A} \longrightarrow \mathbf{A}_1$ into a concretely complete p.m. category $((\mathbf{A}_1, U_1), \mathbf{D}_1)$ such that $F(\mathbf{D}) \subseteq \mathbf{D}_1$ there exists a unique concrete and limit preserving functor $F_c: \mathbf{A}_c \longrightarrow \mathbf{A}_1$ such that

$F(\mathbf{D}_c) \subseteq \mathbf{D}_1$ and $F_c Z = F$.

Proof. (*sketch*) Let $(S, J, \eta, (\ )^*)$ be the SAT associated to $((\mathbf{A}, U), \mathbf{D})$. We construct a new SAT $(S^c, J^c, \eta^c, (\ )^c)$ as follows: for every X in C, $S^c X$ is the universal completion of the poset SX, i.e.

$S^c X = \{A \subseteq SX;\ A \text{ is increasing}\}$ and $A \leq B$ iff $B \subseteq A$ for $A, B \in SX$,

and for every $A \in S^c X$;

$J^c_X(A) = \lim(J_X(j);\ j \in A)$, while

$(\eta^c)^X_A$ is a morphism determined by the family $(\eta^X_j;\ j \in A)$

The operator $(\ )^c$ is defined by the following diagram:

$$\begin{array}{ccc}
J_Y(i(j)) & \xrightarrow{\quad (p^j h)^* \quad} & J_X(j) \\
\uparrow & & \uparrow p^j \ \text{(limit projection)} \\
J^c_Y(A_c) & \xrightarrow{\quad h^c \quad} & J^c_X(A) = \lim(J_X(i): i \in A) \\
\uparrow & \nearrow_{h} & \\
Y & &
\end{array}$$

(where $A_c = \{\ k \in SY;\ k \geq i(j) \text{ for some } j \in A\}$).
The category of S^c-algebras is the required universal completion of $((\mathbf{A}, U), \mathbf{D})$ ([7]).

Remark: Underlying functors of concretely complete p.m. categories over sets are topologically-algebraic ([5]).

In a p.m.category $(\mathbf{A}, U), \mathbf{D})$ having a concrete terminal object, we distinguish a subcategory of *total* objects. An object A in $\mathbf{A}$ is said to be total, iff the unique morphism from A to the terminal object is in $\mathbf{D}$.

Lemma 4.. The full subcategory of total objects is monadic over the base category.

Proof. If total objects exist, spectra of the underlying functor have greatest elements preserved by the operator $(\ )^*$. Observe that total objects are S-algebras (A,i,a) such that i is the greatest element of SA. Now the lemma is obvious.

Under some additional assumptions on base category, (for instance, for p.m. categories over sets) the category of total objects is a reflective subcategory of a p.m.category considered [7].

Examples. Total objects in PalgΩ are total Ω-algebras. In **Reg** and **Lcomp** total objects are compact spaces. **Tich** is the subcategory of these objects in **Reg** which are embeddable into total objects. The category **Lcrng** has no terminal object.

Finitary partially monadic categories over Set. The following definition covers a wide class of p.m.categories over sets.

Definition 3. A class V of partial Ω-algebras (or a full subcategory it determines) is called a weak variety if the following holds:

i.) if $e:A \longrightarrow B$ is strong and $A \in V$, then $B \in V$,

ii.) if $(m^i:A \longrightarrow A^i : i \in I)$ is a monosource containing at least one strong homomorphism and every A^i is in V, then $A \in V$.

Lemma 5. Every weak variety is a p.n.category over sets.

Proof. Recall that for any type Ω, PalgΩ is partially monadic with respect to the subcategory $PalgΩ_s$ of strong homomorphisms. Let $V \subseteq PalgΩ$ be a weak variety. We show that the underlying functor $U:V \longrightarrow Set$ is $(V \cap PalgΩ_s)$-monadic. Observe the following: for every initial segment X_i, if the source $(h:X_i \longrightarrow UA: A \in V)$ contains at least one strong homomorphism, then it has a factorization

$$(X_i \xrightarrow{e_i} X_i^V \xrightarrow{(m_h)} UA : A \in V)$$

such that e_i is strong and surjective and (m_h) is a monosource containing strong homomorphism. Thus X_i^V is in V. It is clear that the family of all X_i^V's is the desired spectrum of U over X. The first condition of Definition 3 guarantees that U satisfies the generalized Beck's criterion.

Let $L^P_{\infty,\infty}(\Omega)$ be an infinitary language with atomic formulas of the following forms ([1]): $\exists t$ (*existential atomic formulas*) and $(t = t_1)$

(*equations*), where t, t_1 are Ω-terms .

A partial Ω-algebra $\mathcal{A}$ satisfies the formula $\exists t$ at a valuation of variables $h: X \longrightarrow A$ ($\mathcal{A} \models \exists t[h]$) iff the term t exists in $\mathcal{A}$ at the valuation h. $\mathcal{A} \models (t = t_1)[h]$ iff $\mathcal{A} \models (\exists t \wedge \exists t_1)[h]$ and the values of t and t_1 at h coincide. Then we extend the validity relation in the usual way (Here we use quantifier free formulas only. Infinite conjunctions ($\bigwedge$) and disjunctions ($\bigvee$) are allowed.).

Proposition 6. A class $\mathbf{V} \subseteq \mathrm{Palg}\Omega$ is a weak variety iff $\mathbf{V} = \mathrm{Mod}\Phi$, where Φ is a class of $L^P_{\infty,\infty}(\Omega)$-formulas of the form:

(1) $\bigvee_{i \in I} (\bigwedge \Phi_i^+ \wedge \bigwedge \Phi_i^-)$ or

(2) $\bigwedge \Phi^+ \Rightarrow \bigwedge \Phi^=$

where every Φ_i^+ and Φ^+ are sets of existential atomic formulas, every Φ_i^- is a set of negations of formulas of this kind while $\Phi^=$ is a set of equations (for the proof we refer the reader to [8]).

Examples. Local commutative rings, small categories, posets, partial monoids are weak varieties of partial algebras of suitable types.

This concept generalizes the notion of a variety of total algebras in the sense that a nonempty weak variety consisting of total algebras only is a variety.

The next theorem indicates the role of weak varieties in the theory of p.m.categories. Throughout $(\mathbf{A},U),D)$ is a p.m. category over **Set** with the associated SAT $(S, J, \eta, (\ )^*)$. For simplicity, we assume that $\mathbf{A}$ contains at most one object with an empty carrier. For a set X by FinX we denote the directed system of all finite subsets of X. For any $i \in SX$ and an embedding $w_Y: Y \longrightarrow X$, where $Y \in \mathrm{Fin}X$, by $i(Y) \in SY$ we denote the element such that

$$(\eta_i^X w_Y)^*: J_Y(i(Y)) \longrightarrow J_X(i).$$

Theorem 7. Assume that the following holds true for every set X:

i.) For every $j \in SX$, $i \le j$ iff $i(Y) \le j(Y)$ for every $Y \in \mathrm{Fin}X$, (i.e. SX is a subset of $\lim(SY : Y \in \mathrm{Fin}X)$ with the induced order),

ii.) For every $i \in SX$, $(J_X(i), ((\eta_i^X w_Y)^*: J_Y(i(Y)) \longrightarrow J_X(i) ;\ Y \in \mathrm{Fin}X)$ is an epi-cocone in **Set**.

Then there exists a weak variety $\mathbf{V}$ of partial algebras of a finitary type together with a concrete isomorphism $H:\mathbf{A}\longrightarrow\mathbf{V}$ such that for every f in **A**, $f\in D$ iff Hf is a strong homomorphism.

Proof.(*sketch*) For every $n\in N$, put $\Omega_n=$ the disjoint union of the family $\{J_n(i); i\in Sn\}$, where $n = \{0,1,\ldots,n-1\}$.
For any $A\in \mathbf{A}$, put $H(A) = (UA,(\tilde{q}))$, where for $q\in J_n(k)$, and $h:n\longrightarrow UA$ with the perfect extension $\tilde{h}:\tilde{J}_n(j)\longrightarrow A$,

$$\tilde{q}(h) \text{ is defined iff } k\le j \text{ and then } \tilde{q}(h) = \tilde{h}\cdot J_n(k\le j)(q).$$

We left the proof of the fact that H(**A**) is a weak variety of partial Ω-algebras to the reader (compare [6]).

In general, weak varieties are not elementary, i.e. they are not always classes of models for first order theories.

A p.m. category ((**A**,U),D) is said to be *strongly finitary* iff it is representable (in the sense of Theorem 7) by an elementary weak variety. A characterization of strongly finitary p.m. categories in terms of their spectra needs more subtle methods.
First we distinguish so called *finitary* p.m.categories over sets as follows: a p.m.category ((**A**,U),D) is finitary iff for every set X:

i^o. $SX = \lim(SY: Y\in FinX)$

ii^o. For every $i\in SX$, $(J_X(i),((\eta_i^X w_Y)^*:J_Y(i(Y))\longrightarrow J_X(i);\ Y\in FinX)$ is a colimit cone (compare Theorem 7).

A p.m. category is finitary iff it has a representation as a weak variety $V = Mod\Phi$ such that sets of variables occurring in formulas in Φ are finite (i.e. every ϕ in Φ is an $L^P_{\infty,\omega}(\Omega)$-formula ([8]).

Strongly finitary p.m.categories form a subclass of finitary p.m. categories. To distinguish them we need the concept of a *Priestley space*. An ordered, compact space $(X,\le,\mathcal{T})$ is called a Priestley space iff it is *totally order disconnected* i.e. for every $i,j\in X$, $i\le j$ iff for every clopen increasing set A, $j\in A$ provided $i\in A$ ([11]).

Theorem 8. A finitary p.m.category ((**A**,U),D) with the associated SAT $(S,J,\eta,(\ )^*)$ is strongly finitary iff for every $n\in N$, Sn carries a structure of a Priestley space $(Sn,\mathcal{T}^n)$ subject to the following conditions:

(below $\mathcal{T}^n_+$ denotes the subtopology of $\mathcal{T}^n$ with an open base consisting of all $\mathcal{T}$-clopen increasing subsets).

i.) There is a presheaf $\bar{J}_n:(\mathcal{T}^n_+)^{op}$——>Set such that every $J_n(i)$ is the stalk of $\bar{J}_n$ at i and every $J_n(i\le j)$ is the stalk-morphism,

ii.) There exists a natural transformation $\bar{\eta}^n:\Delta n$——>$\bar{J}_n$ such that for every $U\in \mathcal{T}^n_+$ and $i\in U$, $\eta^n_i = \bar{\phi}_{Ui}\cdot\bar{\eta}^n_U$

($(\bar{\phi}_{Ui}:\bar{J}_n(U)$——> $J_n(i)$: $U\in \mathcal{T}^n_+$, $i\in U)$ denotes the stalk cone)

iii.) For every $U\in \mathcal{T}^n_+$ and $f:m$——>$\bar{J}_n(U)$ there is a partial morphism of presheaves ($\hat{f}:U$——>Sm, $\tilde{\tilde{f}}:\bar{J}_m$——>$\bar{J}_n\hat{f}^{-1}$) such that for any $B\in \mathcal{T}^m$ and $i\in \hat{f}^{-1}(B)$ the diagram

$$\begin{array}{ccc} J_m(\hat{f}(i)) & \xrightarrow{(\bar{\phi}_{Ui}\cdot f)^*} & J_n(i) \\ \bar{\phi}_{B\hat{f}(i)}\uparrow & & \uparrow\bar{\phi}_{\hat{f}^{-1}(B)i} \\ \bar{J}_m(B) & \xrightarrow{\tilde{\tilde{f}}_B} & \bar{J}_n\hat{f}^{-1}(B) \\ \bar{\eta}^m_B\uparrow & & \uparrow \\ m & \xrightarrow{f} & \bar{J}_n(U) \end{array}$$

is commutative.

Roughly speaking, for strongly finitary p.m. categories spectra may be "recovered" from presheaves defined on associated Priestley spaces.

The following indicates the role of the associated presheaf in the proof of sufficiency: the disjoint sum of $\bar{J}_n(i)$'s serves as a set of n-ary operation symbols (compare Theorem 7). The full proof of this theorem is too long, so we refer the reader to [8].

We hope that the following remarks will clarify the role of Priestley topologies in this theorem. Spectra over finite sets of the functor U^{Ω}:PalgΩ——>Set are algebraic lattices of initial segments. Hence, endowed with the Lawson topologies (with open subbases consisting of all sets $\uparrow n_i=\{n_j:\ n_i\subseteq n_j\}$ and $S^{\Omega}n\setminus\uparrow n_i$,where n_i is finite) are Priestley spaces. For a family of initial segments $\{n_i:i\in I\}\subseteq Sn$ and an ultrafilter $\mathcal{U}$ on I let X be the ultraproduct of this family in PalgΩ. Let $f:n$——>UX be the function determined by the family ($\eta^n_i:n$——>$n_i:i\in I$) and let $n_j\in S^{\Omega}n$ be the domain of the strong extension of the U^{Ω}-morphism ($f:n$——>$U^{\Omega}X,X$). Then n_j is a limit point of the ultrafilter $\mathcal{U}$ in the Lawson space $S^{\Omega}n$.

Thus, if $V\subseteq$ PalgΩ is a weak variety closed under ultraproducts, then its spectra may be identified with closed subsets of lattices of initial segments. Hence they are Priestley spaces. (The associated presheaves will be described in the last section.)

Spectra and implicit operations. Throughout $((\mathbf{A},U),D)$ is a p.m. category over Set with the associated SAT $S = (S,J,\eta,(\ )^*)$.

By a *partial n-ary implicit operation* in $\mathbf{A}$ we mean a family of partial functions $\phi = \{\phi_A : UA^n \longrightarrow UA : A \in \mathbf{A}\}$ such that for every $\mathbf{A}$-morphism $h:A \longrightarrow B$,

$h\cdot\phi_A \le \phi_B \cdot h^n$ and, moreover, $h\cdot\phi_A = \phi_B\cdot h^n$ provided h is in D.

Here we consider only nontrivial operations i.e. having at least one domain nonempty .

Example. In any weak variety of partial algebras every term-operation (defined in the usual sense) is an implicit operation.

Proposition 9. Every n-ary implicit operation ϕ in $\mathbf{A}$ is uniquely determined by the following data:

- an increasing set $X_\phi \subseteq Sn$,
- an element $r_\phi = (r_\phi^i) \in \lim(J_n(i) : i \in X_\phi)$

in the sense that for any $A \in \mathrm{Ob}\mathbf{A}$ and $h:n \longrightarrow UA$, with the perfect extension $\tilde{h}:\tilde{J}_n(j) \longrightarrow A$, $\phi_A(h)$ is defined iff $j \in X_\phi$ and then

$$\phi_A(h) = \tilde{h}(r_\phi^j) ,$$

Proof. Put $X_\phi = \{i \in Sn : \phi(\eta_i^n) \text{ is defined}\}$ and $r_\phi^i = \phi(\eta_i^n)$ for every i in X_ϕ .

An operation ϕ is said to be *irreducible* whenever

$\phi \subseteq \bigcup(\gamma^i : i \in I)$, implies that $\phi \subseteq \gamma^j$ for some $j \in I$

($\phi \subseteq \gamma$ means $\phi_A \subseteq \gamma_A$ for every A in $\mathbf{A}$). Directly from Proposition 9 we get the following.

Corollary 10.i. Every irreducible operation is uniquely determined by a pair $(i \in Sn,\ q \in J_n(i))$ in the sense described in Proposition 9.

ii.Every implicit operation is a join of irreducible operations.

Assume that the p.m.category considered is strongly finitary. We shall follow the notation of Theorem 8. For every pair $(U,\ t \in \bar{J}_n(U))$ where U is $\mathcal{T}$-clopen and increasing we define an n-ary implicit operation $\phi_{(U,t)}$, where

$\phi_{(U,t)}$ is defined on $h:n \longrightarrow UA$ with $\tilde{h}:J_n(j) \longrightarrow A$ provided $j \in U$

and then $\phi_{(U,t)}(h) = \tilde{h}\phi_{Uj}(t)$.

We call these operations *𝒯-operations*. 𝒯-operations form a "dense" subset of implicit operations in the following sense:

Proposition 11. If ϕ is an irreducible n-ary operation in **A** then for every $h:n \longrightarrow UA$, ϕ is defined on h iff for every 𝒯-operation γ such that $\phi \leq \gamma$, $\gamma(h)$ is defined. If ϕ and ϕ' are irreducible operations, then $\phi = \phi'$ iff for every 𝒯-operation γ, $\phi \leq \gamma$ iff $\phi' \leq \gamma$.

Proof. This is a straightforward consequence of the fact that "every $J_n(i)$ is the stalk of the presheaf $\bar{J}_n$ at i".

Theorem 12. The following conditions are equivalent:

i.) every irreducible n-ary implicit operation is a 𝒯-operation,

ii.) for every $i \in Sn$ the set $\uparrow i = \{j \in Sn: i \leq j\}$ is clopen,

iii.) every decreasing chain of clopen increasing sets in Sn stops after finite steps.

Proof is an easy exercise.

Clearly, the meaning of a 𝒯-operation depends of the choice of topologizations of spectra. However, from Theorem 12 we conclude that the following properties of spectra are consequences of the fact that all irreducible operations are 𝒯-operations with respect to some topologization of spectra:

- every spectrum satisfies the ascending chain condition,
- every spectrum has a finite set of minimal elements. Every element of a spectrum is greater then or equal to some minimal element.

Lemma 13. Assume that **A** is finitary and let $\mathbf{A} \cong \mathbf{V} \leq \mathrm{Palg}\Omega$ be a fixed representation of **A**. For every implicit operation ϕ in **A** there is a pair $((\Phi_i : i \in I), (t_i : i \in I))$, where every Φ_i is a set (maybe infinite) of existential atomic formulas with variables in n,
every t_i is an Ω-term with variables in n,
and $\mathbf{V} \leq \mathrm{Mod}(\bigwedge \Phi_i \wedge \bigwedge \Phi_j \rightarrow (t_i = t_j) : i,j \in I)$
such that for any $\mathcal{A}$ in **V** and $h:n \longrightarrow U\mathcal{A}$,

$\phi_{\mathcal{A}}(h)$ is defined iff $\mathcal{A} \models \bigwedge \Phi_i[h]$ for some $i \in I$ and then

$\phi_A(h) = \tilde{h}(t_i)$ (= the value of t_i at h)

Proof follows directly from Proposition 9 and the construction of spectra of weak varieties.

Now assume that **A** is strongly finitary and $\mathbf{A} \cong \mathbf{V} \leq \mathrm{Palg}\Omega$,where **V** is a weak variety closed under ultraproducts. Let $(S^V, J^V, \eta^V, (\ )^V)$ be the associated SAT. Every poset $S^V n$ is a closed subset of the Lawson space $S^\Omega n$ while for every $n_i \in S^V n$, $J^V_n(i)$ is a strong quotient of the initial segment n_i (compare Lemma 5). Let $\mathcal{T}^n$ be a topology on $S^V n \leq S^\Omega n$. We are going to describe implicit $\mathcal{T}$-operations.

The associated presheaf $\bar{J}^V_n : (\mathcal{T}^n_+) \longrightarrow \mathrm{Set}$ may by defined as follows: for any $\mathcal{T}^n$-clopen and an increasing set X in $S^V n$ consider the source

$(n_X \longrightarrow n_i \xrightarrow{\ e_i\ } J^V_n(i) \ : i \in X \leq S^V n\ )$

where n_X is the meet (in $S^\Omega n$) of all initial segments in X.

Then $\bar{J}^V_n(X)$ is a central object of an (epi,monosource)-factorization of this source.

Every $\mathcal{T}$-clopen and increasing set X is a finite sum of sets of the form $X_T = \{\ n_i \in S^V n : T \leq n_i\ \}$, where T is a finite set of Ω-terms with variables in n. Finally we get

Lemma 14. Every n-ary $\mathcal{T}$-operation ϕ in **V** is a "finitely definable restriction of a term operation" i.e. there is a term $t \in \Omega n$ and a finite family $(\Phi_i : i \in I)$ of finite sets of existential atomic formulas describing ϕ accordingly to the rule stated in Lemma 12.

Remark. Let $\hat{J}^V_n$ be the canonical sheaf associated to the presheaf $\bar{J}^V_n$. Every pair $(U,\ q \in \hat{J}^V_n(U))$, where U is $\mathcal{T}$-clopen and increasing, determines an implicit operation $\phi_{(U,q)}$ (in the same way as in the case of $\mathcal{T}$-operations) .Every such operation is a finite join of $\mathcal{T}$-operations. One can call them also finitely definable because every such operation has a logical description $((\Phi_i : i \in I), (t_i : i \in I))$ (Lemma 12) such that the set I and all sets Φ_i are finite.

Example. Let $\Omega = \Omega_1 = \{p,q,r,s\}$ $\mathbf{V} = \mathrm{Mod}(\exists p(x) \wedge \exists q(x) \Rightarrow r(x) = s(x)) \leq \mathrm{Palg}\Omega$. Then $\phi = \{\phi_A : A \in \mathbf{V}\}$, where for every $A \in \mathbf{V}$ and $a \in A$, $\phi_A(a)$ is defined provided p(a) or q(a) is defined in A and then

$$\phi_A(a) = \begin{cases} r(a) & \text{if } p(a) \text{ is defined,} \\ s(a) & \text{if } q(a) \text{ is defined.} \end{cases}$$

is a finitely definable operation in **V**.

Open problem. For a given concrete category (**A**,U) the underlying functor may be partially monadic with respect to different perfect subcategories of **A**. We are able to show that in general there is no the greatest perfect subcategory D of **A** making the underlying functor partially monadic ([7]). We would like to ask the following question:

- is it true that, if there exists any, there exists a maximal perfect subcategory making the functor considered partially monadic?

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