Category Theory at Work
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Algebra $\cup$ Topology

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Abstract

The question "What is Algebra $\bigcup$ Topology?" is asked and answered.

1 The Question

The question "What is Algebra $\bigcup$ Topology?" first requires explanations of the terms "algebra", "topology", and "union".

1.1 Algebra

For the purpose of this note we interpret "algebra" as "algebraic structure", or more technically as "algebraic category"; i.e., as a concrete category $(\mathbf{A}, U)$ over a base category $\mathbf{X}$ with forgetful functor $U: \mathbf{A} \to \mathbf{X}$ that is *algebraic*, or at least is *essentially algebraic*. In order to define these last two concepts without entering into discussions about uninteresting technical details we will assume the following throughout this article

Conventions:

(C1) All categories are (Epi, Mono-Source)-factorizable[1], and

(C2) All functors are faithful and uniquely transportable[2].

Definition 1.1 *A functor $U: \mathbf{A} \to \mathbf{X}$ is called*

(1) essentially algebraic *provided that U is adjoint and reflects isomorphisms.*

(2) algebraic *provided that U is essentially algebraic and preserves extremal epimorphisms.*

(3) regularly monadic *provided that U is monadic and preserves regular epimorphisms and $\mathbf{X}$ has regular factorizations.*

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[1] The restriction (C1) is not as harsh as it looks, since whenever $U: \mathbf{A} \to \mathbf{X}$ is essentially algebraic or topological, then the (Epi, Mono-Source)-factorizability of $\mathbf{A}$ follows from that of the base category $\mathbf{X}$.

[2] Unique transportability is equivalent to being transportable and amnestic.

This work is an original contribution and will not appear elsewhere.

Observe the following facts:

(1) A functor U is essentially algebraic if and only if U is adjoint and U reflects extremal epimorphisms. Cf. [1], Proposition 23.2. Moreover, essentially algebraic functors reflect identities. Cf. [1], Proposition 13.36(3).

(2) If $U: \mathbf{A} \to \mathbf{X}$ is algebraic and $\mathbf{X}$ has regular factorizations, then U preserves regular epimorphisms. Cf. [1], Proposition 23.39.

(3) Every regularly monadic functor is algebraic. Cf. [1], Proposition 23.36.

(4) Every monadic functor $U: \mathbf{A} \to \mathbf{Set}$ is regularly monadic (cf. [1], Theorem 20.35) and hence is algebraic.

1.2 Topology

Likewise we interpret "topology" as "topological structure", or more technically as "topological category"; i.e., as a concrete category $(\mathbf{A}, U)$ over a base category $\mathbf{X}$ with *topological* forgetful functor U.

Definition 1.2 *A functor $U: \mathbf{A} \to \mathbf{X}$ is called (*mono*) *topological* provided that each structured (mono)source $(X \xrightarrow{f_i} UA_i)_I$ has a U-initial lift $(A \xrightarrow{\bar{f}_i} A_i)_I$.*

1.3 Union

Since the formation of complex structures from simpler ones corresponds to the composition of the corresponding forgetful functors, it seems natural to interpret the word "union" in the title as the compositive hull of the set-theoretic union.

Thus the vague question "What is Algebra ∪ Topology?" can be stated more precisely as: "What is the compositive hull of the conglomerate of all functors that are essentially algebraic (resp. algebraic, resp. regularly monadic) or topological?" i.e., what is the smallest conglomerate of functors that is closed under composition and contains all essentially algebraic (resp. algebraic, resp. regularly monadic) functors and all topological functors?

2 The Answer

Possible candidates for answers to the above questions are given by the following concepts:

Definition 2.1 *A functor $U: \mathbf{A} \to \mathbf{X}$ is called*

(1) **topologically algebraic** *provided that each U-structured source $(X \xrightarrow{f_i} UA_i)_I$ has a (Generating, Initial-Source)-factorization*

$$X \xrightarrow{f_i} UA_i = X \xrightarrow{g} UA \xrightarrow{Um_i} UA_i,$$

(2) **solid** *provided that each U-structured sink $(UA_i \xrightarrow{f_i} X)_I$ has a semi-final arrow $X \xrightarrow{e} UA_i$ (which is universal with respect to making all composites $UA_i \xrightarrow{e \circ f_i} UA$ be $\mathbf{A}$-morphisms).*

Theorem 2.2 *For any functor $U: \mathbf{A} \to \mathbf{X}$ the following are equivalent:*

1. *U belongs to the compositive hull of all essentially algebraic and all topological functors,*
2. *U is solid,*
3. *U can be expressed as the composite of two topologically-algebraic functors,*
4. *U can be expressed as the composite $U = T \circ A$ of an essentially-algebraic functor A with a topological functor T.*

Proof:
(2) ⇒ (4). If $\mathbf{A} \xrightarrow{U} \mathbf{X}$ is a solid functor, consider $(\mathbf{A}, U)$ as a concrete category over $\mathbf{X}$, and let $(\mathbf{B}, V)$ be the following concrete category over $\mathbf{X}$: objects are pairs $(X, (e, A))$, consisting of an $\mathbf{X}$-object X and a semifinal U-structured arrow $X \xrightarrow{e} UA$; morphisms $(X, (e, A)) \xrightarrow{f} (X', (e', A'))$ are those $\mathbf{X}$-morphisms $X \xrightarrow{f} X'$ for which there exists a (necessarily unique) $\mathbf{A}$-morphism $A \xrightarrow{g} A'$ with $e' \circ f = Ug \circ e$. The concrete functor $(\mathbf{A}, U) \xrightarrow{E} (\mathbf{B}, V)$ defined by $E(A) = (UA, (id_{UA}, A))$ is a full embedding. Moreover, for each $\mathbf{B}$-object $(X, (e, A))$, the E-structured arrow $(X, (e, A)) \xrightarrow{e} EA$ is E-universal. Hence $\mathbf{A} \xrightarrow{E} \mathbf{B}$ is a full reflective embedding. In $(\mathbf{B}, V)$ every structured source $\mathcal{S} = (X \xrightarrow{f_i} V(X_i, (e_i, A_i)))_I$ has a (not necessarily unique) initial lift. This can be seen as follows: in $(\mathbf{A}, U)$ consider the structured sink $\mathcal{T} = (UA_k \xrightarrow{g_k} X)_K$ that consists of all pairs (A_k, g_k) such that $UA_k \xrightarrow{e_i \circ f_i \circ g_k} UA_i$ is an $\mathbf{A}$-morphism for each $i \in I$. Let the U-structured arrow $X \xrightarrow{e} UA$ be a semifinal arrow for $\mathcal{T}$. It follows that the source $((X, (e, A)) \xrightarrow{f_i} (X_i, (e_i, A_i)))_I$ is a V-initial lift of $\mathcal{S}$. Now $(\mathbf{B}, V)$ may fail to be topological since it may violate Convention (C2) by not being amnestic. Define an equivalence relation for $\mathbf{B}$-objects by setting B and B' to be equivalent if and only if $B \leq B'$ and $B' \leq B$ (where $\leq$ is the order of the appropriate fibre). Let $\mathbf{C}$ be a full subcategory of $\mathbf{B}$ that contains as objects exactly one member of each equivalence class, and let $I: \mathbf{C} \to \mathbf{B}$ be the inclusion. Now amnesticity means that the equivalence classes defined above are singletons, so that $T = V \circ I: \mathbf{C} \to \mathbf{X}$ is amnestic and thus topological. If $H: (\mathbf{B}, V) \to (\mathbf{C}, T)$ denotes the concrete functor that sends each $\mathbf{B}$-object to its unique equivalent $\mathbf{C}$-object, then H is an equivalence, so that $A = H \circ E$ is the composite of two functors that are adjoint and reflect isomorphisms. Consequently, A is essentially algebraic. For each $\mathbf{B}$-object B the objects B and $I \circ H(B)$ are equivalent. Thus $V \circ I \circ H(B) = V(B)$, which implies that $V \circ I \circ H = V$. Therefore $U = V \circ I \circ H \circ E = T \circ A$ is the desired factorization.

(4) ⇒ (3) ⇒ (2) and (4) ⇒ (1) ⇒ (2) are straightforward or well known. (Cf. also Figure 1.) □

Theorem 2.3 *If $\mathbf{X}$ has regular factorizations, then for a functor $U: \mathbf{A} \to \mathbf{X}$ the following are equivalent:*

1. *U belongs to the compositive hull of all algebraic and all topological functors,*
2. *U belongs to the compositive hull of all regularly monadic and all topological functors,*
3. *U is solid and preserves regular epimorphisms,*
4. *U can be expressed as the composite $U = T \circ A$ of a regularly monadic functor A with a topological functor T.*

Proof:
(3) ⇒ (4). We use the factorization $\mathbf{A} \xrightarrow{U} \mathbf{X} = \mathbf{A} \xrightarrow{A} \mathbf{C} \xrightarrow{T} \mathbf{X}$ constructed in the proof of Theorem 2.2. $A = H \circ E$ is an isomorphism-closed full reflective embedding, so it is monadic (cf. [1], Example 20.10(4)). Since $T: \mathbf{C} \to \mathbf{X}$ is topological and has regular factorizations, so has $\mathbf{C}$ (cf. [1], Theorem 21.16(5)). Since H is an equivalence, it preserves regular epimorphisms. It remains to show that E preserves them as well. Let $B \xrightarrow{c} C$ be a regular epimorphism in $\mathbf{A}$, and thus a coequalizer of some pair $A \underset{q}{\overset{p}{\rightrightarrows}} B$ of $\mathbf{A}$-morphisms. Then $UB \xrightarrow{Uc} UC$ is a regular epimorphism in $\mathbf{X}$, and so is a coequalizer of some pair $X \underset{s}{\overset{r}{\rightrightarrows}} UB$ of $\mathbf{X}$-morphisms. Consider the U-structured sink $\mathcal{S} = (UA_i \xrightarrow{f_i} X)_I$ that consists of those U-costructured arrows $UA_i \xrightarrow{f_i} X$, for which $UA_i \xrightarrow{r\circ f_i} UB$ and $UA_i \xrightarrow{s\circ f_i} UB$ are $\mathbf{A}$-morphisms, and let $X \xrightarrow{e} U\overline{A}$ be a semifinal arrow for $\mathcal{S}$. Then there exist $\mathbf{A}$-morphisms $\overline{A} \xrightarrow{\overline{r}} B$ and $\overline{A} \xrightarrow{\overline{s}} B$ with $r = U\overline{r} \circ e$ and $s = U\overline{s} \circ e$. Thus $(X, (e, \overline{A})) \underset{s}{\overset{r}{\rightrightarrows}} E(B)$ is a pair of $\mathbf{B}$-morphisms. It suffices to show that $E(B \xrightarrow{c} C) = E(B) \xrightarrow{Uc} E(C)$ is a coequalizer of this pair in $\mathbf{B}$. Obviously, $Uc \circ r = Uc \circ s$. Let $E(B) \xrightarrow{f} (\overline{X}, (\overline{e}, \overline{B}))$ be a $\mathbf{B}$-morphism with $f \circ r = f \circ s$. Since Uc is a coequalizer of r and s in $\mathbf{X}$, there exists a unique $\mathbf{X}$-morphism $UC \xrightarrow{\overline{f}} \overline{X}$ with $f = \overline{f} \circ Uc$. Hence it suffices to show that $E(C) \xrightarrow{\overline{f}} (\overline{X}, (\overline{e}, \overline{B}))$ is a $\mathbf{B}$-morphism. Since $E(B) \xrightarrow{f} (\overline{X}, (\overline{e}, \overline{B}))$ is a $\mathbf{B}$-morphism, there exists an $\mathbf{A}$-morphism $B \xrightarrow{g} \overline{B}$ with $\overline{e} \circ f = Ug \circ id_{UB}$. Thus $U(g \circ p) = \overline{e} \circ f \circ Up = \overline{e} \circ \overline{f} \circ Uc \circ Up = \overline{e} \circ \overline{f} \circ Uc \circ Uq = \overline{e} \circ f \circ Uq = U(g \circ q)$. Hence $g \circ p = g \circ q$. Since c is a coequalizer of p and q in $\mathbf{A}$, there exists an $\mathbf{A}$-morphism $C \xrightarrow{\overline{g}} \overline{B}$ with $g = \overline{g} \circ c$. Thus the equations $U\overline{g} \circ Uc = Ug = \overline{e} \circ f = \overline{e} \circ \overline{f} \circ Uc$ and the fact that Uc is an epimorphism in $\mathbf{X}$ imply that $U\overline{g} = \overline{e} \circ \overline{f}$, i.e., that $E(C) \xrightarrow{\overline{f}} (\overline{X}, (\overline{e}, \overline{B}))$ is a $\mathbf{B}$-morphism.

The implications (4) ⇒ (2) ⇒ (1) ⇒ (3) are straightforward or well known. (Cf. also Figure 1.) □

3 Examples

As seen above, every functor that can be expressed as a (finite) composite of topological and essentially algebraic functors can already be expressed as a composite $T \circ A$ of an essentially algebraic functor A with a topological functor T. Can it be expressed as a composite $A \circ T$ of a topological functor T and an essentially algebraic functor A as well? The answer is "No". Before we provide examples, let us consider the implications of Figure 1.

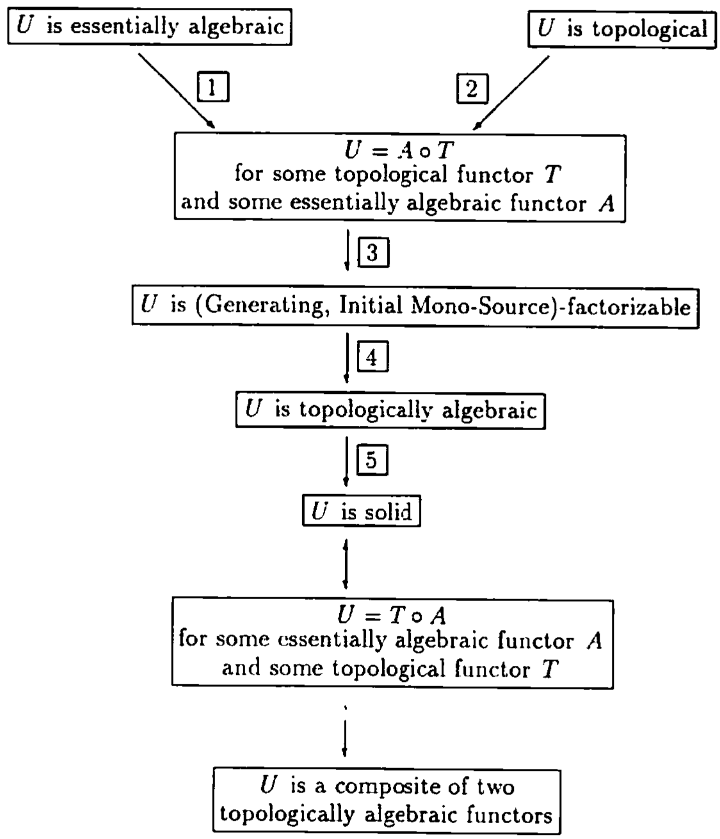


Figure 1

Obviously the implications [1] and [2] cannot be reversed. How about [3], [4], and [5]?

3.1 Example

In view of the following result, [3] cannot be reversed:

Proposition 3.1 *Let $U: \mathbf{B} \to \mathbf{Set}$ be a mono-topological functor such that there exists some* **B**-*object B with* $\mathrm{card}(UB) \geq 2$. *Then the following hold:*

1. *U is (Generating, Initial Mono-Source)-factorizable.*
2. *The following are equivalent:*
   (a) *U is topological.*
   (b) *U has a factorization $U = A \circ T$ with T topological and A essentially algebraic.*

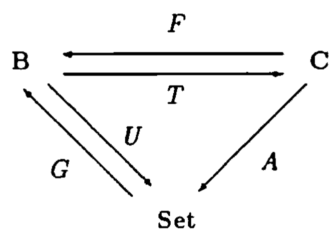
In particular, if U satisfies the hypothesis but is not topological, such a factorization cannot exist. So such a functor U provides a counterexample for the reversal of implication [3].

Proof:

1. For an arbitrary U-structured source $(X \xrightarrow{f_i} UB_i)_I$, let $X \xrightarrow{f_i} UB_i = X \xrightarrow{e} Y \xrightarrow{m_i} UB_i$ be an (Epi, Mono-Source)-factorization in **Set**, and let $(B \xrightarrow{m_i} B_i)_I$ be an initial lift of $(Y \xrightarrow{m_i} B_i)_I$. Then $X \xrightarrow{e} UB \xrightarrow{m_i} UB_i$ is the desired factorization.

2. (a) $\Rightarrow$ (b) is obvious.

   (b) $\Rightarrow$ (a):
   Assume that a factorization $\mathbf{B} \xrightarrow{U} \mathbf{Set} = \mathbf{B} \xrightarrow{T} \mathbf{C} \xrightarrow{A} \mathbf{Set}$ exists with A essentially algebraic and T topological. Let $F: \mathbf{C} \to \mathbf{B}$ denote the discrete functor for $(\mathbf{B}, T)$, which satisfies $T \circ F = id_{\mathbf{C}}$. Now $(\mathbf{B}, U)$ has discrete objects as well: For a set X, consider the source $\mathcal{S} = (X \xrightarrow{f_i} UB_i)_I$ of all U-structured arrows with domain X. Since some UB_i has at least two elements, this is a mono-source. If $(B \xrightarrow{f_i} B_i)_I$ is an initial lift of $\mathcal{S}$, then B is the discrete lift of X. Let $G: \mathbf{Set} \to \mathbf{B}$ denote the corresponding discrete functor that satisfies $U \circ G = id_{\mathbf{Set}}$.



We are done if we can show A to be an isomorphism. Now $A \circ T \circ G = U \circ G = id_{\mathbf{Set}}$, so A has $T \circ G$ as a right inverse. For any **C**-object C, by discreteness of $GA(C)$, $UGA(C) \xrightarrow{id_{A(C)}} UF(C)$ is already a **B**-morphism $GA(C) \xrightarrow{id_{A(C)}} F(C)$. Therefore it is a **C**-morphism $TGA(C) \xrightarrow{id_{A(C)}} TF(C)$. Since A is essentially algebraic, it reflects identities, so we have $TGA(C) = TF(C) = C$. Considering morphisms, we have $ATGA(f) = A(f)$, so $TGA(f) = f$ by the faithfulness of A. Thus $T \circ G$ is the inverse of the functor A. □

Concrete examples that satisfy the above conditions for counterexamples are the forgetful functors of the constructs **Haus** or **PAlg**(Ω). Cf. [1], Examples 21.41.

3.2 Example

In view of Proposition 3.2 below, the implication [5] in Figure 1 cannot be reversed (even though, under mild conditions, it can be; cf. [1], [2] and [3]).

Let PΛ-CCPos be the following construct: Objects are quadruples $\overline{X} = (X, D, \lambda, \leq)$, where $(X, \leq)$ is a chain-complete poset (i.e., suprema of nonempty chains exist) and λ is a partial unary operation on X that has domain D. (More precisely, $\lambda \colon D \to X$ is a function defined on a subset $D \subseteq X$.)
A morphism $f : \overline{X}_1 \to \overline{X}_2$ is a λ-homomorphism that preserves suprema of nonempty chains. [f is called a λ-homomorphism iff $f[D_1] \subseteq D_2$ and the following diagram commutes:]

$$\begin{array}{ccc} D_1 & \xrightarrow{f|_{D_1}^{D_2}} & D_2 \\ {\scriptstyle \lambda_1}\downarrow & & \downarrow{\scriptstyle \lambda_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

Let $V :$ PΛ-CCPos $\to$ Set denote the obvious forgetful functor, and let Λ-CCPos be the full subconstruct of PΛ-CCPos that consists of those objects $\overline{X}$ for which λ is a total operation (i.e., $D = X$). In this way we can identify objects of Λ-CCPos with triples $(X, \lambda, \leq)$. The inclusion functor is denoted by E, and the forgetful functor is denoted by U $(= V \circ E)$.

Notation:
If $f : A \to B$ is a function, $M \subseteq A$, $N \subseteq B$ and $f[M] \subseteq N$, then $f|_M$, $f|^N$, and $f|_M^N$ denote the appropriate restriction, corestriction, and simultaneous restriction and corestriction of f, respectively. For $\overline{X} = (X, \lambda, \leq)$, we define the α-th iteration of λ, where α is any ordinal, by transfinite induction:

(i) $\lambda^0(x) = x$

(ii) $\lambda^{\alpha+1}(x) = \lambda(\lambda^\alpha(x))$

(iii) $\lambda^\alpha(x) = \sup_{\beta<\alpha} \lambda^\beta(x)$, if α is a limit ordinal and the supremum exists.

We are now prepared for the following:

Proposition 3.2

1. *U is solid.*

2. *U is* not *topologically algebraic.*

3. *(a)* Λ-CCPos *is cocomplete and strongly complete.*

   (b) Λ-CCPos *is wellpowered and extremally co-wellpowered.*

   (c) Λ-CCPos *is (Epi, Mono-Source)-factorizable.*

4. *(a)* Λ-CCPos *is* not *strongly cocomplete.*

(b) Λ-**CCPos** *is* **not** *co-wellpowered.*

(c) Λ-**CCPos** *is* **not** *an (Epi,* **M***)-category for any* **M**.

Proof:

1. We show this in two steps:

Step (a): V is topologically algebraic

Step (b): E is topologically algebraic

Then $U = V \circ E$ is the composite of two solid functors, and hence is solid.

Step (a): The concrete product $(\overline{P}, \pi_i)$ of a set-indexed family $(\overline{X}_i)_{i \in I} = ((X_i, D_i, \lambda_i, \leq_i))_I$ of **P**Λ-**CCPos** objects is obtained by letting $\overline{P} = (P, D_P, \lambda_P, \leq_P)$ where

$$\begin{array}{rcl} (P, \pi_i) & = & \Pi_I X_i \text{ in } \mathbf{Set} \\ D_P & = & \bigcap\limits_I \pi_i^{-1}[D_i] \\ \lambda_P & = & \langle \lambda_i \circ (\pi_i|_{D_P}^{D_i}) \rangle : D_P \to P \\ \leq_P & \text{is} & \text{defined to hold iff it holds in all components.} \end{array}$$

If I is a proper class, then $\overline{P}$ can be formally constructed as above — but in a higher universe.

Now for any V-structured source $(X \xrightarrow{f_i} V\overline{X}_i)_I$ let $(\overline{P}, \pi_i)$ be such a (formal) product of $(\overline{X}_i)_I$, and let $f : X \to P$ be the morphism $\langle f_i \rangle$ induced by the product. Now we define Z to be the smallest subconglomerate of P that has the following properties:

(i) $f[X] \subseteq Z$,

(ii) Z is closed under the formation of suprema of nonempty chains (taken in $\overline{P}$).

Claim: Z is small (i.e., is in bijective correspondence with a set).

Proof of Claim:

Let $M = \{\sup(N) \mid N \subseteq f[X] \text{ and } \sup(N) \text{ exists in } \overline{P}\}$. Clearly, $f[X] \subseteq M$, so (i) holds with Z replaced by M. To show (ii), let $C \subseteq M$ be a nonempty chain. For each $c \in C$, choose some $N_c \subseteq f[X]$ with $\sup(N_c) = c$. Let $N = \bigcup_{c \in C} N_c \subseteq f[X]$. Then $\sup(C)$ can be seen to be a supremum for N, so $\sup(C) \in M$. Since Z is the smallest class that satisfies (i) and (ii), we have $Z \subseteq M$. But $\mathrm{card}(M) \leq 2^{\mathrm{card}(f[X])} \leq 2^{\mathrm{card}(X)}$. Hence M is small, so that Z is small, which establishes the claim.

Thus, without loss of generality we may assume that Z is a set. Now let $\overline{Z} = (Z, D_Z, \lambda_Z, \leq_Z)$, where $D_Z = Z \cap \lambda_P^{-1}[Z]$ and λ_Z and $\leq_Z$ are the appropriate restrictions of λ_P and $\leq_P$. Then the inclusion $\overline{Z} \xrightarrow{h} \overline{P}$ formally satisfies the conditions for being a **P**Λ-**CCPos**-morphism. To show that h is formally V-initial, let $Y \xrightarrow{g} Z$ be a function such that $\overline{Y} \xrightarrow{h \circ g} \overline{P}$ is a **P**Λ-**CCPos**-morphism. Then $(h \circ g) \circ \lambda_Y[D_Y] \subseteq Z$, and since $h \circ g$ is a λ-homomorphism, we have $\lambda_P \circ (h \circ g)|_{D_Y}[D_Y] \subseteq Z$, so that

$h \circ g|_{D_Y}[D_Y] \subseteq \lambda_P^{-1}[Z]$. Since h is an inclusion, this yields $g[D_Y] \subseteq D_Z$. The other conditions for g being a $\mathbf{P}\Lambda$-CCPos-morphism are easily checked.

To show that $X \xrightarrow{f|^Z} V\overline{Z}$ is generating, consider morphisms $r, s : \overline{Z} \to \overline{Y}$ with $r \circ f|^Z = s \circ f|^Z$. Now the set $M = \{z \in Z \mid r(z) = s(z)\}$ satisfies conditions (i) and (ii) above, so by the minimality of Z, we have $M = Z$; hence $r = s$. Since concrete product sources are initial, we have that

$$X \xrightarrow{f_i} V\overline{X}_i = X \xrightarrow{f|^Z} V\overline{Z} \xrightarrow{\pi_i \circ h} V\overline{X}_i$$

is a (Generating, Initial-Source)-factorization. Thus Step (a) has been established.

Step (b): Since full reflective embeddings are topologically algebraic, it suffices to show that E is reflective.

For an object $\overline{X} = (X, D, \lambda, \leq)$ in $\mathbf{P}\Lambda$-CCPos, the reflection $\overline{X} \xrightarrow{r} \overline{Y}$ is constructed as follows:

Let $\overline{Y} = (Y, \lambda_Y, \leq_Y)$, where

$$\begin{aligned} Y &= \{(x, n) \in X \times \mathbb{N} \mid x \in D \Rightarrow n = 0\} \\ \lambda_Y(x, n) &= \begin{cases} (\lambda(x), 0) & \text{if } x \in D \\ (x, n+1) & \text{otherwise} \end{cases} \\ \leq_Y &= \{((x_1, 0), (x_2, 0)) \mid x_1 \leq x_2\} \cup \{(y, y) \mid y \in Y\} \end{aligned}$$

That is, we construct $\overline{Y}$ from $\overline{X}$ by adding a countable set $\{\lambda_Y^n(x) \mid n \in \mathbb{N}\}$ for each $x \in X \setminus D$.

The reflection map r is defined by $r(x) = (x, 0)$. To show the universal property, let $\overline{X} \xrightarrow{f} \overline{Z}$ be a morphism with $\overline{Z}$ an object in Λ-CCPos. Then the unique $\hat{f} : \overline{Y} \to \overline{Z}$ with $\hat{f} \circ r = f$ can be defined inductively as follows:

(i) $\hat{f}(x, 0) = \hat{f}(r(x)) = f(x)$. (This is required because $\hat{f} \circ r = f$.)

(ii) $\hat{f}(x, n+1) = \hat{f}(\lambda_Y(x, n)) = \lambda_Z(\hat{f}(x, n))$. (This is required because $\hat{f}$ must be a λ-homomorphism.)

Thus Step (b) has been established, so that Part 1 has been proved.

2. Assume that U is topologically algebraic. We consider the following structured source $\mathcal{S} = (\{0\} \xrightarrow{i_\alpha} U\overline{Y}_\alpha)_{\alpha \in \mathrm{Ord}}$, where Ord is the class of all ordinals, $\overline{Y}_\alpha = (Y_\alpha, \lambda_\alpha, \leq_\alpha)$, with Y_α the set of ordinals that are less than or equal to α, having $\leq_\alpha$ the natural induced order, $\lambda_\alpha(\gamma) = \min(\gamma + 1, \alpha)$, and $i_\alpha(0) = 0$. By the assumption on U, $\mathcal{S}$ must have a (Generating, Initial-Source)-factorization

$$\{0\} \xrightarrow{i_\alpha} U\overline{Y}_\alpha = \{0\} \xrightarrow{g} U\overline{X} \xrightarrow{f_\alpha} U\overline{Y}_\alpha$$

(with, say, $\overline{X} = (X, \lambda_X, \leq_X)$). Now for $\alpha, \beta \in$ Ord, consider $\overline{Z}_\beta \xrightarrow{h_{\alpha\beta}} \overline{Y}_\alpha$, where $\overline{Z}_\beta = (Z_\beta, \mu_\beta, \leq'_\beta)$, with $Z_\beta = \beta + \omega$ (the set of ordinals smaller than $\beta + \omega$), $\leq'_\beta = \leq_\beta \cup \{(z, z) \mid z \in Z_\beta\}$, $\mu_\beta(\gamma) = \gamma + 1$, and $h_{\alpha\beta}(\gamma) = \lambda_\alpha^\gamma(0) = \begin{cases} \gamma, & \text{if } \gamma \leq \alpha \\ \alpha, & \text{if } \alpha < \gamma. \end{cases}$

Claim: For each $\beta \in$ Ord, there is some $\overline{Z}_\beta \xrightarrow{g_\beta} \overline{X}$ with

$$(*) \quad \overline{Z}_\beta \xrightarrow{h_{\alpha\beta}} \overline{Y}_\alpha = \overline{Z}_\beta \xrightarrow{g_\beta} \overline{X} \xrightarrow{f_\alpha} \overline{Y}_\alpha, \text{ and}$$

(**) If $\beta' < \beta$, then $g_{\beta'} = g_\beta|_{Z_{\beta'}}$.

Proof of Claim (by transfinite induction over β):

Case 1: $\beta = 0$.

Take $g_0(\gamma) = \lambda_X^\gamma(g(0))$. Then $f_\alpha(g_0(\gamma)) = f_\alpha(\lambda_X^\gamma(g(0)) = \lambda_\alpha^\gamma(f_\alpha(g(0))) = \lambda_\alpha^\gamma(0) = h_{\alpha 0}(\gamma)$.

Case 2: β is a limit ordinal greater than 0.

By the induction hypothesis, the function $k: \beta \to X$, defined by $k(\gamma) = g_\gamma(\gamma)$ for all $\gamma < \beta$ is monotonic, so $k[\beta]$ is a chain in $\overline{X}$. Now define $g_\beta : \beta + \omega \to X$ by

(i) $g_\beta(\gamma) = k(\gamma)$ for each $\gamma < \beta$. (so that (**) is guaranteed), and

(ii) $g_\beta(\beta + n) = \lambda_X^n(\sup k[\beta])$ for each $n \in \mathbb{N}$.

Now to prove (*), i.e., that $f_\alpha(g_\beta(\gamma)) = \lambda_\alpha^\gamma(0)$ for each $\gamma \in Z_\beta$, we can use the induction hypothesis, if $\gamma < \beta$. Otherwise, we have

$$\begin{aligned} f_\alpha(g_\beta(\beta+n)) &= f_\alpha(\lambda_X^n(\sup k[\beta])) \\ &= \lambda_\alpha^n \sup(f_\alpha \circ k)[\beta] \quad \text{(since } f_\alpha \text{ is a morphism)} \\ &= \lambda_\alpha^n \sup_{\gamma<\beta} f_\alpha(g_\gamma(\gamma)) \\ &= \lambda_\alpha^n \sup_{\gamma<\beta} \lambda_\alpha^\gamma(0) \quad \text{(by the induction hypothesis)} \\ &= \lambda_\alpha^{\beta+n}(0). \end{aligned}$$

Case 3: β is a successor ordinal, say $\beta = \gamma + 1$.

Then let $g_\beta = g_\gamma$ (as a function; note that $Z_\beta = Z_\gamma$).

In each of the above cases, the fact that the source $(\overline{X} \xrightarrow{f_\alpha} \overline{Y}_\alpha)_{\alpha \in \mathbf{Ord}}$ is initial causes g_β to be a Λ-**CCPos**-morphism.

Hence the claim is established.

Now for each β, $Z_\beta \xrightarrow{h_{(\beta+\omega)\beta}} Y_{\beta+\omega}$ is injective, so its first factor $\overline{Z}_\beta \xrightarrow{g_\beta} \overline{X}$ must be injective as well. Therefore, the cardinality of X is greater than that of any ordinal β, which is a contradiction. Consequently, Part 2 is established.

3. (a) This is immediate since solid constructs must be cocomplete and strongly complete. Cf. [1], Corollary 25.16.

   (b) (Λ-**CCPos**, U) is fibre-small and U, being solid, is adjoint. Therefore the well-poweredness of **Set** implies that of Λ-**CCPos**. Cf. [1], Corollary 18.10. To show that Λ-**CCPos** is extremally co-wellpowered, consider an extremal epimorphism

$$(X, \lambda_X, \leq_X) = \overline{X} \xrightarrow{e} \overline{Y} = (Y, \lambda_Y, \leq_Y).$$

Let M be the closure of $e[X]$ under suprema of nonempty chains. As in the proof of the smallness of Z in Part 1, we conclude that $\text{card}(M) \leq 2^{\text{card}(X)}$. Now let $\overline{Z} = (Z, \lambda_Z, \leq_Z)$, with $Z = \{\lambda_Y^n(y) \mid y \in M, n \in \mathbb{N}\}$, $\lambda_Z = \lambda_Y|_Z$, and $\leq_Z = (\leq_Y \cap (M \times M)) \cup \{(z,z) \mid z \in Z\}$. Then $\overline{X} \xrightarrow{e} \overline{Y} = \overline{X} \xrightarrow{e|^Z} \overline{Z} \xrightarrow{m} \overline{Y}$, where m is the obvious inclusion. Since m is a monomorphism and e is an extremal epimorphism, m must be an isomorphism. Thus we have $\text{card}(Y) = \text{card}(Z) \leq \text{card}(M) \cdot \aleph_0 \leq 2^{\text{card}(X)} \cdot \aleph_0$. Consequently, by fibre-smallness and transportability of U, there can be only a set of pairwise non-isomorphic extremal quotient objects of $\overline{X}$.

(c) This follows from (a) and (b). Cf. [1], Theorem 15.25(1). Hence Part 3 is established.

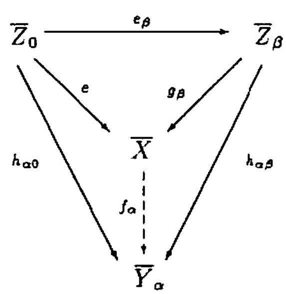
4. (a) Consider the source $M = (\overline{Z}_0 \xrightarrow{e_\beta} \overline{Z}_\beta)_{\beta \in \text{Ord}}$ with $\overline{Z}_\beta$ as defined in Part 2, and with $e_\beta(n) = n$, for each $n \in \mathbb{N}$. To see that each e_β is an epimorphism, let $r, s : \overline{Z}_\beta \to \overline{A}$ be morphisms with $r \circ e_\beta = s \circ e_\beta$. Then for $\gamma \in Z_\beta$:

$$s(\gamma) = s(\mu_\beta^\gamma(e_\beta(0))) = \lambda_A^\gamma(s \circ e_\beta)(0) = \lambda_A^\gamma(r \circ e_\beta)(0) = r(\mu_\beta^\gamma(e_\beta(0))) = r(\gamma).$$

Thus $r = s$.

Now assume that M has a cointersection $\overline{Z}_0 \xrightarrow{e} \overline{X}$. Then e factors through each e_β as, say, $e = g_\beta \circ e_\beta$. Now for each α, $\overline{Z}_0 \xrightarrow{h_{\alpha 0}} \overline{Y}_\alpha$ (as defined in Part 2) factors through each e_β as well: $h_{\alpha 0} = h_{\alpha\beta} \circ e_\beta$. By the universal property of the cointersection, for each α there must be a unique $f_\alpha : \overline{X} \to \overline{Y}_\alpha$ with $f_\alpha \circ g_\beta = h_{\alpha\beta}$.

As in the proof of Part 2, this leads to the contradiction that $U\overline{X}$ has cardinality greater than each ordinal β.



(b) The source M, constructed in 4(a), consists of pairwise non-isomorphic epimorphisms.

(c) This follows from 4(a), since each (Epi, M)-category has cointersections. Cf. [1], Corollary 15.16.

Thus Part 4 is established.

For related examples see [2] and [3].

4 Problems

(1) Is the implication [4] in Figure 1. reversible? [Note that [1], Exercise 16A(b) contains a counterexample that fails, however, to satisfy our Convention (C1).]

(2) Are the following conditions equivalent?

(a) U belongs to the compositive hull of all algebraic functors and all topological functors.

(b) U is solid and U preserves extremal epimorphisms.

(3) Characterize those functors that can be expressed as a composite $A \circ T$ of a topological functor T with an essentially algebraic functor A. Cf. [5] for related results.

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