Category Theory at Work
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# The Art of Pointless Thinking: A Student's Guide to the Category of Locales

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## 1. Introduction

The category of locales, as a substitute for (and in many ways an improvement on) the category of topological spaces, was introduced nearly 20 years ago in a celebrated paper by John Isbell [19]. Of course, the idea of regarding frames (or complete Heyting algebras, or whatever name you prefer) as generalized topological spaces is a good deal older than this: it can certainly be traced back to work of McKinsey and Tarski [46] in the 1940s, and many other authors [49, 16, 9, 50, 12, 2] had worked on variants of the idea. But the reason for citing Isbell's paper as the unequivocal origin of the subject is not just that he introduced the name "locale" (which, although it has since been abused by others [38], is unequivocally the right name — it possesses all the right associations, and is capable of all the needed inflections), but far more that he was the first to take the (literally) revolutionary step of turning the arrows round, and working in the opposite of the concrete category of frames.

Of course, this step is mathematically trivial; the duality principle is something that every student learns near the beginning of a first course on category theory, and which all but the weakest assimilate without a moment's difficulty. However, it is psychologically enormously important: most categories of practical importance in mathematics (with two important classes of exceptions: abelian categories and categories of relations) are very different from their opposites, and our categorical intuition (largely conditioned by our experience of the categories of sets and of topological spaces) is strongly non-self-dual, so that the simple act of turning the arrows round produces a new insight into the nature of the objects and the way in which they interact. And,

This work is an original contribution and will not appear elsewhere.

without that insight, one can't start seriously doing topology with locales: frame theory is lattice theory applied to topology, whereas locale theory is topology itself. (Admittedly, there are eminent authors — Banaschewski, Pultr and others — who would disagree with this assertion, and who continue to write papers about frames rather than locales. However, detailed examination of those papers convinces me that I am right, and I shall try to pass on that conviction in the course of this article.) Thus the title of the article might with equal appropriateness have been "The Art of Backwards Thinking".

A simple illustration will help to clarify the argument of the previous paragraph. The category Frm of frames, like any category of lattices, is enriched over partially ordered sets: that is, its hom-sets have a natural (pointwise) partial ordering, which is stable under composition. Turn the arrows around, and we see that Loc, the category of locales, is also enriched over posets; in fact each locale X has a natural partial ordering (that is, a sublocale of $X \times X$ which is reflexive, transitive and antisymmetric), and properties of this ordering are of importance in locale theory (for example, the requirement that it be discrete provides a "$T_1$ axiom" for locales). To echo the words of Isbell [20], "I have no idea what co-partial order is", but I do understand partial orders: I have enough experience of partially ordered sets and spaces to transfer my intuition about them to any category with finite limits. But the reversal of arrows is <u>necessary</u> if we are to bring this intuition to bear on the category of locales.

There remains the question: why study locales at all? Isn't the category of topological spaces (or perhaps some cartesian closed extension of it, such as is described elsewhere in this volume) good enough for doing topology in? There are several answers to this question. One, which was stressed in my earlier expository article [29] (and which I shall therefore play down in the present context: anyone who wants to know more can go back and read [29]), is that locale theory is inherently constructive: by working with locales one can avoid not only most uses of the axiom of choice in topology, but even most uses of the law of excluded middle, so that topology can be done localically in toposes and other non-classical contexts. (Unfortunately, this has turned out to be slightly less true than I believed when I wrote [29]. Whilst it's true enough at the "topological" level, when one tries to ascend to the "uniform" level it emerges that there are significant differences between the classical and constructive theories [35]. And even my conviction that locale theory is inherently choice-free has recently received a nasty shock [59].)

A second reason, related to the first, is that the category of locales is the right place for doing fibrewise topology: the slice categories Loc/B of locales over a fixed base B are more like Loc itself than the categories Top/B are like Top. (To be technical for a moment, the reason for this is that Loc/B is, up to equivalence, the category of internal locales in the topos of sheaves on B [27].) Again, there are those (notably I.M. James [24]) who would disagree with this statement, but I believe there is now plenty of evidence for its validity. (Some of this evidence will be set out in section 4 below.)

Thirdly, the reason which originally moved Isbell [19] to study locales was the fact that locale products are so much nicer than space products: all sorts of topological properties which fail to be inherited by product spaces behave well for product locales. (Isbell's original example was paracompactness, but one could cite numerous others such as the Lindelöf property — which turns out to be equivalent, for regular locales, to realcompactness [45]. And the Tychonoff theorem for locales requires no use of choice [26]; indeed, when viewed from the right perspective it becomes a triviality [37].) Related to this is the good behaviour of function spaces [20, 18] and Vietoris hyperspaces [30], which both work more smoothly for locales; perhaps the best illustration, in the case of function-spaces, is the remarkable discovery by Moerdijk and Wraith [47] that connected and locally connected locales are path-connected.

But all the reasons adduced in the previous paragraph are properly viewed (I believe) as consequences of the single most important fact which distinguishes locales from spaces: the fact that every locale has a smallest dense sublocale. If you want to "sell" locale theory to a classical topologist, it's a good idea to begin by asking him to imagine a world in which an intersection of dense subspaces would always be dense; once he has contemplated some of the wonderful consequences that would flow from this result, you can tell him that that world is exactly the category of locales. (In a sense which can be made quite precise, it is the least possible modification of the category of spaces which makes the result true: as we shall see in section 3, every locale is not merely a dense sublocale of a space but an intersection of dense spatial sublocales.) It is certainly clear that in order to achieve such a world, we have to abandon the idea of a space as a set of points equipped with some kind of structure; for there will be examples in any category of this type of pairs of dense subspaces of a nontrivial space having no points in common. Thus the topological structure of a locale cannot "live on its points"; from here, it is a small step to make the topological structure (i.e. the frame of open subsets) into the extensional essence of the locale (so that the points, if any, live on the open sets rather than the other way about). This, to me, is the real justification for the effort involved in training oneself to "think pointlessly".

The layout of the rest of the article is as follows. In section 2 we develop the theory of frames, as far as we need it; in section 3 we turn the arrows round and survey the topological aspects of locales; and in section 4 we look briefly at locales over a base. Detailed proofs are not given — the interested reader will find them in the two textbooks so far written on locales [28, 67], or in the original papers to which we provide references — but we have tried to indicate the principal constructions, and the ideas behind them, in some detail.

## 2. Frames

The basic definition is simple enough: a <u>frame</u> is a complete lattice A in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\} \qquad (2.1)$$

holds for all $a \in A$ and $S \subseteq A$. By the Adjoint Functor Theorem, a frame is extensionally the same thing as a complete Heyting algebra, the Heyting implication being given by

$$(a \to b) = \bigvee\{c \in A \mid c \wedge a \leq b\} , \qquad (2.2)$$

but frame homomorphisms (i.e. maps preserving finite meets and arbitrary joins) do not normally preserve the implication. The motivating example of a frame is of course the open-set lattice $\mathcal{O}(X)$ of a topological space X (and the motivating example of a frame homomorphism is the mapping $f^{-1}: \mathcal{O}(Y) \to \mathcal{O}(X)$ induced by a continuous map $f: X \to Y$), but there are others: for example, a complete Boolean algebra is a frame, and is never of the form $\mathcal{O}(X)$ unless it is atomic. Frames also have a natural tendency to turn up [10] as lattices of localizations of well-behaved categories $\mathcal{C}$ (that is, left exact idempotent monads on $\mathcal{C}$), and for this reason they are of interest to (non-commutative) ring theorists [17].

The category **Frm** is (infinitary) algebraic (that is, monadic over **Set**); this follows from the fact that it can be presented by operations and equations (which is implicit in the previous paragraph), plus the fact that the forgetful functor **Frm** → **Set** has a left adjoint. This left adjoint turns out to factor through $\mathcal{O}$: **Top**$^{op}$ → **Frm**, and in fact provides a left adjoint for $\mathcal{O}$ regarded as a functor **Top**$^{op}$ → **Set**. Thus frames can be regarded [63] as "the algebraic part of the theory of topological spaces" — where, however, the latter is understood not as the theory of the forgetful functor **Top** → **Set** (whose algebraic part is trivial), but as that of the functor $\mathcal{O}$: **Top**$^{op}$ → **Set**. (Warning: as a functor on the whole of **Top**$^{op}$, isn't faithful — different spaces can have the same open-set lattice — but one can restrict it to the subcategory **Sob**$^{op}$ of sober spaces, on which it is faithful, without changing anything essential. See section 3 below.)

There are several possible constructions of the free functor **Set** → **Frm**. The earliest, due to Bénabou [9], explicitly constructs the free frame on a set X as the open-set lattice of the power $S^X$, where S is the Sierpiński space, i.e. the two-point space with just one closed point. (The fact that powers of S appear here is no accident; for free frames, like the free objects in any algebraic category, are regular projective — in **Frm** this assertion doesn't even need the axiom of choice, since surjective frame maps have right adjoints which are splittings for them in **Set** — and so one would expect them to be related to Scott's injective spaces [60], which are the retracts of powers of S.)

However, there are other constructions which yield more information by constructing free frames in stages: that is, they factor the forgetful functor **Frm** → **Set** through some simpler algebraic category, and then construct left adjoints for each half of the factorization. In [28], we used the factorization

$$\mathbf{Frm} \to \mathbf{SLat} \to \mathbf{Set}$$

where SLat is the category of meet-semilattices; that is, "first forget the arbitrary joins, and then the finite meets". This has the advantage that the way one builds the free functor corresponds to the way in which one often defines a topology on a set, by first specifying a base for it (closed under finite intersections) and then closing off under arbitrary unions; and indeed this approach readily generalizes to the problem of constructing frames from generators and relations ([28], II 2.11). As a particular application of the latter, we gave a construction of coproducts in Frm which closely parallels the construction of the Tychonoff topology on a product space. (This construction, however, was not new; it had appeared, minus the motivation, in [14].)

Joyal and Tierney [38], on the other hand, exploited the "opposite" factorization

$$\mathbf{Frm} \to \mathbf{CSLat} \to \mathbf{Set}$$

in which one first forgets the finite meets and then the arbitrary joins (CSLat is the category of complete join-semilattices). An advantage of this approach is that, just as (meet-)semilattices may be regarded as a special case of commutative monoids, so frames may be seen as a class of commutative monoids in CSLat, relative to the monoidal closed structure which it acquires as the category of models of a commutative algebraic theory. This too leads to an illuminating construction of coproducts, at least for finite families of frames: the coproduct of two frames A and B is simply their tensor product in CSLat, i.e. the complete join-semilattice generated by symbols $a \otimes b$ ($a \in A$, $b \in B$), freely subject to the bilinearity relations that $a \otimes (-)$ and $(-) \otimes b$ preserve arbitrary joins. (Once again, this description was not new; it had appeared, at least implicitly, in [68].) Moreover, since the extra structure involved in the passage from complete semilattices to frames is finitary, the forgetful functor Frm → CSLat creates filtered colimits; thus one can obtain information about more complicated colimits in Frm from their counterparts in CSLat, which are easier to construct. For example, one has the result that if the transition maps in a filtered diagram of frames are injective, then so are the maps from the vertices of the diagram to its colimit. (Yet again, someone had been there before: in this case Isbell [21].)

A third way of factoring Frm → Set was suggested by Banaschewski [5] and recently exploited by Vickers and the present author [37]: this is the factorization

$$\text{Frm} \to \text{PreFrm} \to \text{Set}$$

where the category PreFrm of preframes has as objects posets with finite meets and directed joins (plus (2.1) restricted to directed joins), with the appropriate morphisms. At first sight, this factorization appears less promising than the others, because the theory of preframes is not algebraic but only "essentially algebraic"; in fact its algebraic part is no more than the theory of meet-semilattices. However, the significant thing is that the theory of preframes is commutative, and so PreFrm, like CSLat, has a closed monoidal structure, with respect to which frames appear as (a special case of) commutative monoids. Thus the coproduct of two frames A and B may also be regarded as their tensor product in PreFrm; that is, it is freely generated by elements $a \otimes b$ ($a \in A$, $b \in B$), subject to preframe bilinearity relations. (These generators are related to the CSLat ones by

$$a \otimes b = (a \otimes 1) \vee (1 \otimes b)$$

and $$a \otimes b = (a \otimes 0) \wedge (0 \otimes b)\ .) \qquad (2.3)$$

This approach yields the simplest proofs yet devised of the "localic Tychonoff theorem" that a coproduct of compact frames is compact [5, 37]. It seems likely that it will yield further results of interest — particularly in connection with theoretical computer science, where directed joins and finite meets are more "natural" operations to consider than finite joins.

For completeness, we should also mention the fourth possible factorization

$$\text{Frm} \to \text{DLat} \to \text{Set}$$

(DLat = category of distributive lattices), where one forgets the infinitary structure first and remembers the finitary part. Here, it is the (non-full) image of the free functor DLat → Frm, the category of coherent frames, that is of most interest: if one restricts the forgetful functor to this subcategory, and then dualizes to coherent locales, one obtains the (choice-free) localic version of Stone duality for distributive lattices ([28], II 3.3). (And if one further restricts from distributive lattices to Boolean algebras, one obtains a duality with a full subcategory of Loc — the compact zero-dimensional locales.) The (full) subcategory of retracts in Frm of coherent frames, the stably continuous frames, is also of interest [4, 65].

Like any algebraic category, Frm has regular epic/monic factorizations, which are stable under pullback. Regular epimorphisms in Frm are just surjective frame homomorphisms, and they are of considerable importance since they correspond to subspaces. The surjective images of a frame A of course correspond to congruences on A; but since a congruence $\theta$ is (in particular) stable under arbitrary joins, each $\theta$-equivalence class has a greatest element, and so $\theta$ is determined by the mapping (j, say) which sends each element of A to the greatest member of its equivalence class. This mapping has the properties

$$a \leqslant j(a) = j(j(a)) \quad \text{and} \quad j(a \wedge b) = j(a) \wedge j(b) \qquad (2.4)$$

(the third identity corresponding to stability of $\theta$ under finite meets): that is, j is a left exact (idempotent) monad on A considered as a category, and we call it a nucleus on A. (It may be shown that nuclei may be characterized by a single identity: they are exactly the maps $j\colon A \to A$ satisfying

$$(a \to j(b)) = (j(a) \to j(b)) \qquad (2.5)$$

for all $a, b \in A$ [25]. However, this seems less useful than the characterization by three separate identities in (2.4).)

The quotient frame $A/\theta$ may be identified with the sub-poset (not a subframe!) $A_j$ of A consisting of the fixed points of j; the quotient map becomes identified with j itself, regarded as a frame homomorphism $A \to A_j$. It is easy to show that a sub-poset B of A is of the form $A_j$ for a nucleus j iff it is closed under arbitrary meets and an exponential ideal (i.e. $a \in A$ and $b \in B$ imply $(a \to b) \in B$); we call such a subset a fixset. Note that an arbitrary intersection of fixsets is a fixset; this yields a construction of meets in the lattice of quotients of A. (Joins in the lattice of quotients correspond, as usual, to intersections of congruences, or equivalently to (pointwise) meets of nuclei.)

With any $a \in A$ we may associate three particular nuclei: the <u>closed</u> nucleus $c(a) = a \vee (-)$, the <u>open</u> nucleus $u(a) = a \rightarrow (-)$ and the <u>quasi-closed</u> nucleus $q(a) = ((-) \rightarrow a) \rightarrow a$. The quotient frames corresponding to the first two are respectively isomorphic to the principal filter $\uparrow(a)$ and the principal ideal $\downarrow(a)$; that corresponding to $q(a)$ is Boolean, and every Boolean quotient frame arises in this way (the element a being determined as the least element of the corresponding fixset). In fact $A_{q(a)}$ is the intersection of all fixsets which contain a; in particular, $q(0)$ is the largest nucleus j such that $0 \in A_j$.

Since nuclei on A are localizations of A considered as a category, it is no surprise that they form a frame (in their pointwise ordering, which corresponds to the inclusion ordering on congruences but is the opposite of the inclusion ordering on fixsets); this was first proved by Isbell [19], but a simpler argument will be found in [28], II 2.5. This frame is denoted N(A), and called the <u>assembly</u> of A. The function $c: A \rightarrow N(A)$ is a frame homomorphism, but u and q are not; in fact, for any $a \in A$, $c(a)$ and $u(a)$ are complementary elements of N(A). Further, the elements $c(a)$ and $u(a)$, $a \in A$, generate N(A) as a frame; in fact for any $j \in N(A)$ we have

$$j = \bigvee\{c(j(a)) \wedge u(a) \mid a \in A\} \tag{2.6}$$

(see [28], II 2.7). From this it follows easily that $c: A \rightarrow N(A)$ is epic as well as monic in **Frm**. (Note: although N(A) is generated as a frame by complemented elements, it does not follow that all its elements are complemented, i.e. that it is a Boolean algebra. An infinite join of complemented elements need not be complemented.)

It can be shown that N(A) is freely generated from A by adjoining complements for all its elements: that is, any frame homomorphism $A \rightarrow B$ whose image is contained in the sublattice of complemented elements of B factors uniquely through $c: A \rightarrow N(A)$. (This was apparently first observed by A. Joyal.) In particular, any homomorphism from A to a complete Boolean algebra factors through c; so we could obtain a left adjoint for the forgetful functor from complete Boolean algebras to frames, if it existed, by iterating the construction $A \mapsto N(A)$ (transfinitely often if necessary) until it converged. However, it is well known that this left adjoint does not exist; so there exist frames A for which all the transfinitely iterated assemblies $N^{\alpha}(A)$ are distinct. But the canonical maps $A \to N^{\alpha}(A)$ are all epic; so we see that Frm is not co-well-powered.

It is of interest to ask what happens if, instead of adjoining complements for all the elements of A, we adjoin complements only for the elements of some subset $B \subseteq A$. At least in the case when B is a subframe of A, this question was answered in [25]: we obtain a subframe $N_B(A)$ of N(A), whose elements are the B-fibrewise closed nuclei which we shall meet again in section 4. Such nuclei are uniquely determined by their restrictions to B, which are exactly the functions $B \to A$ satisfying (2.5) for all a, b ∈ B.

## 3. Locales

We now embark on the task of "turning round" the information about frames in the last section, and viewing it as topological information about locales. Experience suggests that, in order to avoid confusion between a locale and the frame which is extensionally the same thing, it is necessary to adopt a notation which distinguishes between them; so we shall use the letters X, Y, Z, ... for locales and write $\mathcal{O}(X)$ for the frame corresponding to X. (Thus our notation doesn't distinguish between a space X and the locale which is extensionally its frame of open sets. This is reasonable provided we restrict our attention to sober spaces, as we shall see shortly.) Likewise, if f: X → Y is a morphism of Loc, we write $f^*: \mathcal{O}(Y) \to \mathcal{O}(X)$ for the frame homomorphism which is extensionally the same thing.

The (now) nameless functor Top → Loc which is extensionally $\mathcal{O}$: Top → $\text{Frm}^{op}$ has a right adjoint, which we denote $X \mapsto X_s$; we call $X_s$ the spatial part of X, and say X is spatial if the counit $X_s \to X$ is an isomorphism. The adjunction is idempotent (so that spatial locales are exactly those which arise from spaces), and its counit is always regular monic, so we can think of $X_s$ as a sublocale — the equivalent of a quotient frame — of X, namely the smallest sublocale which contains all the points of X, the latter being by definition morphisms from the terminal locale 1 ($\mathcal{O}(1) = 2$) to X. (Frame-theoretically, points correspond to prime elements (that is, elements generating prime principal ideals) of $\mathcal{O}(X)$, and $\mathcal{O}(X_s)$ may be identified with the fixset of elements of $\mathcal{O}(X)$ which are meets of prime elements.)

On the other side of the adjunction, for a space X the unit map $X \to X_s$ need not be either monic or epic; but it is monic iff X is a $T_0$-space (and then it is regular monic), and an isomorphism iff X is sober (which one may think of as a mild strengthening of the $T_0$ separation axiom, implied by $T_2$ but neither implied by nor implying $T_1$). Virtually all the spaces one meets in "everyday life" (that is, apart from counterexamples cooked up for the purpose) are sober, and it does little harm to topology to pretend that all spaces are sober, since the replacement of X by its soberification $X_s$ does not change its open-set lattice, and does very little damage to its topological properties (the $T_1$ axiom being the only significant exception). Henceforth we shall assume all spaces are sober, and identify spaces with spatial locales, so (retrospectively) justifying our notation.

Sublocales of a given locale X correspond to quotient frames of $\mathcal{O}(X)$, as we have already remarked; they also correspond to nuclei on $\mathcal{O}(X)$, but the correspondence is order-reversing. The passage from elements of $\mathcal{O}(X)$ to the sublocales corresponding to open nuclei is thus order-preserving, and we frequently identify $\mathcal{O}(X)$ with the lattice of such cublocales — if X is a space, they are exactly the open subspaces of X. Likewise, the closed sublocales of a space are exactly its closed subspaces; in particular they are all spatial. However, with dense sublocales (as we emphasized in the Introduction), the picture is very different. Of course, we call a sublocale dense if it has nontrivial intersection with every nontrivial open sublocale of X (equivalently, if its closure is the whole of X); this is equivalent to saying that the bottom element 0 of $\mathcal{O}(X)$ belongs to the corresponding fixset, or that the corresponding nucleus j satisfies $j \leq q(0)$. In particular, the sublocale $X_b$ corresponding to q(0) is the smallest dense sublocale of X; it is rarely spatial, although each dense subspace of a space is a dense sublocale.

We digress briefly to justify the claim, made in the Introduction, that every locale is an intersection of dense subspaces of a space. Since free frames are spatial and every frame is a quotient of a free frame, every locale is a sublocale of a space; and since closed sublocales of spatial locales are spatial, we may assume it is dense. Moreover, the formula (2.6) for nuclei tells us that every sublocale is an intersection of sublocales, each of which is the union of an open and a closed sublocale; and such sublocales of a spatial locale, like any union of spatial sublocales, are spatial.

We write $X_d$ for the locale defined by $\mathcal{O}(X_d) = N(\mathcal{O}(X))$, and call it the dissolution of X; we may think of it as something like the discrete modification of a space, except that the construction $X \mapsto X_d$ is not idempotent. More specifically, since the correspondence between sublocales and nuclei is order-reversing, we see that the canonical map $X_d \to X$ (corresponding to c: $\mathcal{O}(X) \to N(\mathcal{O}(X))$) induces a bijection between closed sublocales of $X_d$ and arbitrary sublocales of X; so we can think of $X_d$ as "the result of declaring all sublocales of X to be closed". Moreover, $X_d$ has the same points as X (since $\mathcal{O}(1)$ is Boolean), and its spatial part is often discrete (at least if the spatial part of X satisfies a mild separation condition — the $T_D$ axiom of Aull and Thron [1] — to ensure that all its subspaces are sober and therefore define distinct sublocales). However, $X_d$ is rarely spatial [64]; the "dissolute" spaces which can occur as subspaces of $X_d$ for some X have been studied by Isbell [22].

A large part of "the art of pointless thinking" consists in finding locale-theoretic ways to express familiar topological concepts. For some (notably compactness and connectedness) the task is trivial, since these notions are traditionally defined in terms of properties of the lattice of open sets of a space, but others require more ingenuity. We cannot (sensibly) adopt definitions which talk about arbitrary points of a locale, since locales do not live on their points, but frequently we may use the lattice of all sublocales as an effective substitute for the lattice of all subsets.

For example, we have already remarked that the "classical" $T_1$ axiom for spaces must be sacrificed, since it not only mentions points but behaves badly with respect to soberification; but there are various alternatives that we can use. One such is the "unorderedness" axiom to which we alluded in the Introduction; another approach would be to replace the assertion "Every point is closed" by "Every subset is a union of closed sets", or equivalently "Every subset is an intersection of open sets". For locales, the two latter assertions are not equivalent, since not every sublocale has a complement; but the second of them (the assertion that every sublocale is an intersection of open sublocales — or equivalently, by (2.6), that every closed sublocale is an intersection of open sublocales) provides a useful separation condition for locales, which was christened fitness by Isbell [19] (and more recently rediscovered by Wraith [70]). Isbell observed that it implies the condition that every open sublocale is a union of closed sublocales (but not conversely); he called this condition subfitness, and it has also been studied in frame-theoretic terms by other authors [13, 62, 6], who often call it conjunctivity. For the relationship (or lack of it) between subfitness and unorderedness, see [48]. (Rather oddly, and very much counter to the general philosophy expounded here, the assertion "All points of X are closed" turns out not to be a totally meaningless condition on locales; it defines the epireflective hull of (sober) $T_1$-spaces in Loc [58]. However, it isn't much use as a separation axiom.)

For localic versions of the Hausdorff axiom, the picture is also somewhat confused, but here at least there is a clear "market leader": the assertion that the diagonal $X \to X \times X$ is a closed inclusion. The only failing of this condition is that it isn't equivalent to the classical Hausdorff axiom for spaces, because of the difference between space products and locale products: a space X may be closed in $(X \times X)_s$ without being closed in $X \times X$. For this reason, Isbell [19] called locales satisfying this condition "strongly Hausdorff"; however, most authors now call them simply Hausdorff [15, 34]. Numerous alternative axioms have been proposed [13, 62, 58, 51, 36], most of which have the advantage that they coincide with the classical Hausdorff axiom on spaces, but all are unsatisfactory in other respects.

For regularity, the position is at last clear: there is one axiom which is accepted by everyone, and which has no known drawbacks (unless we count Kříž's surprising observation [40] that it doesn't yield the epireflective hull in Loc of regular spaces). The reason is that the regularity axiom for spaces, though usually stated in a form involving points, is essentially frame-theoretic: it says that each open set is a union of open sets whose closures it contains. Defined thus for locales, it can readily be shown to combine well with compactness and to imply all the other separation axioms we have considered; for this reason, most authors take it (rather than the Hausdorff axiom) as the "basic" separation property for locales. The stronger separation conditions of complete regularity and zero-dimensionality, and the lattice-theoretic notion of normality, are similarly straightforward for locales.

In passing, let us indicate how the difference between space and locale products is forced upon us by the difference in intersections. Let X be a space which is Hausdorff as a locale (e.g. a regular space), and suppose X has dense subspaces Y and Z whose spatial intersection is not dense. The sublocale intersection $Y \cap Z$ is the pullback along $Y \times Z \to X \times X$ of the diagonal $X \to X \times X$, and so closed in $Y \times Z$; hence if $Y \times Z$ were spatial $Y \cap Z$ would be too. This simple observation can be pushed to quite surprising lengths: for example, if X is a localic group and Y, Z are any two dense sublocales of X, then the composite $Y \times Z \to X \times X \to X$ is epic — from which it follows that every localic subgroup of a localic group is closed [23, 32]. (We should perhaps mention that, by exploiting the interaction between the CSLat and PreFrm generators of frame coproducts (2.3), it is possible to define a symmetric monoidal "weak product" structure on Loc, which extends the spatial product on spatial locales [36]. However, this structure appears to have few practical uses.)

As we said in the Introduction, the discrepancies between locale products and space products are almost all to the advantage of the former. However, pullbacks in Loc are not quite so pleasant: one particular problem is that they do not preserve surjections (that is, epimorphisms in Loc). A simple example: let X be a regular space without isolated points; then the canonical morphism $X_{ds} \to X$ from the spatial discrete modification of X is surjective, but its pullback along $X_b \to X$ is not, since the Boolean part $X_b$ has no points. This lends a particular interest to the study of open maps [38] in Loc, since they (and open surjections) are stable under pullback. Of course, a map $f: X \to Y$ is called <u>open</u> if the image under f of any open sublocale of X is an open sublocale of Y; frame-theoretically, this is equivalent to saying that $f^*: \mathcal{O}(Y) \to \mathcal{O}(X)$ has a left adjoint $f_!$, and that the "Frobenius reciprocity" condition

$$f_!(U) \cap V = f_!(U \cap f^*(V)) \tag{3.1}$$

holds for $U \in \mathcal{O}(X)$, $V \in \mathcal{O}(Y)$. (The first condition implies that, for each U, $f_!(U)$ is the smallest open sublocale of Y containing the image of U; so if Y is a fit locale condition (3.1) may be omitted.) A map f: X → Y of (sober) spaces is open in the localic sense iff the spatial image of each open $U \subseteq X$ is "almost open" in Y in the sense that its soberification is open; Dobbertin [11] has constructed an example to separate this from the spatial notion of openness, but the difference disappears if Y is a $T_D$-space.

Open surjections are by no means the only universal (i.e. pullback-stable) surjections in Loc; proper surjections [27] also have this property, as does the canonical map $X_d \to X$ [34] although the latter is never open (unless it is an isomorphism) and rarely proper. But they are undoubtedly the most important class, largely on account of the descent theorem of Joyal and Tierney [38] (which, being topos-theoretic in character, lies rather outside the scope of this article). Having defined open maps, we may go on to define local homeomorphisms as those open f: X → Y for which the diagonal $X \to X \times_Y X$ is also open, and to establish the familiar equivalence between the categories of local homeomorphisms over a fixed locale X and of sheaves on X (that is, sheaves for the canonical Grothendieck topology on $\mathcal{O}(X)$).

As we have mentioned, compactness is easy to define for locales. A locale X is said to be locally compact if $\mathcal{O}(X)$ is a continuous lattice in the sense of [60]; at first sight this may surprise the novice, who might have expected a definition in terms of compact sublocales. However, in a continuous lattice the Scott-open filters separate the points [3], and in a frame each Scott-open filter corresponds to at least one compact quotient [30], so the definition could be recast in this form. (Incidentally, Scott-open filters in a frame A correspond to preframe homomorphisms A → 2; this is the reason why the "preframe approach" to frames is well-adapted to questions about compactness.) Compact regular locales, and coherent locales, are locally compact (in fact the former are all retracts of the latter [29]); if we assume the axiom of choice, then locally compact locales are all spatial [3]. Even without choice, locally compact locales are exactly the exponentiable objects of Loc [18], i.e. those X such that $(-)\times X$ has a right adjoint $(-)^X$. (We note that Loc, like Top, is not cartesian closed. Moreover, unlike Top, it does not even admit a biclosed monoidal structure [41]. The search for a (locally) cartesian closed category into which Loc may be nicely embedded is still relatively unexplored; the standard methods developed for Top and its relatives will not work here, because of the lack of a faithful "set of points" functor.)

Connectedness for locales is also easy to define, but tends to be rather badly behaved [42]; in particular, a product of two (non-spatial) connected locales need not be connected. However, it improves markedly [43, 47] in the presence of local connectedness (which is simply the assertion that each open sublocale is a union of connected open sublocales). We have already referred to the fact that connected and locally connected locales are (globally and semi-locally) path-connected: this means that the "evaluation at end-points" map $X^I \to X \times X$, where I is the closed unit interval, is an open surjection. (Note that $X^I$ need not be spatial when X is; this explains why the preceding result does not conflict with the well-known counterexamples to the corresponding assertion about spaces.) Starting from this result, it is clearly possible to develop a localic version of homotopy theory, although little work has been done in this area yet.

This concludes our brief survey of the fundamental topological properties in the category of locales. Of course, there are many more properties that could be (and have been) studied, but there is no room in this article to describe them. Likewise, we cannot do more than refer in passing to the existence of a well-developed theory of uniform locales, which was (yet again) initiated by Isbell [19], and has been extensively pursued in recent years by Pultr [53 - 57] and others. However, we hope that enough has been said to convince the reader that Loc really is a feasible context in which to develop topological ideas.

## 4. Locales Over a Base

In this section we shall comment briefly on some of the special features of locale theory over a base locale B, that is of the category Loc/B. We have already remarked on the fact that Loc/B is very like Loc, in that it is equivalent to the category of internal locales in the topos of sheaves on B; but the non-classical internal logic of the latter (unless $\mathcal{O}(B)$ is Boolean) gives rise to a number of new features. Of course, the fact (alluded to in the Introduction) that many of the key results of locale theory have constructively valid proofs means that they translate immediately to valid results about Loc/B, though occasionally some work is needed to interpret the definitions in this new context. For example, the exponentiable objects of Loc/B are exactly those $(X \to B)$ which correspond to locally compact internal locales in sheaves on B, since the proof in [18] is constructive; but it takes some work [27] to identify the latter as "locally perfect" maps $X \to B$. However, we are more concerned in this section with features which are simply not present in the classical case, because they involve conditions which are automatically satisfied, or pairs of conditions which are classically equivalent but constructively distinct.

In the first place, the property of openness becomes significant for locales as well as for maps of locales; classically, every morphism $X \to 1$ in Loc is open, but over a (non-Boolean) base B the locales whose projection maps to B are open are distinctly better behaved than those which lack this feature. Locales which are spatial over B (that is, those which are surjective images of discrete locales over B — the latter being defined as local homeomorphisms — or equivalently those which are covered by the images of their sections over open sublocales of B) are necessarily open. (However, spatial locales over B are in general less common than spatial locales over 1. One should in particular beware of the fact that, whilst open sublocales of spatial locales over B are spatial over B, closed sublocales need not be.)

An example of a context in which openness becomes important in fibrewise locale theory is the preservation of separation properties under exponentiation. Classically, if X is a locally compact locale then an exponential $Y^X$ is unordered (resp. Hausdorff, regular) if Y is. For unorderedness, there is no difficulty in constructivizing the proof; but for the other two properties, the proof of the corresponding result in Loc/B requires the extra assumption that X is open over B [27]. The reason for this requirement is worth noting: the proofs of the corresponding results in Top require the idea that every subset of X can be expressed as a union of points, and when we come to recast them in Loc we have to replace this by the idea that any sublocale of X may be covered by arbitrary small, but nonempty, sublocales — i.e. that any covering may be refined to a covering by nonempty sublocales. Of course, this is classically trivial, since we may simply delete the empty sublocale from any covering which contains it, but constructively we do not have such a sharp dichotomy between emptiness and nonemptiness, and the assertion above (the "positive covering lemma") turns out to be equivalent to the openness of X.

Another area where fibrewise locale theory diverges from "classical" locale theory concerns the bifurcation of the notion of closure. It is easy to see, even in the spatial context, that the usual "absolute" notion of closure is ill-adapted to fibrewise topology: if Y is a subspace of a space X over B, then the closure of Y may "spread sideways" into fibres of X (over points of B) which contain no points of Y itself. What one wants to use, in many contexts, is the fibrewise closure: that is, the union, over all points $b \in B$, of the closure of the fibre $Y_b$ in $X_b$. It is quite remarkable that a concept which at first sight appears to be tied so firmly to the existence of points in B can be expressed quite simply in localic terms, and even more so that this new notion of closure retains the crucial feature that distinguishes locales from spaces: an intersection of fibrewise dense sublocales is fibrewise dense.

Explicitly, a sublocale Y of a locale X over B is called B-_fibrewise_ _dense_ [33] if the corresponding fixset in $\mathcal{O}(X)$ contains the image of the frame homomorphism $\mathcal{O}(B) \to \mathcal{O}(X)$. B-_fibrewise_ _closed_ sublocales [25] correspond to the fibrewise closed nuclei defined at the end of section 2; if Y corresponds to a nucleus j on $\mathcal{O}(X)$, then its B-fibrewise closure corresponds to the unique smallest nucleus agreeing with j on the image of $\mathcal{O}(B) \to \mathcal{O}(X)$. The locale $X_d^B$ corresponding to the frame $N_{\mathcal{O}(B)}(\mathcal{O}(X))$ of fibrewise closed nuclei on $\mathcal{O}(X)$ is called the B-_dissolution_ of X; we may think of it as something like the effect of making each fibre of X over B discrete, but otherwise retaining as much as possible of the topology of X.

Fibrewise closure does not spread sideways: that is, a sublocale Y of X and its B-fibrewise closure $\overline{Y}$ have the same image in B. Moreover, Y is open over B iff $\overline{Y}$ is. The fibrewise notions have the properties one would expect under change of base (i.e. pullback along, or composition with, a locale map $B' \to B$), although for some of these results it is necessary to assume that the locales under consideration are open over B. Almost the only unsatisfactory feature of the notion is that fibrewise closed sublocales need not have complements; we do not have a corresponding notion of "fibrewise open" sublocale.

Fibrewise closure was originally introduced [33] in order to prove a fibrewise version of the theorem on closedness of localic subgroups, already mentioned in section 3, together with an extension of the result to localic groupoids. But it has a multitude of other applications: at least in theory, any topological property whose definition for locales involves closed or dense sublocales can spawn two distinct properties for locales over B, one using the absolute closure and the other the fibrewise one. (Naturally, one can expect to find cases where one or other of these properties turns out not to be worth serious study.) Among the concepts which have already been investigated from this point of view are the separation axioms [34] — it had long been a source of embarrassment that discrete locales over B, as we defined them earlier, need not satisfy the "absolute" versions of the Hausdorff or regularity axioms, but they do satisfy all the fibrewise ones — and compactness [66]; but much more remains to be done.

For instance, what can be said about the "fibrewise Boolean" locales over B which occur as the smallest B-fibrewise dense sublocales of arbitrary locales over B? It is clear that such locales have no proper B-fibrewise dense sublocales, and that this property is implied by the condition that all sublocales are B-fibrewise closed; but (at the time of writing) it is not known whether the converse of the last implication holds, although some partial results have been obtained by A. Kock, M. Jibladze and the author. (One difficulty is that fibrewise Booleanness is not stable under change of base. As was recently pointed out by D. Strauss, a locale of the form $X \times X$ is Boolean only if X itself is not only Boolean but spatial; so if X is Boolean and non-spatial then $X \times X$ is not fibrewise Boolean over X.)

There are many aspects of locale theory over a base which we have not even been able to mention in this brief survey. But we hope that the topics which have been mentioned are sufficient to convey some of the fascination of fibrewise locale theory.

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