Category Theory at Work
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The ABC of Order and Topology

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Abstract. The classical equivalence between quasiordered sets and Alexandrov-discrete spaces (A-spaces) extends to one-to-one correspondences between certain categories of generalized ordered sets where the order relations are still idempotent (hence transitive) but not necessarily reflexive, and categories of topological spaces whose topologies have a smallest base (B-spaces) or are completely distributive lattices (C-spaces). We introduce the appropriate types of morphisms making these one-to-one correspondences functorial and extend the results to arbitrary closure spaces which need not be topological. Moreover, we study various reflections, adjoint situations, equivalences and dualities between certain categories whose objects carry an order-theoretical or a topological structure. Our approach will provide the common framework for old and new topological representations of certain types of lattices by suitable spaces, and vice versa. The categorical point of view makes the manifold links between ordered and topological structures more transparent and yields the appropriate background for applications of topology to order theory, and conversely.

1. Introduction

When a beginner in category theory is confronted with adjoint situations for the first time, he will find this concept rather complicated and perhaps a bit artificial. Then, later on, when he recognizes many types of completions, free constructions, structure modifications, Galois connections, equivalences and dualities as special instances of this general concept, he will appreciate MacLane's slogan "*Adjoint functors arise everywhere*".

Mathematical areas at the borderline between order and topology are particularly suitable for an easy understanding of the powerful concept of adjoint situations at an elementary but nevertheless fruitful level. In the present study, it is our main goal to demonstrate how often adjoint functors occur when relationships between order-theoretical and topological structures are investigated. However, the links between ordered and topological structures are so manifold that we can mention here only a few basic ideas. For our purposes, the approach via closed sets is more adequate than that by open sets, because the former allows a natural extension to closure spaces, a central notion not only in topology but also in many other mathematical disciplines, especially when one is dealing with algebraic structures.

(a) *From topology to order*. Every topological space carries a natural quasiorder, the so-called *specialization* which is defined by calling an element x more special than y if x belongs to the closure of y. We shall generalize this procedure to arbitrary set systems (whose members may be interpreted as "properties" or "attributes").

(b) *From order to topology*. An ordered set carries many "intrinsic" topological structures. Classical "two-sided" topologies generalizing the Euclidean topology on the real line are the *interval topology*, the *order topology*, and the *Lawson topology* (see, for example, [16] or [19]). Each of these topologies is defined in terms of the order relation, but "forgets" the latter. "One-sided" topologies like the *upper topology*, the *Scott topology* and the *Alexandrov topology* (see again [16] or [19]) have the drawback that no separation axiom except the weakest, T_0, is fulfilled; on the other hand, these topologies have the advantage that the order relation is "coded" in the topology and can be reconstructed by specialization.

(c) *Order and topology together*. In many concrete mathematical situations, the sets under consideration carry not only an order structure but also a topological one, and these two are related by certain compatibility conditions. For example, one may postulate that the order-theoretical principal ideals have to be closed, or that the order relation has to be closed in the product space. The systematical investigation of such *ordered topological spaces* has been initiated by Nachbin [33] and is today a vital branch of topology.

(d) *The point of pointless topology*. In recent years, increasing attention has been paid to topology without points, that is, to the investigation of typical lattice-theoretical properties which are valid in lattices of open or closed sets and can be expressed without using points. This leads to the theory of *frames* or *locales* (see, for example, [29], [30], and Johnstone's article in this volume).

In the present note, we shall touch upon each of these four themes sporadically, starting with (a) and concluding with (d). A guiding principle will be the categorical point of view: in most cases, suitable *functors* will establish the various links between the ordered and the topological structures.

As we shall see, the well-known one-to-one correspondence between quasiordered sets and topological spaces in which arbitrary intersections of open sets are open, due to Alexandrov [2], extends to an equivalence between a weak type of ordered sets and so-called *locally supercompact spaces*. One of the manifold characterizations of such spaces is of purely lattice-theoretical nature, stating that the lattice of open, respectively closed, sets is completely distributive. This essentially provides the object part of a duality between locally supercompact sober spaces (i.e. *continuous ordered sets* endowed with their Scott topology) and completely distributive lattices, due to Hoffmann [24], Lawson [31], Hofmann and Mislove [28]. Parts of the presented material belong to the folklore of order-oriented topologists and are known at least on the object level; but after so many widespread specific discoveries in the field, we feel that it is about time to discuss various aspects of this theory under a common point of view and with particular emphasis on the right choice of morphisms and functors. For categorical background, we recommend the recent book by Adámek, Herrlich and Strecker [1].

2. The core of topology

Let us start with a few more or less obvious generalizations from topologies to arbitrary set systems. By a *system* on a set S, we mean a collection $\mathfrak{X}$ of subsets of S. The *specialization* $\leq_{\mathfrak{X}}$ is defined by setting $x \leq_{\mathfrak{X}} y$ iff for all $X \in \mathfrak{X}$, $y \in X$ implies $x \in X$. Evidently, this is always a *quasiorder*, i.e. a reflexive and transitive relation. The pair $(S,\mathfrak{X})$ will be called a *(generalized) space*. Although the extension of well-worn notions from topology to this very general type of spaces is unambiguous in most cases, some of them have to be treated with care in the extended context, so that it appears opportune to recall a few typical definitions.

The specialization $\leq_{\mathfrak{X}}$ is antisymmetric, hence a *(partial) order* iff $(S,\mathfrak{X})$ is a *T_0-space*, i.e. for any two distinct points of S, there is a member of $\mathfrak{X}$ containing exactly one of these points. For each subset Y of S, the intersection of all members of $\mathfrak{X}$ containing Y is the *closure* of Y, denoted by Y^-. The sets Y with $Y = Y^-$ are *closed* and their complements *open*. The collection $\mathcal{C}_{\mathfrak{X}}$ of all closed sets is a *closure system* and the pair $(S,\mathcal{C}_{\mathfrak{X}})$ a *closure space*. Notice that $(S,\mathfrak{X})$ itself is a closure space if and only if $\mathfrak{X}$ is *closed under arbitrary intersections*, i.e. identical with $\mathcal{C}_{\mathfrak{X}}$. On the other hand, the system $\mathcal{O}_{\mathfrak{X}}$ of all open sets is a *kernel system*, i.e. *closed under arbitrary unions*. The *interior* of a set Y, denoted by Y°, is the union of all open sets contained in Y. Notions such as *neighborhoods* and *bases* (of open sets, closed sets, or neighborhoods) are defined as for *topological* closure systems (which are closed under finite unions) and *topologies* (i.e. kernel systems which are closed under finite intersections).

For each $y \in S$, the *point closure* $\downarrow y$ is the intersection of all members of $\mathfrak{X}$ (hence of all closed sets) containing the point y. Thus $x \leq_{\mathfrak{X}} y$ means that x belongs to the point closure of y. In other words, the point closure of y is the order-theoretical *principal ideal generated by y*. But we can also form the intersection of all sets which contain y and are *complements* of sets from $\mathfrak{X}$. This intersection will be referred to as the *core* of y (with respect to $\mathfrak{X}$) and denoted by $\uparrow y$. In fact, an easy computation shows that the specialization $\leq_{\mathfrak{X}^*}$ of the complementary system $\mathfrak{X}^* = \{S \setminus X : X \in \mathfrak{X}\}$ is just the dual of the specialization quasiorder $\leq_{\mathfrak{X}}$, and consequently, the core $\uparrow y$ is the *principal dual ideal* generated by y with respect to $\leq_{\mathfrak{X}}$. Notice that $\uparrow y$ is also the intersection of all (open) neighborhoods of y. More generally, the intersection of all open sets containing a fixed subset Y is called the *saturation* of Y and denoted by $\uparrow Y$; indeed, this is the *upper set generated by Y*, with respect to the specialization quasiorder, i.e. the union of all cores $\uparrow y$ with $y \in Y$ (see [27] and Section 3). A set Y with $Y = \uparrow Y$ is said to be *saturated*. Thus Y is saturated iff it contains for each $y \in Y$ all elements x with $y \leq_{\mathfrak{X}} x$. On the other hand, $\downarrow Y$, the *lower set generated by Y*, is the union of all closures of points from Y. Hence Y is a lower set iff it contains with y all elements $x \leq_{\mathfrak{X}} y$. We remark that $\mathfrak{X}$ is contained in $\mathcal{C}_{\mathfrak{X}}$ and that each closed set is a lower set, while $\mathfrak{X}^*$ is contained in $\mathcal{O}_{\mathfrak{X}}$ and each open set is an upper set.

Closure spaces are in one-to-one correspondence with so-called standard completions, where a *standard completion* of a quasiordered set $(S, \leq)$ is a closure system $\mathcal{C}$ of lower sets containing all principal ideals; in this case, the specialization of $\mathcal{C}$ is just the given quasiorder $\leq$ (see [17]; in [13] and [15], the reader can find a systematical investigation of standard completions with particular emphasis on adjunctions and reflections).

Of course, cores will not be open in general. Indeed, the cores of a closure system $\mathcal{C}$ are open if and only if $\mathscr{C}$ is also a kernel system, hence an *Alexandrov-discrete topology* or *A-topology* (alias *complete ring of sets*). In this case, the pair $(S,\mathscr{C})$ is commonly referred to as an *Alexandrov-discrete space*, or *A-space* for short (the "*A-spaces*" studied by Ershov [18] are something different; they are closer related to our "*C-spaces*" introduced later on). Our choice of the letter A points to Alexandrov, but also to the fact that in an A-space, *arbitrary* unions of open sets are open, and that the closed sets of such spaces are precisely the lower sets ("*Abschnitte*" in German; see Theorem 2.1 below): indeed, the closure Y^- agrees with the lower set $\downarrow Y$. Notice that for any A-space $(S,\mathscr{C})$, the pair $(S,\mathscr{C}^*)$ is an A-space, too, so that words like "open" etc. have to be used with care: the open sets of $(S,\mathscr{C})$ are just the closed sets of $(S,\mathscr{C}^*)$.

The notion of continuity is a bit subtle in the framework of general spaces: we mean by a *continuous map* between spaces $(S,\mathfrak{X})$ and $(S',\mathfrak{X}')$ a map $\varphi : S \to S'$ such that $\varphi^{-1}[X'] \in \mathfrak{X}$ for all $X' \in \mathfrak{X}'$. Thus every continuous map between $(S,\mathfrak{X})$ and $(S',\mathfrak{X}')$ is also a continuous map between the closure spaces $(S,\mathscr{C}_{\mathfrak{X}})$ and $(S',\mathscr{C}_{\mathfrak{X}'})$, respectively, between the kernel spaces $(S,\mathcal{O}_{\mathfrak{X}})$ and $(S',\mathcal{O}_{\mathfrak{X}'})$, *but not conversely*. As in the classical topological situation, a map $\varphi: S \to S'$ is continuous as a map between $(S,\mathscr{C}_{\mathfrak{X}})$ and $(S',\mathscr{C}_{\mathfrak{X}'})$ iff $\varphi[Y^-] \subseteq \varphi[Y]^-$ for all $Y \subseteq S$; in particular, $\varphi[\downarrow y] \subseteq \downarrow\varphi(y)$ for all $y \in S$. Hence every continuous function φ *preserves specialization*, that is,

$$x \leq_{\mathfrak{X}} y \text{ implies } \varphi(x) \leq_{\mathfrak{X}'} \varphi(y).$$

In other words, φ is an *isotone* function between the quasiordered sets $(S, \leq_{\mathfrak{X}})$ and $(S', \leq_{\mathfrak{X}'})$. Hence the *specialization functor Q*, assigning to each space $(S,\mathfrak{X})$ the quasiordered set $(S, \leq_{\mathfrak{X}})$ and acting identically on the underlying set maps, is a (concrete) functor from the category of spaces and continuous maps to the category of quasiordered sets and isotone, i.e. order-preserving maps. In the converse direction, we may assign to each quasiordered set $(S, \leq)$ the A-space $L(S, \leq) = (S, \mathscr{C}_{\leq})$, where $\mathscr{C}_{\leq}$ is the system of all *lower sets*, i.e. subsets Y of S such that $y \in Y$ implies $x \in Y$ for all $x \leq y$. In this context, the lower sets have to be regarded as *closed sets*, and their complements, the *upper sets*, as open sets; these constitute the *upper Alexandrov topology*, and the lower sets the *lower Alexandrov topology*. Already in the thirties, it has been observed by Alexandrov [2] that the specialization of the A-space $(S,\mathscr{C}_{\leq})$ is the original quasiorder $\leq$, and conversely, that the closed sets of an arbitrary A-space are the lower sets with respect to the specialization quasiorder. Moreover, the continuous maps between A-spaces are precisely the isotone maps between the corresponding quasiordered sets. In modern categorical terminology, this may be rephrased as a simple but fundamental isomorphism theorem:

THEOREM 2.1. *The specialization functor Q induces a concrete isomorphism between the category of A-spaces and the category of quasiordered sets, with inverse functor L.*

Let us mention briefly a few applications of this categorical isomorphism: Products of quasiordered sets correspond to products of A-spaces, but these are not the usual topological products: one has to take the "box product", where arbitrary products of open sets, without any restriction, form a base for the product space. The supremum of quasiorders on a fixed set corresponds to the intersection of the associated Alexandrov topologies; conversely, the intersection of quasiorders corresponds to the join of the associated topologies. Sums, i.e. disjoint unions of A-spaces, correspond to disjoint unions of quasiordered sets; in particular, a quasiordered set is connected in the relational sense iff the associated A-space is (path-)

connected in the topological sense; normal connected A-spaces correspond to down-directed quasiordered sets, regular A-spaces to equivalence relations, and so on.

Of course, we may interpret the lower set functor L as a functor from the category of quasiordered sets to the category of spaces and continuous maps. This functor is no longer inverse to the specialization functor Q in the converse direction, but it is still *right inverse and left adjoint to* Q, because for any quasiordered set $(S, \leq)$ and any space $(S', \mathcal{X}')$, the isotone maps between $(S, \leq)$ and $Q(S', \mathcal{X}')$ are precisely the continuous maps between the spaces $L(S, \leq)$ and $(S', \mathcal{X}')$ (cf. [15] and [27]). More generally, one verifies easily that *a map* φ *between spaces* $(S, \mathcal{X})$ *and* $(S', \mathcal{X}')$ *preserves specialization iff for all* $X' \in \mathcal{X}'$, *the inverse image* $\varphi^{-1}[X']$ *is a lower set.*

Furthermore, L restricts to a functor from the category of *(partially) ordered sets* to the category of *T_0-spaces*, i.e. spaces in which distinct points have distinct closures. Similarly, Q restricts to a functor in the other direction, and both functors together establish a concrete isomorphism between the category of ordered sets and the category of T_0-A-spaces.

Although the one-to-one correspondence between A-spaces and quasiordered sets (respectively, between T_0-A-spaces and ordered sets) is quite nice and provides the source for a broad spectrum of applications, in particular for elegant solutions of problems concerning finite sets (where every topology is already an A-topology), not too many topologists pay much attention to A-spaces because of their rare occurence in "real analysis". Hence the question arises: How can we enlarge the category of A-spaces on the one hand and the category of quasiordered sets on the other hand, so that we still keep an equivalence between the topological and the order-theoretical structures, but many more interesting "classical" topologies are included in the extended definition?

A rather satisfactory answer to this question is suggested by the observation that every point in an A-space has a smallest neighborhood; in other words, that the cores are open and form a smallest base for the open sets; in particular, each point has a neighborhood base consisting of a single core. Now we call a closure space a *basic space* if it has an open base of cores, and a *core space* if each point y has a neighborhood base of cores (which neither have to be open nor generated by y). Explicitly, this means that for each open set U containing y, there is an element $x \in U$ and an open set V such that $y \in V \subseteq \uparrow x$. By a *B-space*, we mean a topological basic space, and by a *C-space*, a topological core space. Although B-spaces and C-spaces have been investigated by various authors (see, e.g., [3], [6], [11], [17], [18], [23]), no common nomenclature seems to have been established for them. The letter B refers to the word *base*, but also to Balbedat who called such spaces *monotope* (cf. [6] and [27]). Of course, the letter C reminds us of *cores*, but it also points to the fact that the closure systems of core spaces may be characterized by the purely lattice-theoretical property of being *completely distributive* (see Proposition 2.2.C).

Another, more topological description of basic spaces and of core spaces (respectively, of B- and C-spaces) is obtained by the following remark. Notice that every core (regardless whether open or not) is a *supercompact* (or *monogeneous*) set, that is, a set C with the property that for any collection $\mathcal{U}$ of open sets with $C \subseteq \bigcup \mathcal{U}$, there is some $U \in \mathcal{U}$ with $C \subseteq U$ (this interpretation of the word *supercompact* has been used by Grimeisen and others; some topologists, for example members of de Groot's school, use the word *supercompact* in a wider sense). Moreover, *the cores are precisely the supercompact saturated sets*: indeed, if a set

Y is saturated but not a core then for each $x \in Y$ there is an open set U with $x \in U$ and $Y \nsubseteq U$ (otherwise $Y = \uparrow x$): but then Y cannot be supercompact. In particular, the open cores are precisely the supercompact open sets. Thus exactly those closure spaces which have a base of supercompact open sets are basic, while core spaces are characterized by the property that each point has a neighborhood base of supercompact sets. Hence, C-spaces may be called *locally supercompact* as well. Interesting for topologists is the fact that (locally) supercompact spaces are both (locally) connected and (locally) compact, but it should be emphasized that for us, (local) compactness does *not* include the Hausdorff separation property. In modern topology, non-Hausdorff locally (quasi)compact spaces play a considerable rôle (see, for example, [26] and [27]). In fact, a restriction to higher separation axioms would make our studies on core spaces rather ineffective because the only core spaces with closed points are the discrete ones. In particular, it is clear that real Euclidean spaces cannot be C-spaces. However, the *upper topology* on $\mathbb{R}$, consisting of all "open rays" $]y) = \{x \in \mathbb{R}: x > y\}$ together with $\mathbb{R}$ and the empty set, makes $\mathbb{R}$ a C-space which is not a B-space, and the same holds for the dually defined *lower topology*. The Euclidean topology is generated by these two, and conversely, the upper (resp. lower) topology may be reconstructed from the Euclidean topology and the natural order on $\mathbb{R}$ by taking all open upper (resp. lower) sets. A thorough investigation of this phenomenon in the more general context of ordered topological spaces leads to an interesting synthesis of order and topology (see [33]), and in particular, to the theory of compact pospaces (see e.g. [19]).

The notions of compactness and of supercompactness extend in an obvious manner to pointless topology and, more generally, to lattice theory: an element x of a complete lattice L is called *compact* if for all directed subsets Y of L whose join (supremum) dominates x, there is an element $y \in Y$ with $x \leq y$; the element x is said to be *(finitely) join-prime*, written $\vee$-*prime*, if this condition holds for all finite sets Y, and x is called *supercompact* or *completely join-prime* if this condition is fulfilled for arbitrary subsets Y of L. Furthermore, a complete lattice is called *(super)compactly generated* or *(super)algebraic* if each of its elements is a join of (super)compact elements. It is easy to see that the property of being superalgebraic is selfdual (cf. [34]). A complete lattice in which every element is a join of $\vee$-prime elements is called $\vee$-*primely generated*, *cospatial*, or a *T-lattice*, with regard to the fact that such lattices are, up to isomorphism, the lattices of closed sets in topological spaces (see Proposition 5.6). Dually, lattices isomorphic to topologies are called *spatial*.

If a closure system $\mathscr{C}$ is considered as a complete lattice, the notion of (super)compactness has to be treated with care: a closed set C is compact in the topological sense iff for each collection $\mathscr{Y}$ of *open* sets with $C \subseteq \bigcup\mathscr{Y}$, there is a finite $\mathscr{F} \subseteq \mathscr{Y}$ with $C \subseteq \bigcup\mathscr{F}$, while C is a compact member of $\mathscr{C}$ in the lattice-theoretical sense if for each collection $\mathscr{Y}$ of *closed* sets with $C \subseteq (\bigcup\mathscr{Y})^-$ (the closure is essential!) there is a finite $\mathscr{F} \subseteq \mathscr{Y}$ with $C \subseteq (\bigcup\mathscr{F})^-$.

Next, let us turn to certain "infinite" distributive laws which are of particular interest in connection with the previous definitions and facts. A complete lattice L is called *completely distributive* iff for any collection $\mathscr{Y}$ of subsets and for the "crosscut system"

$$\mathscr{Y}^{\#} = \{Z \subseteq L: Y \cap Z \neq \emptyset \text{ for all } Y \in \mathscr{Y}\},$$

one has the identity

$$\text{(D)} \quad \bigwedge\{\bigvee Y: Y \in \mathscr{Y}\} = \bigvee\{\bigwedge Z: Z \in \mathscr{Y}^{\#}\}.$$

Notice that this definition avoids choice functions and enables us to prove many results about such lattices without using the axiom of choice. For example, observing that $\mathcal{Y}^{\#\#}$ is the upper set generated by $\mathcal{Y}$ in the power set of L, it is very easy to see that complete distributivity is a selfdual property, while the usual proofs using choice functions are rather tedious. Also, the crucial fact that A-topologies are completely distributive is an immediate consequence of our definition: for the nontrivial inclusion $\bigcap\{\bigcup Y : Y \in \mathcal{Y}\} \subseteq \bigcup\{\bigcap Z : Z \in \mathcal{Y}^{\#}\}$, notice that each element x of the left hand side belongs to the set $\bigcap Z$, where $Z = \{X \in \bigcup \mathcal{Y} : x \in X\} \in \mathcal{Y}^{\#}$.

The complete lattices enjoying the distributive law (D) at least for all families of *directed* subsets are just the *continuous lattices* in the sense of Scott (see [19]). We note the well-known fact that every superalgebraic lattice is completely distributive, and every algebraic lattice is continuous (the proof is an easy exercise).

Of course, in every completely distributive lattice, the infinite distributive law

(d) $x \wedge \bigvee Y = \bigvee\{x \wedge y : y \in Y\}$

and its dual are fulfilled for arbitrary subsets Y. Complete lattices satisfying (d) are called $\bigvee$-*distributive*, *frames* or *locales*. Thus, for example, every spatial lattice is a frame, and every cospatial lattice is a dual frame.

Now let us collect a few alternative characterizations of A-, B- and C-spaces (respectively, of basic spaces and of core spaces). For A, these characterizations are straightforward and belong to the "folklore" of topology. For B and C, we could refer to [11], [17], [22], and [24]; but for the readers convenience, we give short ad-hoc proofs.

PROPOSITION 2.2.A. *The following conditions on a closure space* $(S,\mathscr{C})$ *are equivalent:*

(a) $(S,\mathscr{C})$ *is an A-space.*

(b) *Each point has a smallest neighborhood.*

(c) *All cores are open.*

(d) *The closure operator of* $\mathscr{C}$ *preserves arbitrary unions.*

(e) $\mathscr{C}$ *is a topology.*

PROPOSITION 2.2.B. *The following conditions on a closure space* $(S,\mathscr{C})$ *are equivalent:*

(a) $(S,\mathscr{C})$ *is a basic space.*

(b) *The open cores form the smallest base for the open sets.*

(c) $(S,\mathscr{C})$ *has an A-subspace whose closure system is isomorphic to* $\mathscr{C}$ *via relativization.*

(d) $\mathscr{C}$ *is isomorphic to an A-topology.*

(e) $\mathscr{C}$ *is a superalgebraic lattice.*

Furthermore, a topological space is a B-space iff it has a smallest resp. minimal base.

PROOF. (a) $\Rightarrow$ (b): Being supercompact, an open core belongs to every base of the given space.
(b) $\Rightarrow$ (c): Setting $B = \{y \in S : \uparrow y \text{ is open}\}$, we obtain an A-subspace $(B,\mathscr{C}_{|B})$ (cf. 3.2), and the relativization map $X \mapsto X \cap B$ is a lattice isomorphism between $\mathscr{C}$ and $\mathscr{C}_{|B}$.

The implications (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are clear.

(e) $\Rightarrow$ (a): Since $\mathscr{C}$ is dually isomorphic to the lattice of open sets, the latter is superalgebraic, too. Hence every open set is a union of superalgebraic open sets, i.e. of open cores.

Finally, in a topological space possessing a minimal open base $\mathcal{B}$, each member of $\mathcal{B}$ is supercompact (otherwise, it could be omitted). Hence the members of $\mathcal{B}$ are open cores. $\square$

(*Caution*: a non-topological closure space possessing a smallest base need not be basic; indeed, each finite closure space has a smallest base, but finite basic spaces are rather rare!)

PROPOSITION 2.2.C. *The following conditions on a closure space* $(S,\mathscr{C})$ *are equivalent:*

(a) $(S,\mathscr{C})$ *is a core space.*

(b) $(S,\mathscr{C})$ *is locally supercompact.*

(c) *The closure operator preserves intersections of lower sets (with respect to specialization).*

(d) $y \in (\{x \in S : y \in (\uparrow x)^\circ\})^-$ *for all* $y \in S$.

(e) $\mathscr{C}$ *is a completely distributive lattice.*

Hence a topological space is a C-space iff its topology is a completely distributive lattice.

PROOF. (a)$\Leftrightarrow$(b): This equivalence has already been explained earlier.

(a)$\Rightarrow$(c): For any collection $\mathscr{Y}$ of lower sets, we must prove the inclusion $\bigcap\{Y^- : Y \in \mathscr{Y}\} \subseteq (\bigcap\mathscr{Y})^-$. For each y in the complement of $(\bigcap\mathscr{Y})^-$, there exists an open set U containing y and disjoint from $\bigcap\mathscr{Y}$. Choosing an $x \in U$ with $y \in (\uparrow x)^\circ$, we find some $Y \in \mathscr{Y}$ not intersecting the core $\uparrow x$ (otherwise $x \in \downarrow Y = Y$ for all $Y \in \mathscr{Y}$, in contrast to $U \cap \bigcap\mathscr{Y} = \emptyset$). Hence $y \in (\uparrow x)^\circ \subseteq (X \setminus Y)^\circ = X \setminus Y^-$.

(c)$\Rightarrow$(d): As the lower sets are precisely the complements of upper sets, we may reformulate (c) in terms of open sets by saying that the interior operator preserves arbitrary unions of upper sets. In particular, assuming $y \in U = S \setminus (\{x \in S : y \in (\uparrow x)^\circ\})^-$, we would obtain $y \in U^\circ = (\bigcup\{\uparrow x : x \in U\})^\circ = \bigcup\{(\uparrow x)^\circ : x \in U\}$ and therefore $y \in (\uparrow x)^\circ$ for some $x \in U$, a contradiction.

(d)$\Rightarrow$(a): If U were a neighborhood of an element y such that $y \notin (\uparrow x)^\circ$ for all $x \in U$ then $y \in (\{x \in S : y \in (\uparrow x)^\circ\})^- \subseteq (S \setminus U)^- = S \setminus U$, a contradiction.

(c)$\Rightarrow$(e): The closure operator of $\mathscr{C}$ induces a complete homomorphism from the completely distributive A-topology of all lower sets onto $\mathscr{C}$ (observe that $(\bigcup\mathscr{Y})^- = \bigvee\{Y^- : Y \in \mathscr{Y}\}$).

(e)$\Rightarrow$(c): For any system $\mathscr{Y}$ of lower sets, the complete distributive law yields:

$\bigcap\{Y^- : Y \in \mathscr{Y}\} = \bigcap\{\bigvee\{\downarrow x : x \in Y\} : Y \in \mathscr{Y}\} = \bigvee\{\bigcap\mathscr{X} : \text{for all } Y \in \mathscr{Y}, \text{ there is an } x \in Y \text{ with } \downarrow x \in \mathscr{X}\} = (\bigcup\{\bigcap\mathscr{X} : \text{for all } Y \in \mathscr{Y}, \text{ there is an } x \in Y \text{ with } \downarrow x \in \mathscr{X}\})^- = (\bigcap\{\bigcup\{\downarrow x : x \in Y\} : Y \in \mathscr{Y}\})^- = (\bigcap\mathscr{Y})^-$. □

In the last section, we shall show that conversely, every completely distributive lattice is isomorphic to the topology (respectively, to the closure system) of a C-space.

An interesting generalization of C-spaces has been studied by Hoffmann [21], namely what he called "*spaces admitting a dual*". These are topological spaces having a base of $\vee$-prime, alias *strongly connected* [32] or *ultraconnected* [36] open sets. (A space is ultraconnected iff each closed subspace is connected). In lattice-theoretical terms, such spaces are characterized by the condition that the lattice of closed sets is not only cospatial, but also spatial. With the help of the previous results, one can show that *the locally supercompact topological spaces are precisely the locally compact topological spaces admitting a dual.*

From the viewpoint of categorical topology, it is important to know under what circumstances the (topological) product of A-, B- and C-spaces, respectively, is again such a space. The answer is easily given with the help of a general result due to Hoffmann [21]. Let P be any class of topological spaces which is closed under the formation of continuous images and products. The members of P are referred to as *P-spaces;* furthermore, call a topological space *P-basic* if it has an open base of subsets which are P-spaces, and *P-local* if each point has a neighborhood base consisting of subsets which are P-spaces. Then *a product of nonempty topological spaces is P-basic (respectively, P-local) iff all factors are P-basic (re-*

spectively, P-local) and all but a finite number of the factors are P-spaces. This result applies, for example, to the following special classes P:

P-spaces	P-basic spaces	P-local spaces
compact spaces	spaces with a compact-open base	locally compact spaces
connected spaces	locally connected spaces	locally connected spaces
ultraconnected spaces	spaces having a dual	locally ultraconnected spaces
supercompact spaces	B-spaces	C-spaces

3. The core of order theory

Our next aim is a description of B-spaces and C-spaces by means of certain generalized order relations, in the same vein as A-spaces are described by quasiorders.

What are typical properties of an order relation? Doubtless transitivity, and perhaps, in more restricted situations, antisymmetry and linearity. That order relations frequently are assumed to be reflexive is just a matter of convenience. In fact, the irreflexive approach is tantamount and sometimes even preferred to the reflexive one. A quasiorder ρ has not only the property of *transitivity*,

$x\rho y$ and $y\rho z$ for some y implies $x\rho z$,

but, being reflexive, it also satisfies the converse implication, sometimes referred to as the *interpolation property:*

$x\rho z$ implies $x\rho y$ and $y\rho z$ for some y,

which has also been discovered in many non-reflexive situations (see, for example, [19], [20], and [35]). If $<$ denotes the "strictly less" relation associated with a reflexive order relation $\leq$ then the interpolation property of $<$ is often referred to as *density*. Both properties together, transitivity and interpolation property, characterize *idempotent relations*, that is, relations ρ with $\rho\rho = \rho$, where the *product* $\rho\sigma$ of two relations ρ and σ is defined by

$x\rho\sigma z$ iff $x\rho y$ and $y\sigma z$ for some y.

A wide field of applications shows that idempotency plays a crucial rôle in order theory as well as in many other mathematical theories. Therefore, we will be mainly concerned with idempotent relations in the subsequent paragraphs.

Given an arbitrary relation ρ, the following notations will be convenient:

$\rho y = \{x : x\rho y\}$, $\rho Y = \bigcup\{\rho y : y \in Y\}$,

and dually,

$y\rho = \{x : y\rho x\}$, $Y\rho = \bigcup\{y\rho : y \in Y\}$.

If $\leq$ is a quasiorder, we shall use the more common (and perhaps more suggestive) notations introduced in Section 2. Thus we write $\downarrow y$ for $\leq y$, $\downarrow Y$ for $\leq Y$, $\uparrow y$ for $y\leq$, and $\uparrow Y$ for $Y\leq$.

Generalizing the notion of bases for topological spaces, we call a subset B of a quasiordered set $Q = (S, \leq)$ *join-dense* if each element of S is a join of elements from B; in other words, if for all $x, z \in S$ with $x \not\leq z$, there exists a $y \in B$ such that $y \leq x$ but $y \not\leq z$. By a *base* of Q, we mean a join-dense subset B such that $z \in B$ and $z \le y \le z$ imply $y \in B$. Of course, in case of ordered sets, the latter condition is automatically fulfilled, by antisymmetry of the relation $\le$.

To have a name for relations corresponding to T_0-spaces, we call a relation ρ *separating* (in [17]: *identitive*) if $\rho x = \rho y$ implies $x = y$. Thus the orders are just the separating quasiorders. However, neither is a separating relation always antisymmetric, nor conversely. Any relation ρ on a set S induces a "lower" quasiorder $\le_\rho$ and an "upper" quasiorder $\le^\rho$ on S by

$$x \le_\rho y \ \text{ iff } \ \rho x \subseteq \rho y, \qquad x \le^\rho y \ \text{ iff } \ y\rho \subseteq x\rho,$$

and $\le_\rho$ is an order iff ρ is separating. Notice also that ρ is transitive iff it is contained in $\le_\rho$ (resp. in $\le^\rho$), and that ρ itself is a quasiorder iff it coincides with $\le_\rho$ (resp. with $\le^\rho$). By a *lower relation*, we mean a relation ρ such that each of the sets ρy is a lower set with respect to the quasiorder $\le_\rho$. This is tantamount to saying that $x \le_\rho y$ implies $x \le^\rho y$.
By definition, every relation ρ satisfies the equation $\rho = \rho \le_\rho$, while the equation $\rho = \le_\rho \rho$ is necessary and sufficient for ρ to be a lower relation. An idempotent lower relation ρ on a set S will be called a *core quasiorder* and the pair (S,ρ) a *cordially quasiordered set*. If, in addition, ρ is separating then we refer to ρ as a *core order* and to the pair (S,ρ) as a *cordially ordered set*. Although this sounds a bit ironically, we believe that these notions actually lie in the core of order theory, and a further topological motivation for this nomenclature will be given later on (suggestions for a better name with the initial c are welcome).
Any idempotent relation has the following *approximation property*:

LEMMA 3.1. *If ρ is an idempotent relation on a set S then each element y of S is a least upper bound of the set ρy (with respect to $\le_\rho$).*

PROOF. $x \in \rho y$ implies $x \le_\rho y$ by transititivity of ρ, and if z is any upper bound of ρy then we have $\rho x \subseteq \rho z$ for all $x \in \rho y$, and consequently, $\rho y = \rho\rho y = \bigcup\{\rho x : x \rho y\} \subseteq \rho z$, i.e. $y \le_\rho z$. □

Of course, the (quasi)orders are precisely the reflexive core (quasi)orders. Moreover, quasiorders trivially have the following property:

For all x,z with $x\rho z$, there exists some y with $x\rho y\rho y\rho z$.

Core quasiorders with this "double interpolation property" will be called *basic quasiorders* and the corresponding pairs (S,ρ) *basically quasiordered sets*. If, in addition, the relations in question are separating then we speak of *basic orders* and *basically ordered sets*. These notations are justified by

PROPOSITION 3.2. *Assigning to each base B of a (quasi)ordered set $Q = (S, \le)$ the relation $\rho_B = \{(x,z) \in S \times S : x \le y \le z$ for some $y \in B\}$, one obtains a one-to-one correspondence between the bases of Q and the basic (quasi)orders with induced lower (quasi)order $\le$.*

PROOF. Suppose B is a base for Q. Then ρ_B is transitive. The equivalence

$$x \le y \iff B \cap {\downarrow}x \subseteq B \cap {\downarrow}y \iff \rho_B x \subseteq \rho_B y$$

shows that the lower quasiorder induced by ρ_B is the original quasiorder $\le$. Clearly, each $\rho_B y$ is a lower set with respect to $\le$; thus ρ_B is a lower relation. Further, $x \rho_B z$ means $x \le y \le z$ for some $y \in B$, whence $x\rho_B y \rho_B y \rho_B z$. In all, we conclude that ρ_B is a basic quasiorder with $B = \{y \in S : y\rho_B y\}$, since $y \le z \le y$ for some $z \in B$ implies $y \in B$.

(*Remark.* The proof of the last statement given in [17] contains a little gap in the general case of *quasi*ordered sets; we have adapted the definition of bases to this case.)
Conversely, let ρ be any basic quasiorder with induced lower quasiorder $\leq$, and $B=\{y\in S: y\rho y\}$. For $x\not\leq z$ we find an element w with $w\rho x$ but not $w\rho z$, and then a $y\in B$ with $w\rho y\rho x$, whence $\rho y\subseteq\rho\rho x=\rho x$ and $\rho y\not\subseteq\rho z$. Thus $y\leq x$ but $y\not\leq z$.
Furthermore, $z\in B$ and $z\leq y\leq z$ imply $y\leq z\in\rho z\subseteq\rho y$, and as ρy is a lower set, $y\rho y$, i.e. $y\in B$. Hence B is a base for Q, and

$$x\rho_B z \Leftrightarrow x\leq y\rho y\leq z \text{ for some } y \Leftrightarrow x\rho y\rho y\rho z \text{ for some } y \Leftrightarrow x\rho z. \ \square$$

For the intended topological interpretation of generalized order relations, we need some further definitions. Let ρ be a relation on a set S. A subset D of S is *directed* by ρ if for each finite subset F of D, there exists some $z\in D$ with $F\subseteq\rho z$ (in particular, D must be nonempty; in case of a *transitive* relation ρ, a set D is directed by ρ iff $D\neq\emptyset$ and for $x,y\in D$, there is some $z\in D$ with $x\rho z$ and $y\rho z$). By an *ideal* relation, we mean a relation ρ on S such that each of the sets ρy $(y\in S)$ is a directed lower set ("*ideal*" in the sense of [19]) with respect to the *lower* quasiorder $\leq_\rho$. On the other hand, we call a relation ρ on S *topological* if for each $y\in S$, the set ρy is directed by the *upper* quasiorder $\leq^\rho$; that is, for each finite subset F of ρy, there is an element $z\in\rho y$ with $z\rho\subseteq x\rho$ for all $x\in F$. The next lemma yields the elementary relationships between these definitions and explains the name "*topological relation*".

LEMMA 3.3. (1) *A relation ρ is topological iff the sets $x\rho$ form the base of a topology.*
(2) *Every ideal relation is topological.*
(3) *A core quasiorder ρ is ideal iff it is topological iff each ρy is directed by ρ.*

PROOF. (1) By definition, a relation ρ on S is topological iff for all finite sets $F\subseteq S$ and for all $y\in\bigcap\{x\rho: x\in F\}$, there exists a $z\in S$ with $y\in z\rho\subseteq\bigcap\{x\rho: x\in F\}$. But this is just the well-known criterion for the collection $\mathcal{B}=\{x\rho: x\in S\}$ to be a base of a topology.
(2) This is clear since for a lower relation ρ, $x\leq_\rho y$ implies $x\leq^\rho y$.
(3) A transitive relation ρ is contained in $\leq_\rho$; hence, if ρy is directed by ρ then also by $\leq_\rho$. It remains to show that for a topological core quasiorder ρ, each of the sets ρy is directed by ρ. For a finite subset F of ρy, we find some $z\in\rho y$ with $z\rho\subseteq x\rho$ for all $x\in F$. The interpolation property yields an element z' with $z\rho z'\rho y$, whence $z'\in z\rho\subseteq x\rho$ for all $x\in F$, i.e. $F\subseteq\rho z'$ and $z'\in\rho y$. $\square$

Now, by a *B-(quasi-)order* we mean a topological basic (quasi-)order, and by a *C-(quasi-) order* a topological core (quasi-)order. Lemma 3.3 provides the following short description: a lower relation ρ on S is a C-quasiorder iff for all $y\in S$ and finite $F\subseteq S$,

$$F\subseteq\rho y \Leftrightarrow F\subseteq\rho z \text{ for some } z\in\rho y,$$

and it is a B-quasiorder iff, in addition, one can choose z such that $z\rho z$.

Let us mention that by 3.1, the C-orders inducing a given order $\leq$ on S are precisely the so-called *approximating auxiliary orders* for the ordered set $(S,\leq)$ in the sense of [19]. These play a key rôle in the theory of *continuous ideal completions and compactifications* developed by Gierz and Keimel [20]; for a different aspect of C-orders, see Ershov [18].

EXAMPLES 3.4. (1) The standard example of a C-order is the usual strict order $<$ on $\mathbb{R}$ and, more generally, the componentwise strict order on $\mathbb{R}^n$. The restriction of $<$ to the unit interval $[0,1]$ is a core order but not a C-order, as the set <0 is empty. Of course, neither on $\mathbb{R}$ nor on $[0,1]$, the relation $<$ is basic.

(2) A complete lattice L is continuous iff the *way-below relation* $\ll$, defined by $x \ll y$ iff x belongs to each ideal whose join dominates y, is the smallest C-order inducing the order of L; hence, by 3.1, L is continuous iff $x = \bigvee \ll x$ for all $x \in L$ (see [19]).
On the real unit interval, one has $x \ll y$ iff $x < y$ or $x = 0$.
Similarly, L is completely distributive iff the relation ρ, defined by $x \rho y$ iff x belongs to each lower set whose join dominates y, is the smallest core order inducing the given order on L; hence L is completely distributive iff $x = \bigvee \rho x$ for all $x \in L$ (see [35]). However, in general, ρ fails to be an ideal relation.

(3) A complete lattice L is algebraic iff its way-below relation is the smallest B-order inducing the order of L. The corresponding base $B = \{y \in L : y \ll y\}$ is the set of all compact elements. Similarly, L is superalgebraic iff the relation ρ, as defined in (2), is the smallest basic order inducing the order of L, and the corresponding base is the set of all supercompact elements.

(4) The topology $\mathcal{O}$ of any locally compact space is a continuous frame, and the way-below relation is given by $U \ll V$ iff $U \subseteq C \subseteq V$ for some compact subset C. Hence $\ll$ is the smallest C-order whose induced lower order is the set inclusion on $\mathcal{O}$ (see [19], [26] and [27]).

(5) For any normal (Hausdorff) space, the relation $U \lhd V$ iff $U \subseteq A \subseteq V$ for some closed set A defines a C-order on the topology $\mathcal{O}$, and again, the induced lower quasiorder on $\mathcal{O}$ is the set inclusion. However, $\lhd$ need not be the *smallest* C-order with this property: for example, in the locally compact Euclidean topology on the reals, we have $\mathbb{R} \lhd \mathbb{R}$ but not $\mathbb{R} \ll \mathbb{R}$.

(6) Similar C-orders occur in the theory of *proximity spaces* (so-called *strong containment* or *subordination orders*: see, for example, [20]).

We are now prepared to formulate the object part of the announced functorial equivalence between certain categories of closure spaces on the one hand and certain categories of generalized ordered sets on the other hand. Given any closure space $(S,\mathcal{C})$, we define the *core relation* $\rho_{\mathcal{C}}$ by setting $x \rho_{\mathcal{C}} y$ iff y belongs to the interior of the core of x, i.e. $y \in (\uparrow x)^{\circ}$. In terms of closures, the core relation is described by

$$x \rho_{\mathcal{C}} y \text{ iff for all } Y \subseteq S,\ y \in Y^{-} \text{ implies } x \in \downarrow Y.$$

In this way, we assign to each closure space $(S,\mathcal{C})$ a pair $R(S,\mathcal{C}) = (S,\rho_{\mathcal{C}})$. Core relations are always transitive; moreover, a few simple verifications lead to the following fundamental coincidence, justifying the name "*core quasiorder*" (see Theorem 3.4 below):

The core quasiorders are precisely the core relations of core spaces.

On account of 2.2.C, core spaces are characterized by the property $y \in (\rho_{\mathcal{C}} y)^{-}$ for all elements y, while A-spaces enjoy the stronger property $y \in \rho_{\mathcal{C}} y$.

Starting in the converse direction from any relation ρ on a set S, we have a kernel system

$$\mathcal{O}_{\rho} = \{Y\rho : Y \subseteq S\}$$

and a closure system

$$\mathcal{C}_{\rho} = \mathcal{O}_{\rho}^{*} = \{S \setminus Y\rho : Y \subseteq S\}.$$

In particular, we may assign to each correlated set (S,ρ) the closure space $L(S,\rho) = (S,\mathcal{C}_{\rho})$. The twofold use of the symbols $\mathcal{C}_{<}$ and L is unambiguous because for a quasiordered set

$(S,\leq)$, the system $\mathcal{O}_{\leq}$ consists of all upper sets, and consequently, the closure system $\mathcal{C}_{\leq}$ is actually the collection of all lower sets. On the other hand, A-spaces $(S,\mathcal{C})$ may be characterized by the condition that the core relation $\rho_{\mathcal{C}}$ coincides with the specialization $\leq_{\mathcal{C}}$, whence R agrees with the specialization functor Q on A-spaces.

Already in the fifties, it was shown by Raney [35] that for any idempotent relation ρ on S, the kernel system $\mathcal{O}_{\rho}$ is completely distributive; his observation is an immediate consequence of 2.2.C, because $(S,\mathcal{C}_{\rho})$ is obviously a core space. More precisely, Zareckiĭ [37] has shown that $(S,\mathcal{C}_{\rho})$ is a core space iff the relation ρ is *regular*, i.e. $\rho=\rho\sigma\rho$ for some relation σ.

The subsequent crucial connection between C-spaces and C-quasiorders has been discovered independently by R.-E. Hoffmann [24] and the author [10]. For a more detailed discussion, including the facts on core spaces and basic spaces, see [17].

THEOREM 3.5. *Assigning to each closure space its core relation, one obtains bijections between*
(A) *A-spaces and quasiorders,*
(B) *basic spaces and basic quasiorders (respectively, B-spaces and B-quasiorders),*
(C) *core spaces and core quasiorders (respectively, C-spaces and C-quasiorders).*
Furthermore, the specialization quasiorders of the involved spaces coincide with the lower quasiorders induced by the associated core relations. Hence the spaces are T_0 iff the corresponding relations are separating. In particular, T_0-B-spaces correspond to B-orders, and T_0-C-spaces to C-orders.

PROOF. (A) For A-spaces, the assertion is clear since the core relations of A-spaces agree with the specialization quasiorders.

(B) If $(S,\mathcal{C})$ is a basic space with specialization $\leq$ then the set $B=\{y\in S: y\rho_{\mathcal{C}}y\}$ turns out to be a base for the quasiordered set $(S,\leq)$: for $x\not\leq z$, there exists an open core $\uparrow y=y\rho_{\mathcal{C}}$ containing x but not z; thus we have $y\in B$, $y\in\rho_{\mathcal{C}}x$ and $y\notin\rho_{\mathcal{C}}z$, and it follows that $y\leq x$ but $y\not\leq z$. Hence, by Proposition 3.2, $\rho_{\mathcal{C}}$ is a basic quasiorder inducing the specialization quasiorder $\leq$.

Conversely, if ρ is a basic quasiorder then $B=\{y\in S: y\rho y\}$ is a base for the quasiordered set $(S,\leq_{\rho})$, and $(S,\mathcal{C}_{\rho})$ is a basic space with a base of open cores $\uparrow y=y\rho$, $y\in B$.

The equations $\rho_{\mathcal{C}_{\rho}}=\rho$ and $\mathcal{C}_{\rho_{\mathcal{C}}}=\mathcal{C}$ will be established in Part (C).

(C) Let $(S,\mathcal{C})$ be a core space. Then the lower quasiorder induced by the core relation $\rho_{\mathcal{C}}$ is the specialization $\leq_{\mathcal{C}}$: in fact, $\rho_{\mathcal{C}}x\subseteq\rho_{\mathcal{C}}y$ means $x\in z\rho_{\mathcal{C}}$ for all elements z with $y\in z\rho_{\mathcal{C}}$, and as the sets $z\rho_{\mathcal{C}}=(\uparrow z)^{o}$ form an open base, the latter condition is equivalent to $x\leq_{\mathcal{C}}y$. The interpolation property is clear by definition of core spaces, which ensures that for $z\in x\rho_{\mathcal{C}}$, there is some $y\in x\rho_{\mathcal{C}}$ with $z\in y\rho_{\mathcal{C}}$. Furthermore, $x\in\rho_{\mathcal{C}}y$ and $w\leq_{\mathcal{C}}x$ imply $y\in(\uparrow x)^{o}\subseteq(\uparrow w)^{o}$, hence $w\in\rho_{\mathcal{C}}y$. This shows that $\rho_{\mathcal{C}}$ is a lower relation. Since the sets $x\rho_{\mathcal{C}}$ form a base for the open sets in the space $(S,\mathcal{C})$, the closure system $\mathcal{C}$ coincides with $\mathcal{C}_{\rho_{\mathcal{C}}}$.

On the other hand, if ρ is a core quasiorder then by idempotency of ρ, $(S,\mathcal{C}_{\rho})$ is a core space, and its specialization is the lower quasiorder $\leq_{\rho}$, on account of the following equivalences:

$$x\leq_{\mathcal{C}_{\rho}}y \iff y\in U \text{ for all } U\in\mathcal{O}_{\rho} \text{ with } x\in U \iff y\in z\rho \text{ for all } z \text{ with } x\in z\rho \iff \rho x\subseteq\rho y.$$

Similarly, we have:

$$x\rho_{\mathcal{C}_{\rho}}y \iff y\in U\subseteq\uparrow x \text{ for some } U\in\mathcal{O}_{\rho} \iff y\in z\rho\subseteq\uparrow x \text{ for some } z \iff x\rho y$$

(for the last equivalence, observe that $x\rho\subseteq\uparrow x$, and that $z\rho\subseteq\uparrow x$ implies $z\rho=z\rho\rho\subseteq x\leq_{\rho}\rho=x\rho$).

Lemma 3.3 completes the proof for the topological case. □

Observing that any finite topological space is an A-space, we infer from 3.5 that the B- resp. C-quasiorders on a finite set are just the usual quasiorders on this set.

Let us conclude this section with a remark on the frequency of C-spaces. While finite topological spaces are always C-spaces, the number of C-space structures on an *infinite* set S of cardinality m is rather small, compared with the number of all topological structures on S: indeed, it is well known that there are 2^{2^m} topologies on S, whereas there are only $2^{m^2} = 2^m$ relations on this set; as every C-space is uniquely determined by its core relation, there are not more than 2^m C-space structures on S, and this upper bound is exact, because there are 2^m quasiorders, hence A-topologies on a set of infinite cardinality m.

4. The fundamental isomorphisms

The bijections established in 3.5 become concrete functorial isomorphisms if we take *isotone*, i.e. order-preserving maps as morphims between the objects in question: on the side of correlated sets, the term "*isotone*" refers to the induced lower quasiorders, and on the side of spaces, it means *specialization preserving*. Thus we have the following pairs of isomorphic categories:

A	(actually) quasiordered sets	AS	A-spaces
B	basically quasiordered sets	BS	basic spaces
C	cordially quasiordered sets	CS	core spaces

For each subcategory X of the category C, we denote by XO that full subcategory whose objects carry separating relations, hence core orders. Similarly, for any subcategory Y of the category S of spaces, we denote by YO that full subcategory whose objects are T_0-spaces. By Theorem 3.4, we have isomorphisms between the following pairs of categories:

AO	(actually) ordered sets	ASO	T_0-A-spaces
BO	basically ordered sets	BSO	basic T_0-spaces
CO	cordially ordered sets	CSO	T_0-core spaces

and similar isomorphisms hold for the topological versions:

BT	B-quasiordered sets	BTS	B-spaces
CT	C-quasiordered sets	CTS	C-spaces
BTO	B-ordered sets	BTSO	T_0-B-spaces
CTO	C-ordered sets	CTSO	T_0-C-spaces

For most purposes, the morphism class of isotone maps is too large and not very interesting from the topological point of view. Hence we are confronted with the question which stronger types of maps should be chosen as morphisms between generalized ordered sets and between core spaces, on the other side. For spaces, certainly the most obvious choice is that of continuous maps, and in case of sets endowed with certain relations, it is most natural to postulate that the morphisms have to preserve these relations. Thus, given a relation ρ on S and a relation ρ' on S', we call a map $\varphi: S \to S'$ *relation-preserving* if $x \rho y$ implies $\varphi(x) \rho' \varphi(y)$, and *isotone* if $x \leq_\rho y$ implies $\varphi(x) \leq_{\rho'} \varphi(y)$. Neither of these two properties implies the other; but, of course, in case of (quasi)orders, *relation-preserving* as well as *isotone* means the same as *order-preserving*. Unfortunately, on the level of cordially quasiordered sets and core spaces, relation-preserving maps do *not* correspond to continuous maps, so we must look for suitable restrictions of the morphism classes on both sides. By definition, a map φ is relation-preserving iff

$$\varphi[\rho y] \subseteq \rho' \varphi(y)$$

for all elements y in the domain of φ. If ρ' happens to be a lower relation then the above inclusion is equivalent to

$$\downarrow\varphi[\rho y] \subseteq \rho' \varphi(y),$$

where $\downarrow$ refers to the quasiorder $\leq_{\rho'}$, and if ρ' is transitive, it follows that

$$\rho' \varphi[\rho y] \subseteq \rho' \varphi(y).$$

The converse inclusion

$$\rho' \varphi(y) \subseteq \rho' \varphi[\rho y]$$

states that for all $y \in S$ and all $x' \in S'$ with $x' \rho' \varphi(y)$, there is an $x \in S$ with $x \rho y$ and $x' \rho' \varphi(x)$. Maps with this property will be called *interpolating*. Indeed, a relation ρ on S has the interpolation property iff the identity map $id_S : (S,\rho) \to (S,\rho)$ is interpolating. It is easy to see that in case of a *core quasiorder* ρ', a map $\varphi : (S,\rho) \to (S',\rho')$ is interpolating iff

$$\rho' \varphi(y) \subseteq \downarrow\varphi[\rho y].$$

Thus we arrive at

PROPOSITION 4.1. *A map φ between cordially quasiordered sets (S,ρ) and (S',ρ') is both relation-preserving and interpolating iff $\rho'\varphi(y) = \downarrow\varphi[\rho y]$ for all $y \in S$. Any such map is isotone.*

If $x \leq_\rho y$ implies $\varphi(y) \leq_{\rho'} \varphi(x)$ then we speak of an *antitone* map φ between (S,ρ) and (S',ρ'). As a matter of curiosity, such a map is interpolating, provided ρ is transitive and $\rho y \neq \emptyset$ for all y (a condition that is certainly fulfilled for C-quasiorders), while the composition of two antitone maps is isotone but not always interpolating. For example, the real functions φ and ψ with $\varphi(x) = 0$ for $x < 0$, $\varphi(x) = -1$ for $x \geq 0$, and $\psi(x) = -x$, are antitone, but the composite map $\psi \circ \varphi$ is not interpolating for the core order $<$. Indeed, a real function is isotone and interpolating iff it is continuous with respect to the upper topology on $\mathbb{R}$.

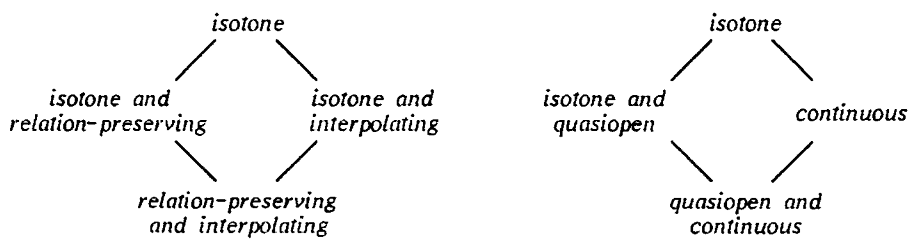
We shall see below that in the general situation of cordially quasiordered sets, the isotone interpolating maps are the continuous maps between the corresponding core spaces, whence this class of maps is closed under composition. However, as the above example demonstrates, the composition of two interpolating maps need not produce an interpolating map, while relation-preserving maps and isotone maps are obviously stable under composition.

Recall that a map between closure spaces is *continuous* iff inverse images of closed sets are closed, or equivalenty, inverse images of open sets are open. Following Hofmann and Mislove [28], we call a map between closure spaces *quasiopen* if the image of each open set has an open saturation, i.e. a least neighborhood. We notice that in contrast to the situation for continuous maps, the class of quasiopen maps is not closed under composition. However, the composition of any two isotone (i.e. specialization-preserving) quasiopen maps is again isotone and quasiopen. The next result is central for the intended isomorphism theorems.

PROPOSITION 4.2. *Let* φ *be a map between cordially quasiordered sets* (S,ρ) *and* (S',ρ').
(1) φ *is isotone and interpolating iff it is a continuous map between the spaces* $(S,\mathcal{C}_\rho)$ *and* $(S',\mathcal{C}_{\rho'})$.
(2) *If* φ *is relation-preserving then it is a quasiopen map between* $(S,\mathcal{C}_\rho)$ *and* $(S',\mathcal{C}_{\rho'})$. *Conversely, if* φ *is isotone and quasiopen then it is relation-preserving.*
(3) φ *is relation-preserving and interpolating iff it is a quasiopen and continuous map between* $(S,\mathcal{C}_\rho)$ *and* $(S',\mathcal{C}_{\rho'})$.

PROOF. (1) Suppose φ is isotone and interpolating. Then $y \in \varphi^{-1}[x'\rho']$ means $x'\rho'\varphi(y)$, and this implies $x'\rho'\varphi(x)$ for some $x \in \rho y$, whence $y \in x\rho$ and $x\rho \subseteq \varphi^{-1}[x'\rho']$; indeed, $x\rho z$ entails $\rho x \subseteq \rho z$, and as φ is isotone, $x' \in \rho'\varphi(x) \subseteq \rho'\varphi(z)$. Thus inverse images of the basic open sets $x'\rho'$ are open. Conversely, if φ is continuous then it preserves specialization, i.e. the lower quasiorders induced by ρ and ρ', respectively. By continuity of φ and the core space property of $(S,\mathcal{C}_\rho)$, $x'\rho'\varphi(y)$ implies $y \in x\rho \subseteq \uparrow x \subseteq \varphi^{-1}[x'\rho']$ for some x, and it follows that $x\rho y$ and $x'\rho'\varphi(x)$.
(2) If φ is relation-preserving, we must show that for each $U \in \mathcal{O}_\rho$, the saturation $\uparrow\varphi[U]$ belongs to $\mathcal{O}_{\rho'}$. For $x' \in \uparrow\varphi[U]$, we find a $y \in U$ with $\rho'\varphi(y) \subseteq \rho'x'$ and an $x \in U$ with $x\rho y$ (as $U = U\rho$, by idempotency of the relation ρ). It follows that $\varphi(x) \in \rho'\varphi(y) \subseteq \rho'x'$, i.e. $x' \in \varphi(x)\rho' \in \mathcal{O}_{\rho'}$. On the other hand, $x \in U$ implies $\varphi(x)\rho' \subseteq \varphi[U]\rho' \subseteq \uparrow\varphi[U]$, by transitivity of ρ'.
Conversely, assume φ is quasiopen and isotone. For $x\rho z$, we obtain $\varphi(z) \in \varphi[x\rho] \subseteq \uparrow\varphi[x\rho]$, and since the latter set is open in the core space $(S',\mathcal{C}_{\rho'})$, we find elements $y \in x\rho$ and $x' \in S'$ with $\varphi(y) \leq_{\rho'} x'$ and $\varphi(z) \in x'\rho'$. But this implies $x \leq_\rho y$ (by transitivity of ρ) and then $\varphi(x) \leq_{\rho'} \varphi(y) \leq_{\rho'} x'$. Now we use the fact that ρ' is a lower relation to conclude that $\varphi(z) \in x'\rho' \subseteq \varphi(x)\rho'$. In all, this proves the desired implication $x\rho z \Rightarrow \varphi(x)\rho'\varphi(z)$.
(3) is an immediate consequence of (1), (2), and the last statement in 4.1. □

In all, we see that the following classes of maps may serve as morphisms for categories whose objects are cordially quasiordered sets, respectively, core spaces:



For any category X of cordially quasiordered sets, we denote by $X^{\rightarrow}$ the subcategory with the same objects but relation-preserving morphisms, by $X^{\leftarrow}$ the subcategory with the same objects but interpolating morphisms, and by $X^{\leftrightarrow}$ the intersection of $X^{\rightarrow}$ and $X^{\leftarrow}$. Similarly, for any subcategory Y of S, the category of spaces and isotone (!) maps, we denote by $Y^{\rightarrow}$ the subcategory with the same objects and quasiopen morphisms, by $Y^{\leftarrow}$ the subcategory with the same objects and continuous morphisms, and by $Y^{\leftrightarrow}$ the intersection of $Y^{\rightarrow}$ and $Y^{\leftarrow}$. Then we may summarize our previous results as follows:

THEOREM 4.3. *Let* X *be any full subcategory of the category* C *of cordially quasiordered sets and isotone maps, and denote by* XS *that full subcategory of* S *whose objects corres-*

pond to X-objects under the bijection between core quasiorders and core spaces. Then the functors L and R with $L(S,\rho) = (S,\mathcal{C}_\rho)$ and $R(S,\mathcal{C}) = (S,\rho_\mathcal{C})$ induce mutually inverse concrete isomorphisms between the following categories:

$$\mathrm{X} \underset{L}{\overset{R}{\rightleftarrows}} \mathrm{XS}\ , \quad \mathrm{X}^{\rightarrow} \underset{L}{\overset{R}{\rightleftarrows}} \mathrm{XS}^{\rightarrow}, \quad \mathrm{X}^{\leftarrow} \underset{L}{\overset{R}{\rightleftarrows}} \mathrm{XS}^{\leftarrow}, \quad \mathrm{X}^{\leftrightarrow} \underset{L}{\overset{R}{\rightleftarrows}} \mathrm{XS}^{\leftrightarrow}.$$

In particular, the category of C-spaces with continuous (and quasiopen) maps is isomorphic to the category of C-quasiordered sets with isotone interpolating (and relation-preserving) maps, and similar isomorphisms exist for A and B instead of C.

On the level of A-spaces, these results are not very exciting, since *every* map between A-spaces is quasiopen and *every* map between quasiordered sets is interpolating. However, for B- and C-spaces, these relationships between topological and order-theoretical morphisms are less evident and facilitate many considerations, for example in the theory of continuous ordered sets (see the end of Section 5).

An interesting problem in connection with Theorem 4.3 is the question of whether the involved categories are complete, cocomplete, topological, cartesian closed etc. By reasons of limited space, we only touch upon one aspect, namely that of products and coproducts. While coproducts are simply obtained by forming disjoint unions, the case of products is more subtle. At the end of Section 2, we have remarked that finite *topological* products of A-, B- or C-spaces are again such spaces, but for infinite products, almost all factors must be supercompact (on the order-theoretical side, supercompactness is equivalent to the existence of least elements). Of course, this does not mean that *categorical* products would not exist in the infinite case. Indeed, as mentioned earlier, the *box product* of an arbitrary number of A-spaces is again an A-space, and the corresponding product of quasiordered sets is quasiordered (componentwise). However, a closer inspection shows that infinite products of B- and C-quasiordered sets need not be B- or C-quasiordered, respectively, while the finite case behaves well (directedness of the sets ρy is crucial).

5. Adjunctions, equivalences and dualities

As explained in [14], there is a nice categorical adjunction between certain categories of closure spaces, in particular, of topological spaces, on the one hand, and of certain types of complete lattices, on the other hand. Before we are going to discus this theme in detail, it appears convenient to recall the *order-theoretical* notion of adjoint pairs (a rather simple instance of the corresponding categorical notion).

Given quasiordered sets $(S,\leq)$ and $(S',\leq')$, a map $\varphi : S \to S'$ is *lower adjoint* or *left adjoint* to a map $\psi : S' \to S$, and ψ is *upper adjoint* or *right adjoint* to φ, iff for all $x \in S$ and $y' \in S'$, the inequality $\varphi(x) \leq' y'$ is equivalent to $x \leq \psi(y')$. Furthermore, a map is called *residuated* (resp. *residual*) if inverse images of principal (dual) ideals are principal (dual) ideals. It is well-known that a map is residuated iff it has an upper adjoint, and residual iff it has a lower adjoint. In case of ordered sets, upper and lower adjoints determine each other uniquely, so we write $\psi = \Delta\varphi$ and $\varphi = \nabla\psi$ if ψ is upper adjoint to φ, hence φ lower adjoint to ψ. Every lower adjoint map preserves joins (to the extent they exist), and every upper adjoint map preserves meets; in particular, adjoint maps are isotone. Moreover, the join-

(resp. meet-)preserving maps between complete lattices are precisely the residuated (resp. residual) ones. For this and more background on order-theoretical adjunctions and their connections with certain categorical adjunctions and reflections, see [13], [15] and [19]. We call a map with a residuated upper adjoint *doubly residuated*, and a map with a residual lower adjoint *doubly residual*. Since these classes of maps are closed under composition, we may introduce, for any subcategory X of the category of ordered sets and isotone maps, the following subcategories:

category of ordered sets	*morphisms*
X_∇	residuated maps
X^Δ	residual maps
X^∇_∇	doubly residuated maps
X^Δ_Δ	doubly residual maps
X^Δ_∇	residuated and residual maps

In particular, for the category CL of complete lattices and isotone maps, we obtain the following frequently used subcategories providing the background for many dualities:

category of complete lattices	*morphisms*
CL_∇	join-preserving maps
CL^Δ	meet-preserving maps
CL^∇_∇	maps possessing join-preserving upper adjoints
CL^Δ_Δ	maps possessing meet-preserving lower adjoints
CL^Δ_∇	complete homomorphisms (join- and meet-preserving maps)

Passing from lower to upper adjoints and vice versa, one obtains mutually inverse concrete dual isomorphisms Δ and ∇ between the following pairs of categories:

$$X_\nabla \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} X^\Delta \quad , \qquad X^\nabla_\nabla \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} X^\Delta_\nabla \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} X^\Delta_\Delta \ .$$

Any quasiordered set $(S,\le)$ may be regarded as a space $E(S,\le) = (S,\mathcal{E}_\le)$, where $\mathcal{E}_\le$ is the collection of all principal ideals. More to the point for us, any residuated map φ between quasiordered sets $(S,\le)$ and $(S',\le')$ is a continuous map between the spaces $E(S,\le)$ and $E(S',\le')$. Hence we have a *concrete full embedding functor* E from the category of ordered sets with residuated maps to the category of spaces with continuous maps, and also from the category CL_∇ of complete lattices with join-preserving maps to the category $CLS^{\leftarrow}$ of closure spaces and continuous maps. Hence CL_∇ may be considered as a full subcategory of $CLS^{\leftarrow}$, and it turns out that, up to isomorphism, CL_∇ is even a *reflective* subcategory of $CLS^{\leftarrow}$. To make this statement more precise, we observe that for each closure space $(S,\mathcal{C})$, there is a natural map

$$\eta_S = \eta_S^{\mathcal{C}}: S \to \mathcal{C},\ x \mapsto {\downarrow}x,$$

which is continuous as a map from $(S,\mathcal{C})$ to $E(\mathcal{C},\subseteq)$. Now one proves easily (cf. [14] and [28]):

PROPOSITION 5.1. *A map φ between closure spaces $(S,\mathscr{C})$ and $(S',\mathscr{C}')$ is continuous iff there exists a (unique) residuated map $\overline{\varphi}: \mathscr{C} \to \mathscr{C}'$ such that $\overline{\varphi} \circ \eta_S = \eta_{S'} \circ \varphi$.*
Furthermore, $\overline{\varphi}$ is doubly residuated iff φ is continuous and quasiopen.

In fact, the unique "extension" $\overline{\varphi}$ of a continuous map φ is the map

$$\overline{\varphi}: \mathscr{C} \to \mathscr{C}',\ X \mapsto \varphi[X]^-,$$

and its upper adjoint is the inverse image map

$$\varphi^{-1}: \mathscr{C}' \to \mathscr{C},\ X' \mapsto \varphi^{-1}[X'].$$

Moreover, φ is quasiopen iff the map

$$\psi: \mathscr{C} \to \mathscr{C}',\ X \mapsto S' \setminus \uparrow\varphi[S \setminus X]$$

is well-defined and upper adjoint to φ^{-1}.

The key observation is now that the natural maps η_S are not only continuous but also *universal* inasmuch as every continuous map φ from a closure space $(S,\mathscr{C})$ into a complete lattice L, considered as a closure space EL, factorizes uniquely through η_S and a residuated, i.e. join-preserving map $\varphi^\vee$, namely

$$\varphi^\vee: \mathscr{C} \to L,\ X \mapsto \bigvee \varphi[X].$$

This map has the upper adjoint

$$\varphi^\wedge: L \to \mathscr{C},\ x' \mapsto \varphi^{-1}[\downarrow x'].$$

Clearly, $\varphi^\vee$ is the only join-preserving map satisfying the equation $\varphi = \varphi^\vee \circ \eta_S$. □

Now we have gathered the necessary ingredients for the intended reflection theorem (cf. [14]). The covariant *completion functor* C associates with any closure space $(S,\mathscr{C})$ the complete lattice $(\mathscr{C},\subseteq)$, and with any continuous map $\varphi: (S,\mathscr{C}) \to (S',\mathscr{C}')$ the residuated (i.e. join-preserving) map $\overline{\varphi}: \mathscr{C} \to \mathscr{C}'$; the contravariant *completion functor* ΔC agrees with C on objects and sends a morphism φ to the inverse image map $\varphi^{-1}: \mathscr{C}' \to \mathscr{C}$. Then the reflection theorem reads as follows:

THEOREM 5.2. *The completion functor C: $\mathrm{CLS}^{\leftarrow} \longrightarrow \mathrm{CL}_\nabla$ is left adjoint to the embedding functor E : $\mathrm{CL}_\nabla \longrightarrow \mathrm{CLS}^{\leftarrow}$, and these two functors induce an adjoint situation between the categories $\mathrm{CLS}^{\leftrightarrow}$ and $\mathrm{CL}^{\nabla}_{\nabla}$. Hence, up to isomorphism, CL_∇ (resp. $\mathrm{CL}^{\nabla}_{\nabla}$) is a full reflective subcategory of $\mathrm{CLS}^{\leftarrow}$ (resp. $\mathrm{CLS}^{\leftrightarrow}$). Furthermore, the contravariant completion functor ΔC: $\mathrm{CLS}^{\leftarrow} \longrightarrow \mathrm{CL}^{\Delta}$ is dually adjoint to the contravariant embedding $E\nabla$: $\mathrm{CL}^{\Delta} \longrightarrow \mathrm{CLS}^{\leftarrow}$, and these functors induce a dual adjunction between the categories $\mathrm{CLS}^{\leftrightarrow}$ and CL_∇.*

In the next proposition, we shall show that the completion functor C "preserves adjointness". Call a map φ between spaces $(S,\mathscr{X})$ and $(S',\mathscr{X}')$ *quasiclosed* iff $X \in \mathscr{X}$ implies $\downarrow\varphi[X] \in \mathscr{X}'$; in other words, iff φ is a quasiopen map between the spaces $(S,\mathscr{X}^*)$ and $(S',\mathscr{X}'^*)$. Quasiclosed maps play a central rôle in the theory of standard extensions and completions (see [13] and [15]).

PROPOSITION 5.3. *Let $(S,\mathscr{C})$ and $(S',\mathscr{C}')$ be closure spaces, endowed with their specialization quasiorders, and let $\varphi: S \to S'$ be lower adjoint to $\psi: S' \to S$. Then:*

(1) *ψ is continuous iff φ is quasiopen.*

(2) *φ is continuous iff ψ is quasiclosed iff $\overline{\varphi}: \mathscr{C} \to \mathscr{C}'$ is lower adjoint to $\overline{\psi}: \mathscr{C}' \to \mathscr{C}$.*

These assertions are easily verified with the help of the following two equations:

$\psi^{-1}[\uparrow Y] = \uparrow\varphi[Y]$ for $Y \subseteq S$,

$\varphi^{-1}[\downarrow Y'] = \downarrow\psi[Y']$ for $Y' \subseteq S'$.

Now let us return to the study of basic spaces and core spaces. By Proposition 2.2.C. we have a restricted completion functor C from the category $CS^{\leftarrow}$ of core spaces and continuous maps to the category CD_{∇} of completely distributive lattices and join-preserving maps. Conversely, every completely distributive lattice L gives rise to a core space EL, because the lattice of closed sets, i.e. the collection of all principal ideals, is isomorphic to L (see again 2.2.C). Similarly, C maps the category $BS^{\leftarrow}$ of basic spaces to the category BD_{∇} of superalgebraic lattices, i.e. completely distributive lattices with a smallest base. In the converse direction, E transforms superalgebraic lattices into basic spaces (see 2.2.B).

COROLLARY 5.4. *The functors C and E induce adjoint situations between the following pairs of categories:*

$$BS^{\leftarrow} \rightleftarrows BD_{\nabla}, \quad BS^{\leftrightarrow} \rightleftarrows BD^{\nabla}_{\nabla}, \quad CS^{\leftarrow} \rightleftarrows CD_{\nabla}, \quad CS^{\leftrightarrow} \rightleftarrows CD^{\nabla}_{\nabla}.$$

Similarly, the functors ΔC and $E\nabla$ induce dual adjunctions:

$$BS^{\leftarrow} \rightleftarrows BD^{\Delta}, \quad BS^{\leftrightarrow} \rightleftarrows BD^{\Delta}_{\nabla}, \quad CS^{\leftarrow} \rightleftarrows CD^{\Delta}, \quad CS^{\leftrightarrow} \rightleftarrows CD^{\Delta}_{\nabla}.$$

Perhaps, the reader might ask for an analogous result on A-spaces. But such an adjunction does not exist, because the restriction $C: AS^{\leftarrow} \to BD_{\nabla}$ is surjective on objects (see 5.6), and the only superalgebraic lattices which are transformed into A-spaces under the embedding E are complete dually well-ordered chains.

It was an important discovery of lattice-theoretical topology that the restriction of the completion functor C to topological (closure) spaces has a left adjoint, namely the $\vee$-*spectrum functor* S from the category $CL^{\vee}_{\nabla}$ of complete lattices and join-preserving maps whose upper adjoint preserves *finite* joins, to the category of topological spaces. This functor associates with any morphism between complete lattices its restriction to the $\vee$-spectra, where the $\vee$-*spectrum* SL of a complete lattice L is the set P of $\vee$-prime elements, topologized by the closure system of all sets $P \cap \downarrow x$ $(x \in L)$. Moreover, C and S induce an equivalence between the category $TSOB^{\leftarrow}$ of *sober spaces*, i.e. topological T_0-spaces in which the point closures are precisely the $\vee$-prime closed sets, and the category $TD^{\vee}_{\nabla}$ of cospatial lattices (= T-lattices) and maps preserving arbitrary joins and $\vee$-spectra. Thus the contravariant functors ΔC and $S\nabla$ induce a duality between $TSOB^{\leftarrow}$ and the category $TD^{\Delta}_{\vee}$ of T-lattices together with dual frame homomorphisms, i.e. maps preserving finite joins and arbitrary meets (see [19] and [27] for the more common, but in the present context also a bit more complicated, dual approach via open sets).

The previous remarks are covered by the following general approach, discussed in [5] and [12]. Let $\mathfrak{Z}$ be any function assigning to each (quasi-)ordered set Q a certain collection $\mathfrak{Z}Q$ of subsets such that for any isomorphism $\varphi: Q \to Q'$, $Z \in \mathfrak{Z}Q$ implies $\varphi[Z] \in \mathfrak{Z}Q'$. The members of $\mathfrak{Z}Q$ are referred to as $\mathfrak{Z}$-*sets* of Q. An element p of a complete lattice L is called $\mathfrak{Z}$-*(join-)prime* if for all $Z \in \mathfrak{Z}L$, $p \leq \bigvee Z$ implies $p \leq z$ for some $z \in Z$; and L is referred to as a $\mathfrak{Z}$-*lattice* if each of its elements is a join of $\mathfrak{Z}$-primes. Furthermore, a set system $\mathcal{X}$ is said to be $\mathfrak{Z}$-*union complete* if $\mathcal{Y} \in \mathfrak{Z}\mathcal{X}$ implies $\bigcup\mathcal{Y} \in \mathcal{X}$. Closure systems with this property may be characterized alternately as follows:

LEMMA 5.5. *The following three statements on a closure system $\mathcal{C}$ are equivalent:*

(a) *$\mathcal{C}$ is $\mathfrak{Z}$-union complete.*

(b) *Each point closure is $\mathfrak{Z}$-prime in $\mathcal{C}$.*

(c) *Each closed set is a union of $\mathfrak{Z}$-prime members of $\mathcal{C}$.*

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are evident.

(c) $\Rightarrow$ (a): For $\mathcal{Y} \in \mathfrak{Z}\mathcal{C}$, the join $\bigvee\mathcal{Y}$ is the union of a set $\mathcal{X}$ of $\mathfrak{Z}$-prime members of $\mathcal{C}$. Hence, for each $x \in \bigvee\mathcal{Y}$, we find some $X \in \mathcal{X}$ with $x \in X$ and some $Y \in \mathcal{Y}$ with $X \subseteq Y$. This shows that $\bigcup\mathcal{Y} = \bigvee\mathcal{Y}$ belongs to $\mathcal{C}$. □

In view of this lemma, it appears reasonable to introduce the following generalization of the topological notion of soberness: a T_0-closure space $(S, \mathcal{C})$ or its closure system $\mathcal{C}$ is called *$\mathfrak{Z}$-sober* if the $\mathfrak{Z}$-prime members of $\mathcal{C}$ are precisely the point closures. Let us collect a few typical instances of the previous general $\mathfrak{Z}$-definitions.

EXAMPLES.

$\mathfrak{Z}$	*members of $\mathfrak{Z}L$*	*$\mathfrak{Z}$-prime*	*$\mathfrak{Z}$-lattice*	*$\mathfrak{Z}$-union complete closure system*	*$\mathfrak{Z}$-sober space*
$\mathcal{A}$	arbitrary subsets	supercompact	superalgebraic lattice	A-topology	T_0-A-space
$\mathcal{B}$ $\mathcal{F}$	binary subsets finite subsets	$\vee$-prime	cospatial lattice	topological closure system	sober space
$\mathcal{C}$ $\mathcal{D}$	nonempty chains directed sets	compact	algebraic lattice	algebraic closure system	ideal space *
$\mathcal{E}$	1-element subsets	arbitrary	complete lattice	closure system	principal ideal space **

* $\vee$-semilattice with least element, endowed with the system of ideals.

**complete lattice, endowed with the system of principal ideals.

The following set representation for $\mathfrak{Z}$-lattices encompasses well-known facts on (super-)algebraic lattices etc. (see [7] and [34] for early references to this subject):

PROPOSITION 5.6. *A complete lattice L is a $\mathfrak{Z}$-lattice iff it is isomorphic to a $\mathfrak{Z}$-union complete, respectively $\mathfrak{Z}$-sober, closure system. In particular:*

(1) *L is superalgebraic iff L is isomorphic to a (T_0-)A-topology.*

(2) *L is cospatial iff L is isomorphic to a (sober) topological closure system.*

(3) *L is algebraic iff L is isomorphic to an algebraic closure system (ideal system).*

PROOF. For any element x of a $\mathfrak{Z}$-lattice L, let $\Sigma_L^{\mathfrak{Z}}(x)$ denote the set of all $\mathfrak{Z}$-prime elements $p \leq x$. It is then easy to see that $\mathcal{C}_{\mathfrak{Z}}L = \{\Sigma_L^{\mathfrak{Z}}(x) : x \in L\}$ is a closure system and that $\Sigma_L^{\mathfrak{Z}}: L \to \mathcal{C}_{\mathfrak{Z}}L$ is an isomorphism, inducing a bijection between the $\mathfrak{Z}$-prime elements of L and the point closures of $\mathcal{C}_{\mathfrak{Z}}L$. Hence $\mathcal{C}_{\mathfrak{Z}}L$ is $\mathfrak{Z}$-sober, a fortiori $\mathfrak{Z}$-union complete.
Conversely, any $\mathfrak{Z}$-union complete closure system is a $\mathfrak{Z}$-lattice, by Lemma 5.5.

The closure system of all ideals of a join-semilattice with 0 is $\mathcal{D}$-sober, because the compact ideals are precisely the principal ideals. Conversely, if $\mathcal{C}$ is any $\mathcal{D}$-sober closure system on a set S, then $(S, \leq_{\mathcal{C}})$ is a join-semilattice with 0, and $\mathcal{C}$ is the system of its ideals (the proof of this fact is left as an exercise). □

The set of all $\mathfrak{X}$-prime elements of a complete lattice L, equipped with the closure system $\mathcal{C}_{\mathfrak{X}}L$, is called the *$\mathfrak{X}$-spectrum* of L and denoted by $S_{\mathfrak{X}}L$. As the proof of 5.6 shows, $S_{\mathfrak{X}}L$ is a $\mathfrak{X}$-sober closure space for any $\mathfrak{X}$-lattice L. We say a function Φ between complete lattices L and L' *preserves $\mathfrak{X}$-spectra* if $\Phi[S_{\mathfrak{X}}L] \subseteq S_{\mathfrak{X}}L'$. Now we are in a position to formulate Proposition 5.6 in categorical terms:

THEOREM 5.7. *The restriction to $\mathfrak{X}$-spectra provides a functorial equivalence $S_{\mathfrak{X}}$ between the category of $\mathfrak{X}$-lattices with maps preserving joins and $\mathfrak{X}$-spectra, and the category of $\mathfrak{X}$-sober spaces and continuous maps. The inverse equivalence C is obtained by assigning to each $\mathfrak{X}$-sober space the lattice of closed sets. In particular:*

(1) *The category of superalgebraic lattices is equivalent to the category of T_0-A-spaces.*
(2) *The category of cospatial lattices is equivalent to the category of sober spaces.*
(3) *The category of algebraic lattices is equivalent to the category of $\vee$-semilattices with 0.*

PROOF. For any join-preserving map Φ between $\mathfrak{X}$-lattices L and L' with $\Phi[S_{\mathfrak{X}}L] \subseteq S_{\mathfrak{X}}L'$, let $S_{\mathfrak{X}}\Phi$ denote the restriction to the $\mathfrak{X}$-spectra. If Ψ denotes the upper adjoint of Φ then, for $x' \in L'$, we obtain $(S_{\mathfrak{X}}\Phi)^{-1}[\Sigma_{L'}^{\mathfrak{X}}(x')] = \Sigma_{L}^{\mathfrak{X}}(\Psi(x'))$, proving continuity of the map $S_{\mathfrak{X}}\Phi$.
Going the other way around, we infer from 5.6 that the completion functor C transforms $\mathfrak{X}$-sober spaces into $\mathfrak{X}$-lattices. For any continuous map φ between $\mathfrak{X}$-sober spaces $(S,\mathcal{C})$ and $(S',\mathcal{C}')$, the map $C\varphi = \varphi$ preserves joins and sends point closures to point closures (see 5.1). By $\mathfrak{X}$-soberness of the underlying spaces, the latter means that $C\varphi$ preserves $\mathfrak{X}$-spectra. Furthermore, $S_{\mathfrak{X}}C(S,\mathcal{C})$ is naturally isomorphic to $(S,\mathcal{C})$, via the principal ideal map $\eta_S\colon x \mapsto {\downarrow}x$. On the other hand, we have shown in the proof of 5.6 that for any $\mathfrak{X}$-lattice L, there is a (natural) isomorphism $\Sigma_L^{\mathfrak{X}}$ between L and the closure system $\mathcal{C}_{\mathfrak{X}}L = CS_{\mathfrak{X}}L$.
The specializations (1) and (2) are now clear. For (3), observe that the homomorphisms between $\vee$-semilattices with 0, i.e. those maps which preserve finite joins, are precisely the *ideal-continuous* ones, i.e. maps such that inverse images of ideals are ideals. □

If $\mathfrak{X}$-sets are mapped onto $\mathfrak{X}$-sets under meet-preserving maps (e.g. for $\mathfrak{X} = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$, or $\mathcal{F}$) then the above equivalence can be transformed into a duality, with the help of the following observation whose proof is left as an exercise (see [5]):

LEMMA 5.8. *Let $\varphi: L \to L'$ be a join-preserving map between $\mathfrak{X}$-lattices, and let $\psi: L' \to L$ denote the upper adjoint of φ. If φ preserves $\mathfrak{X}$-spectra then ψ preserves $\mathfrak{X}$-joins, i.e. $\psi(\bigvee Z) = \bigvee \psi[Z]$ for $Z \in \mathfrak{X}L'$. The converse implication holds whenever $Z \in \mathfrak{X}L'$ implies $\psi[Z] \in \mathfrak{X}L$. In particular:*

(1) *A map between superalgebraic lattices preserves arbitrary joins and supercompactness iff it has an upper adjoint preserving arbitrary joins.*
(2) *A map between cospatial lattices preserves arbitrary joins and $\vee$-spectra iff it has an upper adjoint preserving finite joins.*
(3) *A map between algebraic lattices preserves arbitrary joins and compactness iff its has an upper adjoint preserving directed joins.*

Thus, if $\mathfrak{X}$-sets are preserved by meet-preserving maps then we have a concrete dual isomorphism between the category of $\mathfrak{X}$-lattices with maps preserving joins and $\mathfrak{X}$-spectra on the one hand, and $\mathfrak{Z}$-lattices with maps preserving meets and $\mathfrak{Z}$-joins on the other hand. Composing this dual isomorphism with the equivalence in Theorem 5.7, we arrive at

COROLLARY 5.9. *Suppose $\mathfrak{Z}$-sets are preserved under meet-preserving maps. Then the category of $\mathfrak{Z}$-lattices with maps preserving meets and $\mathfrak{Z}$-joins is dual to the category of $\mathfrak{Z}$-sober spaces and continuous maps. In particular:*

(1) *The category of superalgebraic lattices with maps preserving arbitrary meets and joins is dual to the category of T_0-A-spaces.*
(2) *The category of cospatial lattices with maps preserving arbitrary meets and finite joins is dual to the category of sober spaces.*
(3) *The category of algebraic lattices with maps preserving arbitrary meets and directed joins is dual to the category of $\vee$-semilattices with 0.*

These three special duality theorems are well-known (see e.g. [19]); but it seems that their common feature via $\mathfrak{Z}$-soberness is a new "topologically flavoured" aspect. Moreover, the general $\mathfrak{Z}$-theory opens a broad spectrum of further adjunctions and interesting applications. We shall return to this topic in a forthcoming paper.

Our next aim is to show that the duality between cospatial lattices and sober spaces restricts to a duality between completely distributive lattices and sober C-spaces. To make this statement correct, we must ensure that in fact every completely distributive lattice is (spatial and) cospatial. More precisely, we prove the following:

PROPOSITION 5.10. *The completely distributive lattices are the cospatial continuous lattices. Similarly, the superalgebraic lattices are the cospatial algebraic lattices.*

PROOF. Clearly, every completely distributive lattice L is continuous. Since complete distributivity is a selfdual property, it will suffice to prove that L is spatial, i.e. that the $\wedge$-primes form a dual base. As in 3.4(2), let us denote by ρx the smallest lower set Y with $x \leq \bigvee Y$. Then $x = \bigvee \rho x$. Hence, for any element y with $x \not\leq y$, there exists an element $x_1 \in \rho x$ with $x_1 \not\leq y$; repeating this argument, we obtain a decreasing sequence (x_n) such that $x_0 = x$, $x_{n+1} \in \rho x_n$, and $x_n \not\leq y$. The lower set $Y = \{z \in L: x_n \not\leq z$ for all $n\}$ has a join p, and this is in fact the greatest element of Y (otherwise $x_n \leq p$ for some n, and $x_{n+1} \in \rho x_n$ would imply $x_{n+1} \in Y$). Thus we have $x = x_0 \not\leq p$ but $y \leq$ p. For any two elements $u,v \not\leq p$, i.e. $u,v \notin Y$, there exists an n with $x_n \leq u \wedge v$, whence $u \wedge v \notin Y$ and therefore $u \wedge v \not\leq p$. This shows that p is a $\wedge$-prime element with $x \not\leq p$ and $y \leq p$, as desired.
Conversely, assume L is a cospatial continuous lattice. Denote by $\ll$ the way-below relation and by Y_x the set of all $\vee$-prime elements $p \ll x$. Then x is the join of Y_x, being the join of elements $y \ll x$ (see again 3.4(2)) which are joins of $\vee$-primes. For any other lower set Y with $x \leq \bigvee Y$, the ideal I formed by finite joins of elements from Y has the same join as Y and must therefore contain Y_x. But then any $\vee$-prime element $p \ll x$ belongs not only to I but to Y. Thus $Y_x \subseteq Y$, and consequently, Y_x is contained in ρx. Now the equation $x = \bigvee Y_x$ entails $x = \bigvee \rho x$, showing that L is completely distributive. □

COROLLARY 5.11. *A lattice is completely distributive iff it is isomorphic to the lattice of closed sets in its $\vee$-spectrum iff it is (dually) isomorphic to the topology of a C-space.*

For related results, see [24], [28] and [31]. Similar arguments shows that *every continuous distributive lattice is spatial, and that the continuous distributive lattices are, up to isomorphism, the topologies of locally compact (sober) spaces* (see [19]).

Recall that BD is the category of superalgebraic lattices and isotone maps. By AD, we denote that full subcategory AD of BD whose objects are strongly coatomic, where a lattice is said to be *strongly (co-)atomic* if each interval with at least two elements has a (co-) atom (cf. [8]). The rôle of strong (co-)atomicity in the present context is clarified by

LEMMA 5.12. *For a T_0-A-space $(S,\mathcal{C})$, the following three statements are equivalent:*
(a) *$(S,\mathcal{C})$ is sober.*
(b) *The ordered set $(S, \leq_{\mathcal{C}})$ satisfies the ascending chain condition.*
(c) *The lattice of open (resp. closed) sets is strongly atomic (resp. coatomic).*

PROOF. (a)$\Leftrightarrow$(b): It is well-known and easy to see that the ascending chain condition (which forbids properly ascending sequences) holds iff each directed (lower) set has a greatest element. But the directed lower sets of a quasiordered set are precisely the $\vee$-prime members of the lattice of all closed (i.e. lower) sets in the corresponding A-space (see [22]). Hence, for T_0-A-spaces, soberness means that each directed lower set is a principal ideal.
(b) $\Rightarrow$(c): Given two lower sets $X \subset Y$, we may choose a maximal element y in $Y \setminus X$. Then $Y \setminus \{y\}$ is a lower set covered by Y, hence a coatom in the interval $[X,Y]$.
(c) $\Rightarrow$(b): If Y is a nonempty lower set then we find a maximal lower set Z properly contained in Y. Any element $x \in Y \setminus Z$ must be maximal in Y: otherwise, if $x < y \in Y$ then Z would be a proper subset of the lower set $Y \setminus \uparrow y$ (since $Z \subseteq Y \setminus \uparrow x \subset Y \setminus \uparrow y$). Thus each nonempty (lower) set has a maximal element, and $(S, \leq_{\mathcal{C}})$ satisfies the ascending chain condition. $\square$

Now let us denote by ASOB, BSOB, and CSOB the categories of sober A-, B- and C-spaces, respectively, together with isotone maps as morphisms. Then CSOB is a full subcategory of LSOB, the category of locally compact sober spaces, and on the lattice-theoretical part, CD is a full subcategory of LD, the category of *lower (i.e. dually) continuous* distributive lattices.
The notations $X^{\leftarrow}$ for the subcategories with the same objects but continuous morphisms etc. remain valid in the present situation. Furthermore, given any category Y of complete lattices (and isotone maps), we denote by $Y^{\vee}$ the subcategory with the same objects as Y and morphisms preserving $\vee$-spectra, i.e. sending $\vee$-primes to $\vee$-primes, and by $Y_{\vee}$ the category with the same objects but morphisms preserving finite joins. Then we have the following equivalence and duality theorem (the necessary details are easily supplied with the help of the previous results):

THEOREM 5.13. *The completion functor C induces equivalences, and the functor Δ induces dual isomorphisms between the following pairs of categories, where* X *stands for one of the letters* A, B, C, L *or* T*:*

$$\mathrm{XSOB}^{-} \xrightarrow{\ C\ } \mathrm{XD}^{\vee}_{\triangledown} \xrightarrow{\ \Delta\ } \mathrm{XD}^{\vartriangle}_{\vee} \quad , \quad \mathrm{XSOB}^{\leftrightarrow} \xrightarrow{\ C\ } \mathrm{XD}^{\triangledown}_{\triangledown} \xrightarrow{\ \Delta\ } \mathrm{XD}^{\vartriangle}_{\triangledown} \ .$$

Hence the composite functor ΔC induces dualities between the categories of sober spaces on the left side and the categories of complete lattices on the right side of the arrows.

Thus, for example, the category of sober B- (resp. A-)spaces is dual to the category of (strongly coatomic) superalgebraic lattices, and the category of sober C-spaces is dual to the category of completely distributive lattices. As one nice application of this duality, one obtains Jakubik's result that *each completely distributive lattice is representable as a (cartesian) product of product-indecomposable ones*: indeed, any C-space is the sum of its components, being locally connected, and this decomposition into "additively indecomposable" parts is transferred by the duality functor to a product decomposition into "multiplicatively indecomposable" factors. Product decompositions of this kind are of importance for the so-called "*fuzzy set theory*", where power sets 2^X are replaced with arbitrary powers L^X of completely distributive lattices L.

At the end of our considerations, it remains to explain the order-theoretical counterparts of the categories CSOB etc. The key result for this object is due to Hoffmann [24] and Lawson [31]: it states that *the sober C-spaces, endowed with their specialization order, are just the continuous ordered sets, topologized by the closure system of Scott closed sets* (i.e. lower sets which are closed under directed joins; cf. [19]). In accordance with the corresponding definition of continuous lattices, an ordered set $(S, \leq)$ is said to be *continuous* if it is *(up-)complete*, i.e. each directed subset has a join, and for each element $y \in S$, the set $\ll y$ is directed and has join y, where the *way-below relation* $\ll$ is defined by

$$x \ll y \text{ iff for all directed lower subsets } D \text{ of } S,\ y \leq \bigvee D \text{ implies } x \in D.$$

These definitions fit perfectly into the framework of generalized order relations developed in Section 3: the way-below relation of a continuous ordered set is not only the smallest C-order inducing the given order but also the core relation of the closure system of Scott-closed sets. Hence the one-to-one correspondence between C-ordered sets and T_0-C-spaces restricts to the afore-mentioned correspondence between continuous ordered sets (C-ordered by the way-below relation) and sober C-spaces.

Similarly, a complete ordered set is *algebraic*, i.e. every element is a join of a directed set of compact elements, iff its way-below relation is a B-order inducing the given order. Hence algebraic ordered sets correspond to sober B-spaces (see [28]). Finally, by 5.12, ordered sets satisfying the *ascending chain condition* correspond to sober A-spaces. Thus, denoting by

ACO the category of ordered sets with ascending chain condition,
BCO the category of algebraic ordered sets ("base of compact elements"!),
CCO the category of (complete) continuous ordered sets,

each with isotone maps as morphisms, we arrive at our final isomorphism theorem:

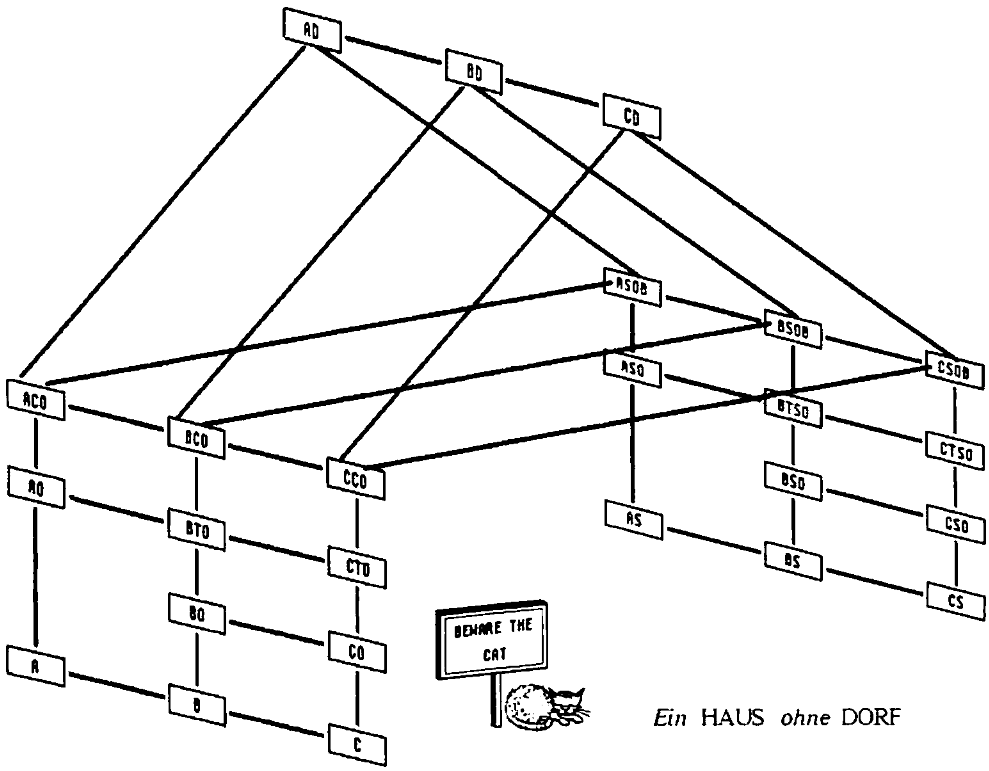
THEOREM 5.12. *The following pairs of categories are concretely isomorphic under the functorial isomorphism between correlated sets and core spaces:*

$$\text{ACO} \rightleftarrows \text{ASOB}, \qquad \text{BCO} \rightleftarrows \text{BSOB}, \qquad \text{CCO} \rightleftarrows \text{CSOB}.$$

Further isomorphisms are obtained if these pairs of categories X *and* Y *are replaced with the following pairs of subcategories:*

$X^{\rightarrow}$ *(isotone maps preserving the way-below relation) and* $Y^{\rightarrow}$ *(quasiopen isotone maps),*
$X^{\leftarrow}$ *(interpolating isotone maps = maps preserving directed joins) and* $Y^{\leftarrow}$*(continuous maps),*
$X^{\leftrightarrow}$ *(interpolating way-below-relation preserving maps) and* $Y^{\leftrightarrow}$ *(continuous quasiopen maps).*

We hope that after this categorically trimmed introduction into the ABC of order and topology, the reader will appreciate the relevance of non-Hausdorff topology.



Ein HAUS *ohne* DORF

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