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Internal Description of Hulls: A Unifying Approach

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Introduction

There is a number of categorical properties—e.g., cartesian closedness—whose presence is useful and convenient, in particular, in a topological setting. The fact that many important categories lack some of these desirable properties has led to the investigation of extensions possessing such properties. Of particular interest are extensions that are as small as possible, i.e., *hulls*.

Natural problems arising in connection with hulls are existence, uniqueness, and description. We will address only one aspect of the last of these, namely the *internal description* of hulls: how can a certain hull of a category be described in terms of the originally given category alone? We present a method that produces internal descriptions of various hulls in a uniform and almost mechanical way.

Numerous papers are devoted to the problem of describing hulls of well-known categories; a paradigmatic presentation for the category of topological spaces can be found in these proceedings [15]. For *external descriptions* of hulls as the injective objects of certain quasi-categories, we refer to the recent article [7] and its references.

Our use of terminology will be as in [15]; in particular, by a *construct*, we will always understand a concrete category over **Set** which is small-fibred and has constant morphisms; the forgetful functor to **Set** will usually be denoted by $|-|$. A convenient reference for background on categorical topology is [2].

The investigations presented in the following are restricted to the important special case of constructs. Similar results can be obtained for concrete categories over suitable base categories other than **Set**.

1. Some fundamentals on hulls

The following properties of constructs are useful in a variety of connections, and hence considered desirable by (categorical) topologists: being

- topological (initial structures exist w.r.t. arbitrary families of maps [15: 1.1]),
- monotopological (initial structures exist w.r.t. point-separating families of maps [17: p. 1362]),
- cartesian closed topological (existence of canonical function spaces [15: 2.14]),
- cartesian closed monotopological (cartesian closedness in monotopological constructs is characterized in the same way as in topological constructs by the existence of canonical function spaces [18: 3.1]),
- extensional topological (existence of one-point extensions for the representability of partial morphisms [15: 1.3]),
- a topological universe (all of the above [15: 3.14]).

This work is an original contribution and will not appear elsewhere.

Since many constructs lack at least some of these properties, a reasonable approach to remedy the situation is to look for extensions which have such properties, and among them for those which are "minimal".

1.1 Definition [12], [15: 2.6, 3.8, 3.17]. Let P denote any of the above-mentioned properties. A construct B satisfying P is called a *P-hull* of a construct A if B is a finally dense extension of A such that every finally dense embedding of A into a construct with the property P can be uniquely extended to B. (We assume embeddings to be concrete, i.e., the underlying set of an object and its image coincide; moreover, embeddings and subcategories are assumed to be full.)

The final density of the extensions plays a crucial rôle—it ensures that the constructions that are characteristic concerning any of the properties P remain intact: finally dense embeddings preserve initial sources and initial monosources [16: Prop. 10], canonical function spaces [16: Prop. 5], [18: 3.4], and one-point extensions [19: 3.4].

There are three types of questions that arise immediately: Do P-hulls always exist? Are they unique? How can they be described?

From the uniqueness of the extension of the embedding in Definition 1.1, it is clear that P-hulls are unique up to (concrete) isomorphism. Generally, they need not exist; counterexamples and conditions for the existence are given in [1] (topological hulls), [3] (cartesian closed topological hulls), [8: 4.3], [22: 3.2.5] (cartesian closed monotopological hulls), [14: 4.2] (extensional toplogical hulls), [4: Remark 5, Example 1] (topological universe hulls).

We will, for the rest of this paper, address the question of descriptions of P-hulls, more specifically: *internal descriptions* in terms of the construct whose P-hull is to be formed.

The starting point for our investigation is the following construction principle: assuming that a construct has a finally dense extension with the property P, then the P-hull can be formed inside this extension as the bireflective or surjective-reflective hull of a specific subclass.

1.2 Theorem. *Assume that* B *is a construct with the property P, and* A *a finally dense subcategory of* B. *Then the P-hull of* A *exists and is obtained as the bireflective hull in* B *of the class* $\mathrm{D} \subset \mathrm{B}$ *specified in the following table:*

P	D	
topological	A	[11: 1.4]
cartesian closed topological	$\{ [X,Y] \mid X,Y \in \mathrm{ObA} \}$	[16]
extensional topological	$\{ Y^{\natural} \mid Y \in \mathrm{ObA} \}$	[12], [13]
topological universe	$\{ [X,Y^{\natural}] \mid X,Y \in \mathrm{ObA} \}$	[20: 3.6], [4: Remark 1(b)]

and as the surjective-reflective hull in B *of* D *in the following cases:*

P	D	
monotopological	A	
cartesian closed monotopological	$\{ [X,Y] \mid X,Y \in \mathrm{ObA} \}$	[8], [22: 3.1]

where $[X,Y]$ and $Y^{\natural}$ denote canonical function spaces and one-point extensions in B, *respectively. (Recall that a* B*-object X is in the bireflective hull of* D *in* B *if and only if there is an initial source in* B *with domain X and codomains in* D*; for the surjective-reflective hull, "initial source" has to be replaced by "initial monosource".)*

Proof. All cases can be proved following the same pattern. We give a detailed proof—different from the original one by Herrlich—for the extensional topological hull, since the investigation of this concept has only recently begun. In addition, we provide a few hints on how to modify the steps of this proof to obtain a proof for the most complex case, the topological universe hull.

So assume **B** is an extensional topological construct, and **A** a finally dense subcategory of **B**. Denote by **C** the bireflective hull of $\{ Y^\sharp \mid Y \in \mathrm{Ob}\mathbf{A} \}$ in **B**.

(1) **C** is topological: Given a structured source $(f_i : S \longrightarrow Y_i \mid i \in I)$ of maps from a set S to **C**-objects Y_i, each Y_i is the domain of an initial source in **B** with codomains in $\{ Y^\sharp \mid Y \in \mathrm{Ob}\mathbf{A} \}$. Composition of these sources yields a structured source whose domain is the set S and whose codomains are in $\{ Y^\sharp \mid Y \in \mathrm{Ob}\mathbf{A} \}$. Since **B** is topological, this structured source can be lifted to an initial source with domain $X \in \mathrm{Ob}\mathbf{B}$. By definition of **C**, it follows that $X \in \mathrm{Ob}\mathbf{C}$. Moreover, $(f_i : X \longrightarrow Y_i \mid i \in I)$ is initial in **B**, and consequently also in **C**.

(2) For every $Y \in \mathrm{Ob}\mathbf{A}$, Y is a subspace of $Y^\sharp$, and consequently, $Y \in \mathrm{Ob}\mathbf{C}$. Hence $\mathbf{A} \subset \mathbf{C}$; it is clear that the embedding is finally dense.

(3) For every $Y \in \mathrm{Ob}\mathbf{B}$, there is an initial morphism $g : Y^{\sharp\sharp} \longrightarrow Y^\sharp$: Since Y is a subspace of $Y^{\sharp\sharp}$, the identity $1_Y : Y \longrightarrow Y$ is a partial morphism from $Y^{\sharp\sharp}$ to Y. Hence the map $(1_Y)^{Y^{\sharp\sharp}} : Y^{\sharp\sharp} \longrightarrow Y^\sharp$ is a **B**-morphism. Put $g = (1_Y)^{Y^{\sharp\sharp}}$. Now consider the composition of g with the inclusion $j : Y^\sharp \longrightarrow Y^{\sharp\sharp}$. Obviously, $g \circ j$ is the identity on $Y^\sharp$. If X is the **B**-object with $|X| = |Y^{\sharp\sharp}|$ such that $g : X \longrightarrow Y^\sharp$ is initial, then $Y^{\sharp\sharp} \leq X$, i.e., $1_{|X|} : Y^{\sharp\sharp} \longrightarrow X$ is a **B**-morphism. On the other hand, since $1_{Y^\sharp} = g \circ j : Y^\sharp \longrightarrow Y^\sharp$ is initial, $j : Y^\sharp \longrightarrow X$ is an initial morphism, i.e., $Y^\sharp$ is a subspace of X. It follows that $X \leq Y^{\sharp\sharp}$, because $Y^{\sharp\sharp}$ carries the greatest structure such that $Y^\sharp$ is a subspace [19: 2.6]. We conclude $X = Y^{\sharp\sharp}$, which implies that $g : Y^{\sharp\sharp} \longrightarrow Y^\sharp$ is initial.

(4) **C** is closed under one-point extensions in **B**, and consequently, **C** is extensional: For every $Z \in \mathrm{Ob}\mathbf{C}$, there is an initial source $(f_i : Z \longrightarrow Y_i^\sharp \mid i \in I)$ with all $Y_i \in \mathbf{A}$. Since the functor $(-)^\sharp : \mathbf{B} \longrightarrow \mathbf{B}$, assigning to each **B**-object its one-point extension and to each **B**-morphism $f : X \longrightarrow Y$ its extension $f^\sharp = f^{X^\sharp} : X^\sharp \longrightarrow Y^\sharp$, preserves initial sources [19: 2.13], we obtain that $(f_i^\sharp : Z^\sharp \longrightarrow Y_i^{\sharp\sharp} \mid i \in I)$ is initial. By (3), there exist initial morphisms $g_i : Y_i^{\sharp\sharp} \longrightarrow Y_i^\sharp$. Hence the composition $(g_i \circ f_i^\sharp : Z^\sharp \longrightarrow Y_i^\sharp \mid i \in I)$ is an initial source, and $Z^\sharp \in \mathrm{Ob}\mathbf{C}$.

(5) To prove that **C** has the universal property of 1.1, we consider first the case that **A** be topological. Let **K** be an extensional topological construct which contains **A** as a finally dense subcategory; one-point extensions of **K**-objects Y are denoted by Y^+. Define $H : \mathbf{C} \longrightarrow \mathbf{K}$ as follows: For $Z \in \mathrm{Ob}\mathbf{C}$, HZ is the **K**-object with the same underlying set as Z such that

$$(f : HZ \longrightarrow Y^+ \mid Y \in \mathrm{Ob}\mathbf{A}, f : Z \longrightarrow Y^\sharp \in \mathrm{Mor}\mathbf{C})$$

is an initial source in **K**; for a **C**-morphism $h : Z_1 \longrightarrow Z_2$, Hh coincides with h as a map (and will be denoted by h subsequently). We have to show that $h : HZ_1 \longrightarrow HZ_2$ is a **K**-morphism whenever $h : Z_1 \longrightarrow Z_2$ is a **C**-morphism. By definition of HZ_2, it is sufficient that $f \circ h : HZ_1 \longrightarrow Y^+$ is a **K**-morphism whenever $f : Z_2 \longrightarrow Y^\sharp$ is a **C**-morphism with $Y \in \mathrm{Ob}\mathbf{A}$. Since $f \circ h : Z_1 \longrightarrow Y^\sharp$ is a **C**-morphism, this follows immediately from the definition of HZ_1. Consequently, $H : \mathbf{C} \longrightarrow \mathbf{K}$ is a concrete functor. We will prove that H is an embedding with $HX = X$ for $X \in \mathrm{Ob}\mathbf{A}$. Let us first show that for $X, Y \in \mathrm{Ob}\mathbf{A}$,

$$h : X \longrightarrow Y^\sharp \text{ is a } \mathbf{C}\text{-morphism} \iff h : X \longrightarrow Y^+ \text{ is a } \mathbf{K}\text{-morphism}.$$

We prove the implication "$\Rightarrow$"; the converse implication is shown analogously. If X and Y are **A**-objects and $h : X \longrightarrow Y^\sharp$ is a **C**-morphism, then the restriction $\bar{h}$ of h to the subspace $h^{-1}(Y)$ of X is also a **C**-morphism, and since $h^{-1}(Y)$ and Y are **A**-objects and **A** is a full subcategory of **C**, $\bar{h}$ is an **A**-morphism. It follows that $\bar{h}$ is a partial morphism from X to Y in **K**; consequently, its extension $\bar{h}^X = h : X \longrightarrow Y^+$ is a **K**-morphism.

Since X and HX have the same underlying set, the definition of HX, together with the equivalence just shown, implies that $X \leq HX$. On the other hand, the inclusion $j : X \longrightarrow X^\sharp$ is a C-morphism, and consequently, $j : HX \longrightarrow X^+$ is a K-morphism. The latter implies $HX \leq X$, because $j : X \longrightarrow X^+$ is initial. Hence $X = HX$.

H is injective on objects: Since initial structures are unique, it suffices to prove that for $Z \in \mathrm{Ob}\mathbf{C}$, $Y \in \mathrm{Ob}\mathbf{A}$,

$$h : Z \longrightarrow Y^\sharp \text{ is a } \mathbf{C}\text{-morphism} \iff h : HZ \longrightarrow Y^+ \text{ is a } \mathbf{K}\text{-morphism}.$$

The implication "$\Rightarrow$" is clear. For the converse implication, it is sufficient to show that $h \circ f : X \longrightarrow Y^\sharp$ is a C-morphism whenever $f : X \longrightarrow Z$ is a C-morphism with $X \in \mathrm{Ob}\mathbf{A}$, because **A** is finally dense in **C**. Now $f : HX \longrightarrow HZ$ and $h : HZ \longrightarrow Y^+$ are K-morphisms, and so is their composition $h \circ f : X = HX \longrightarrow Y^+$. It follows that $h \circ f : X \longrightarrow Y^\sharp$ is a C-morphism.

To prove the uniqueness of the embedding, suppose $H_1, H_2 : \mathbf{C} \longrightarrow \mathbf{K}$ are finally dense embeddings that coincide on **A**-objects. Finally dense embeddings between extensional topological constructs preserve initial sources [16: Prop. 10] and one-point extensions [19: 3.4]. If $Z \in \mathrm{Ob}\mathbf{C}$ and $\left(f_i : Z \longrightarrow Y_i^\sharp \mid i \in I\right)$ is an initial source in **C** with all Y_i in **A**, then $\left(f_i : H_1 Z \longrightarrow (H_1 Y_i)^+ \mid i \in I\right)$ and $\left(f_i : H_2 Z \longrightarrow (H_2 Y_i)^+ \mid i \in I\right)$ are initial sources in **K**; since H_1 and H_2 coincide on $\mathrm{Ob}\mathbf{A}$, and initial structures are unique, it follows that $H_1 Z = H_2 Z$.

Finally, let us drop the condition that **A** be topological. Denote by **A**′ the bireflective hull of **A** in **B**. The bireflective hull **C** of $\{ Y_i^\sharp \mid Y_i \in \mathrm{Ob}\mathbf{A} \}$ in **B** coincides with the bireflective hull of $\{ Y_i^\sharp \mid Y_i \in \mathrm{Ob}\mathbf{A}' \}$ in **B**, and every finally dense embedding of **A** into an (extensional) topological construct can be uniquely extended to **A**′. By the above proof, **C** is the extensional topological hull of **A**′. It follows that **C** fulfils the defining properties of an extensional topological hull of **A**.

We conclude with remarks concerning the topological universe hull. A proof can be given by performing steps analogous to the ones above; we use the same numbering to point out some of the necessary modifications.

(2) Denote by 1 a singleton **B**-object. Then $Y \cong [1, Y]$. Since Y is a subspace of $Y^\sharp$, it follows that $[1, Y]$ is a subspace of $[1, Y^\sharp]$. Since the constant map $Y \longrightarrow 1$ is final, there is an initial morphism $[1, Y^\sharp] \longrightarrow [Y, Y^\sharp]$. Composition yields an initial morphism $Y \longrightarrow [Y, Y^\sharp]$.

(3) By [20: 3.1] and the proof of (3) above, there are initial morphisms $[X, Y^\sharp]^\sharp \longrightarrow [X, Y^{\sharp\sharp}] \longrightarrow [X, Y^\sharp]$.

(4) Use (3) and the fact that $(-)^\sharp : \mathbf{B} \longrightarrow \mathbf{B}$ preserves initial sources to show that $Z \in \mathrm{Ob}\mathbf{C}$ implies $Z^\sharp \in \mathrm{Ob}\mathbf{C}$. To prove that $X \in \mathrm{Ob}\mathbf{B}$, $Z \in \mathrm{Ob}\mathbf{C}$ implies $[X, Z] \in \mathrm{Ob}\mathbf{C}$, observe that $[X, -] : \mathbf{B} \longrightarrow \mathbf{B}$ preserves initial sources, apply the adjunction $- \times X \dashv [X, -]$, and use the fact that $[-, Y] : \mathbf{B} \longrightarrow \mathbf{B}$ transforms final episinks into initial sources.

(5) In the proof of the implication

$$h : W \longrightarrow [X, Y^\sharp] \text{ is a } \mathbf{C}\text{-morphism} \implies h : W \longrightarrow [X, Y^+]_{\mathbf{K}} \text{ is a } \mathbf{K}\text{-morphism}$$

for $W, X, Y \in \mathrm{Ob}\mathbf{A}$ (where $[-,-]_{\mathbf{K}}$ denotes function spaces in **K**), use first the adjunction $- \times X \dashv [X, -]$ to obtain a **C**-morphism $h_1 : W \times X \longrightarrow Y^\sharp$, then restrict h_1 to get an **A**-morphism $\bar{h}_1 : h_1^{-1}(Y) \longrightarrow Y$.

To show that $HX \leq X$ for $X \in \mathrm{Ob}\mathbf{A}$, replace the inclusion j (in the proof for the extensional topological hull) by the initial morphism $X \longrightarrow [X, X^\sharp]$ of step (2). ∎

1.3 Remark. As an immediate consequence of Theorem 1.2, one obtains that *the P-hull of a construct* **A** *is characterized, among the finally dense extensions of* **A** *satisfying property P, by the initial density (resp. initial monodensity) of the class* **D** *specified in* 1.2. Indeed, assume

B is a finally dense extension of A with property P. Form the bireflective (resp. surjective-reflective) hull C of D in B. By 1.2, C is the P-hull of A. If B is the P-hull of A, then B = C, and D is initially (mono)dense in B. Conversely, if D fulfils the initial density condition, then B = C, and B is the P-hull of A.

2. The quasi-category of structured sinks

We will in the following explain a method by which it is possible to obtain internal descriptions of the P-hulls of a construct A in a uniform way; moreover, this method produces the descriptions more or less automatically, without the need for particular ingenuity. Compare also the articles [23] and [12].

The basic approach is this: First find an "umbrella" construct B which has all the properties P and can be described in terms of A; then perform the construction of the P-hulls inside B, as indicated in Theorem 1.2.

Unfortunately, such a B generally does not exist—at least, if B is required to be a construct. However, the outlined approach is successful if the definition of a category is relaxed, and the collection of B-objects is only required to form a conglomerate (an entity that may be larger than any class). This means that B is allowed to be a *quasi-category*. For foundational aspects on conglomerates and quasi-categories, the interested reader is referred to [2: Section 2 and 3.49 ff]; but a naïve, "set-like" treatment will suffice for our purposes.

All this may sound somewhat intimidating; but actually, the idea behind the definition of our umbrella B is no more difficult than Hausdorff's construction of the completion of a metric space [10]: If a metric space fails to be complete, then there are non-convergent Cauchy sequences. Now the very objects that cause the trouble, the Cauchy sequences, are taken as the points of the completion (more precisely, equivalence classes of Cauchy sequences). In our case, in order to achieve "topological completeness", the new quasi-category is made up of the "structured sinks", a concept which originates from [9].

2.1 Definition. For a construct A, the quasi-category SA of *structured sinks over* A is defined as follows: Objects are pairs $(X, \mathcal{S})$ where X is a set and $\mathcal{S}$ a *structured sink* (i.e., $\mathcal{S}$ is a family $(f_i : A_i \longrightarrow X \mid i \in I)$ of maps from A-objects A_i to the set X), subject to the following conditions:

(1) $\mathcal{S}$ is closed under composition with A-morphisms, i.e., if $g : A \longrightarrow A_i$ is an A-morphism and $f_i : A_i \longrightarrow X \in \mathcal{S}$, then $f_i \circ g : A \longrightarrow X \in \mathcal{S}$;

(2) $\mathcal{S}$ contains all constant maps (including the empty map) from A-objects to X.

Morphisms of SA are *sink maps* $h : (X, \mathcal{S}) \longrightarrow (X', \mathcal{S}')$, i.e., maps $h : X \longrightarrow X'$ with $h \circ f \in \mathcal{S}'$ for all $f \in \mathcal{S}$.

Observe that SA is defined in terms of A.

If A is a proper class, then the structured sinks over A are also proper classes, and the collection of SA-objects forms a proper conglomerate; hence SA is, in general, only a quasi-category. We will show that, except for this hugeness, SA has all desirable properties [23]; to avoid ponderous formulations, we will speak of the cartesian closedness of SA, the extensionality of SA, etc., even though these notions are only defined for categories.

2.2 Proposition. SA *has initial structures.*

Proof. The initial structure with respect to a family $(h_i : X \longrightarrow (X_i, \mathcal{S}_i) \mid i \in I)$ of maps h_i from a set X to SA-objects $(X_i, \mathcal{S}_i)$ is given by

$$\mathcal{S} = (f : A \longrightarrow X \mid \text{for all } i \in I,\ h_i \circ f \in \mathcal{S}_i). \qquad ∎$$

It follows immediately from 2.1(2) that the only structured sink on the empty set and on each singleton set is the sink consisting of all constant maps with domains in A; so we have:

2.3 Proposition. *The sets with at most one element carry exactly one* SA*-structure.*

The key fact, which ensures that we do not move too far away from **A** when passing to SA, is final density. (Actually, the requirement of final density enforces that one cannot expand beyond SA, since SA is the "largest" finally dense extension of **A** which has constants, cf. [11].)

2.4 Proposition. **A** *can be regarded as a finally dense subcategory of* SA.

Proof. For an **A**-object A, denote by $\mathcal{T}(A)$ the structured sink determined by the total sink from **A** to A, i.e.,

$$\mathcal{T}(A) = \left(f : B \longrightarrow |A| \;\middle|\; f : B \longrightarrow A \in \operatorname{Mor}\mathbf{A} \right).$$

Then the assignment $A \longmapsto (|A|, \mathcal{T}(A))$, $h : A \longrightarrow B \;\longmapsto\; h : (|A|, \mathcal{T}(A)) \longrightarrow (|B|, \mathcal{T}(B))$ is an embedding of **A** into SA.

Observe that for any structured sink $\mathcal{S}$ on a set X and any **A**-object A,

$$(*) \qquad f : A \longrightarrow X \in \mathcal{S} \iff f : (|A|, \mathcal{T}(A)) \longrightarrow (X, \mathcal{S}) \text{ is a sink map.}$$

The implication "$\Rightarrow$" is clear by 2.1(1). The converse implication follows immediately from $1_A \in \mathcal{T}(A)$.

Finally, the embedding is finally dense: For any SA-object $(X, \mathcal{S})$,

$$\left(f : (|A|, \mathcal{T}(A)) \longrightarrow (X, \mathcal{S}) \;\middle|\; f : A \longrightarrow X \in \mathcal{S} \right)$$

is a final episink by $(*)$. ∎

We will, in the following, generally identify an **A**-object A with its image $(|A|, \mathcal{T}(A))$ in SA.

We have seen in 2.2 that, without any further conditions on the construct **A**, the quasi-category SA has initial structures, in particular, subspaces and finite (concrete) products. Therefore it makes sense to ask whether SA is extensional or cartesian closed. However, unless **A** has subspaces resp. finite concrete products, we cannot expect that one-point extensions and function spaces can be described via features of **A**.

2.5 Proposition. *For any construct* **A** *with finite concrete products, the quasi-category* SA *is cartesian closed.*

Proof. Let us first have a look at products in SA. Consider SA-objects $(W, \mathcal{S}_W)$ and $(X, \mathcal{S}_X)$. Denote by $\mathcal{S}_W \times \mathcal{S}_X$ the structured sink of the product $(W, \mathcal{S}_W) \times (X, \mathcal{S}_X)$. If $g : A \longrightarrow W \in \mathcal{S}_W$ and $f : B \longrightarrow X \in \mathcal{S}_X$, then $g : A \longrightarrow (W, \mathcal{S}_W)$ and $f : B \longrightarrow (X, \mathcal{S}_X)$ are sink maps by $(*)$. Consequently, $g \times f : A \times B \longrightarrow (W, \mathcal{S}_W) \times (X, \mathcal{S}_X)$ is a sink map, and, again by $(*)$, $g \times f : A \times B \longrightarrow W \times X \in \mathcal{S}_W \times \mathcal{S}_X$. (Notice that, by final density, the embedding of **A** into SA preserves finite products.) On the other hand, if $f : A \longrightarrow W \times X \in \mathcal{S}_W \times \mathcal{S}_X$, then the compositions $p_W \circ f$ and $p_X \circ f$ of f with the projections belong to $\mathcal{S}_W$ and $\mathcal{S}_X$, respectively, and $f = \big((p_W \circ f) \times (p_X \circ f)\big) \circ \delta$, where $\delta : A \longrightarrow A \times A$, $\delta(a) = (a, a)$, is the diagonal morphism. We obtain:

$$h : (W, \mathcal{S}_W) \times (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y) \in \operatorname{Mor} \mathrm{SA} \iff \text{for all } g \in \mathcal{S}_W \text{ and all } f \in \mathcal{S}_X,\; h \circ (g \times f) \in \mathcal{S}_Y.$$

Now let us define the canonical function space structure $\mathcal{P}$ on $\operatorname{Hom}((X, \mathcal{S}_X), (Y, \mathcal{S}_Y))$:

$$g : A \longrightarrow \operatorname{Hom}((X, \mathcal{S}_X), (Y, \mathcal{S}_Y)) \in \mathcal{P} \iff \text{for all } f \in \mathcal{S}_X,\; \operatorname{ev} \circ (g \times f) \in \mathcal{S}_Y$$

(where ev denotes the usual evaluation map). $\mathcal{P}$ is closed under composition with **A**-morphisms: For let $g : A \longrightarrow \mathrm{Hom}((X, \mathcal{S}_X), (Y, \mathcal{S}_Y)) \in \mathcal{P}$ and $g' : A' \longrightarrow A$ be an **A**-morphism. Then for each $f : B \longrightarrow X \in \mathcal{S}_X$, we have $\mathrm{ev} \circ (g \times f) \in \mathcal{S}_Y$. Since $g' \times 1_B$ is an **A**-morphism, $\mathrm{ev} \circ ((g \circ g') \times f) = \mathrm{ev} \circ (g \times f) \circ (g' \times 1_B) \in \mathcal{S}_Y$, and consequently, $g \circ g' \in \mathcal{P}$.

That $\mathcal{P}$ contains all constant maps is seen as follows: The case of the empty map is trivial. Given a non-empty constant map $g : A \longrightarrow \mathrm{Hom}((X, \mathcal{S}_X), (Y, \mathcal{S}_Y)) \in \mathcal{P}$, denote by h its single value. If $f : B \longrightarrow X \in \mathcal{S}_X$, then $\mathrm{ev} \circ (g \times f) = h \circ f \circ p_B \in \mathcal{S}_Y$, since h is a sink map and the projection p_B is an **A**-morphism.

It is clear that $\mathcal{P}$ is the largest structured sink on $\mathrm{Hom}((X, \mathcal{S}_X), (Y, \mathcal{S}_Y))$ such that the evaluation map is a sink map. It remains to be shown that for any sink map $h : (W, \mathcal{S}_W) \times (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$, its transpose $h^* : (W, \mathcal{S}_W) \longrightarrow (\mathrm{Hom}((X, \mathcal{S}_X), (Y, \mathcal{S}_Y)), \mathcal{P})$ is also a sink map. By definition of $\mathcal{P}$ and sink maps, this means: If $g \in \mathcal{S}_W$ and $f \in \mathcal{S}_X$, then $\mathrm{ev} \circ ((h^* \circ g) \times f) \in \mathcal{S}_Y$. The latter follows from the facts that h is a sink map, and $\mathrm{ev} \circ ((h^* \circ g) \times f) = h \circ (g \times f)$ at the set level. ∎

2.6 Proposition. *For any construct* **A** *with subspaces, the quasi-category* SA *is extensional.*

Proof. From the definition of initial structures, it is easily seen that $(X, \mathcal{S}_X)$ is a subspace of $(Y, \mathcal{S}_Y)$ if and only if $\mathcal{S}_X$ consists of the restrictions of the structured maps in $\mathcal{S}_Y$, i.e.,

$$\mathcal{S}_X = \left(\bar{g} : g^{-1}(X) \longrightarrow X \;\middle|\; g : B \longrightarrow Y \in \mathcal{S}_Y \right),$$

where $g^{-1}(X)$ is considered as a subspace of B and $\bar{g}$ is the restriction of g to $g^{-1}(X)$. Now define the one-point extension $(Y, \mathcal{S}_Y)^\sharp$ of $(Y, \mathcal{S}_Y)$ as follows: $(Y, \mathcal{S}_Y)^\sharp = (Y^\sharp, \mathcal{S}_Y{}^\sharp)$, where $Y^\sharp = Y \cup \{\infty\}$ with $\infty \notin Y$ and

$$g : B \longrightarrow Y^\sharp \in \mathcal{S}_Y{}^\sharp \iff \bar{g} : g^{-1}(Y) \longrightarrow Y \in \mathcal{S}_Y.$$

Since restrictions of constant maps are constant, it is obvious that $\mathcal{S}_Y{}^\sharp$ fulfils condition (2) of 2.1.

Now let $g : B \longrightarrow Y^\sharp \in \mathcal{S}_Y{}^\sharp$. The restriction $\overline{g'} : (g')^{-1}(g^{-1}(Y)) \longrightarrow g^{-1}(Y)$ of any **A**-morphism g' with codomain B is an **A**-morphism. Since $\bar{g} \in \mathcal{S}_Y$ by definition, it follows that $\overline{g \circ g'} = \bar{g} \circ \overline{g'} \in \mathcal{S}_Y$. Hence $g \circ g' \in \mathcal{S}_Y{}^\sharp$, and $\mathcal{S}_Y{}^\sharp$ is closed under composition with **A**-morphisms.

It is clear that $\mathcal{S}_Y{}^\sharp$ is the largest structured sink on $Y^\sharp$ such that $(Y, \mathcal{S}_Y)$ is a subspace of $(Y^\sharp, \mathcal{S}_Y{}^\sharp)$. We need to show that $(Y^\sharp, \mathcal{S}_Y{}^\sharp)$ represents partial morphisms to $(Y, \mathcal{S}_Y)$. Let $h : (Z, \mathcal{S}_Z) \longrightarrow (Y, \mathcal{S}_Y)$ be a partial morphism from $(X, \mathcal{S}_X)$ to $(Y, \mathcal{S}_Y)$. If $f : A \longrightarrow X \in \mathcal{S}_X$, then $\bar{f} : f^{-1}(Z) \longrightarrow Z \in \mathcal{S}_Z$. Since $h : (Z, \mathcal{S}_Z) \longrightarrow (Y, \mathcal{S}_Y)$ is a sink map, we have $h \circ \bar{f} \in \mathcal{S}_Y$. Consequently, the fact that $\overline{h^X \circ f} = \overline{h^X} \circ \bar{f} = h \circ \bar{f}$ implies $h^X \circ f \in \mathcal{S}_Y{}^\sharp$, as required. ∎

From 2.5 and 2.6, we obtain immediately:

2.7 Corollary. *For any construct* **A** *with subspaces and finite concrete products, the quasi-category* SA *is a topological universe.*

3. Internal descriptions of the P-hulls

In order to obtain internal descriptions of the P-hulls of a construct **A**, we now carry out the constructions described in Theorem 1.2 inside the quasi-category SA of structured sinks over **A**. This involves a two-step "translation process": By 1.2, an SA-object $(X, \mathcal{S})$ is contained in the P-hull of **A** if and only if the total source from $(X, \mathcal{S})$ to the class **D** specified in 1.2 is an initial source (resp., initial monosource, in case of the monotopological and the cartesian closed monotopological hulls). In the first step, we describe the total source from $(X, \mathcal{S})$ to **D**, and translate this description into one in terms of **A**; the condition in each of these cases will be that certain maps are **A**-morphisms. In the second step, the initial SA-structure on X with respect to the source of step 1 is described, and the same translation process is used to put the description in terms of **A**. Then $(X, \mathcal{S})$ is in the P-hull of **A** if and only if $\mathcal{S}$ coincides with the initial structure found in step 2, which yields the desired internal description.

We will give the details for the (mono)topological and cartesian closed (mono)topological hulls, to carefully demonstrate the procedure outlined above; the other results can be obtained analogously, and are only formulated.

Of course, it only makes sense to describe a P-hull if it exists; in the following, let us assume that the given construct **A** fulfils conditions which ensure the existence of the P-hull in question.

Let us start with the simplest case, to get the flavor of the method.

3.1 The topological hull *(MacNeille completion)* [11]. An SA-object $(X, \mathcal{S})$ is in the topological hull of **A** if and only if the total source from $(X, \mathcal{S})$ to **A** is initial.

Step 1. A map $h : (X, \mathcal{S}) \longrightarrow B$ from $(X, \mathcal{S})$ to an **A**-object B is a sink map exactly if for each $g : A \longrightarrow X \in \mathcal{S}$, the composition $h \circ g : A \longrightarrow |B|$ belongs to $\mathcal{T}(B)$. The latter condition can be translated into: $h \circ g : A \longrightarrow B$ is an **A**-morphism. Now define

$$\mathcal{T}(X, \mathcal{S}) = \big(h : X \longrightarrow |B| \;\big|\; B \in \mathrm{Ob}\mathbf{A} \text{ and for all } f \in \mathcal{S},\ h \circ g \in \mathrm{Mor}\mathbf{A} \big).$$

$\mathcal{T}(X, \mathcal{S})$ is a source in **Set** which represents the total source from $(X, \mathcal{S})$ to **A**.

Step 2. By 2.2, the inital SA-structure $I\mathcal{T}(X, \mathcal{S})$ on X with respect to the family $\mathcal{T}(X, \mathcal{S})$ is given by

$$\begin{aligned} I\mathcal{T}(X, \mathcal{S}) &= \big(g : A \longrightarrow X \;\big|\; A \in \mathrm{Ob}\mathbf{A} \text{ and for all } h : X \longrightarrow |B| \in \mathcal{T}(X, \mathcal{S}), \\ &\qquad\qquad h \circ g : A \longrightarrow |B| \in \mathcal{T}(B) \big) \\ &= \big(g : A \longrightarrow X \;\big|\; A \in \mathrm{Ob}\mathbf{A} \text{ and for all } h \in \mathcal{T}(X, \mathcal{S}),\ h \circ g \in \mathrm{Mor}\mathbf{A} \big). \end{aligned}$$

Now the total source from $(X, \mathcal{S})$ to **A** is initial if and only if $\mathcal{S}$ coincides with the initial structure with respect to the total source, i.e., $\mathcal{S} = I\mathcal{T}(X, \mathcal{S})$; or equivalently: $\mathcal{S} \supset I\mathcal{T}(X, \mathcal{S})$, since the converse inclusion is always true. Structured sinks with this property are called *closed sinks*. Observe that the condition is formulated solely in terms of sets, maps, **A**-objects, and **A**-morphisms. A formulation in words, rather than formulae, would read as follows: A structured sink $\mathcal{S}$ on X is closed if $g : A \longrightarrow X \in \mathcal{S}$ whenever $h \circ g : A \longrightarrow B$ is an **A**-morphism for every map $h : X \longrightarrow |B|$ with the property that for each $f \in \mathcal{S}$, $h \circ f$ is an **A**-morphism.— So we have obtained that *the topological hull of* **A** *is the category of closed sinks.* ∎

3.2 The monotopological hull [22: 3.3.6]. An SA-object $(X, \mathcal{S})$ is in the monotopological hull of **A** iff the total source from $(X, \mathcal{S})$ to **A** is an initial monosource, i.e. iff it is initial and separates points. With the notations of 3.1, this means: $\mathcal{S} \supset I\mathcal{T}(X, \mathcal{S})$ and $\mathcal{T}(X, \mathcal{S})$ separates points. Closed sinks with the property that $\mathcal{T}(X, \mathcal{S})$ separates points (or equivalently, $\mathcal{T}(X, \mathcal{S})$ is a monosource in **Set**) are called *mono-induced*. Hence *the monotopological hull of* **A** *is the category of closed, mono-induced sinks.* ∎

Now let us advance to a slightly more complex case:

3.3 The cartesian closed topological hull [6: 3.1]. Let **A** be a construct with finite concrete products; then SA is cartesian closed by 2.5, and the cartesian closed topological hull of **A** can be obtained inside SA by the construction in 1.2. Hence an SA-object $(X, \mathcal{S})$ is in the cartesian closed topological hull of **A** if and only if the total source from $(X, \mathcal{S})$ to the class of function spaces $\mathbf{D} = \{ [B,C] \mid B,C \in \mathrm{Ob}\mathbf{A} \}$ is initial (where the function spaces are formed in SA).

Step 1. A map $h : (X,\mathcal{S}) \longrightarrow [B,C]$ is a sink map iff for each $g : A \longrightarrow X \in \mathcal{S}$, the composition $h \circ g : A \longrightarrow \mathrm{Hom}(B,C)$ belongs to the function space structure $\mathcal{P}$ on $\mathrm{Hom}(B,C)$. By the description of $\mathcal{P}$ in the proof of 2.5, this means that for any $f \in \mathcal{T}(B)$, we have $\mathrm{ev} \circ ((h \circ g) \times f) \in \mathcal{T}(C)$. Since $\mathcal{T}(C)$ is closed under composition with A-morphisms, and $\mathrm{ev} \circ ((h \circ g) \times f) = \mathrm{ev} \circ ((h \circ g) \times 1_B) \circ (1 \times f)$, the last condition can be rewritten as: $\mathrm{ev} \circ ((h \circ g) \times 1_B) \in \mathcal{T}(C)$, or, translated into terms of **A**: $\mathrm{ev} \circ ((h \circ g) \times 1_B) : A \times B \longrightarrow C$ is an **A**-morphism. Hence

$$T(X,\mathcal{S}) = \big(h : X \longrightarrow \mathrm{Hom}(B,C) \mid B,C \in \mathrm{Ob}\mathbf{A} \text{ and for all } g : A \longrightarrow X \in \mathcal{S}, \\ \mathrm{ev} \circ ((h \circ g) \times 1_B) : A \times B \longrightarrow C \text{ is an } \mathbf{A}\text{-morphism} \big)$$

represents the total source from $(X,\mathcal{S})$ to **D**. Observe that the definition of $T(X,\mathcal{S})$ does not involve the function space structures in SA any more.

Step 2. A map $g : A \longrightarrow X$ is in the initial SA-structure $IT(X,\mathcal{S})$ with respect to $T(X,\mathcal{S})$ iff $h \in T(X,\mathcal{S})$ implies that $h \circ g$ is in the function space structure of $[B,C]$. From step 1, we know already that this translates into: $\mathrm{ev} \circ ((h \circ g) \times 1_B) : A \times B \longrightarrow C$ is an **A**-morphism. Consequently,

$$IT(X,\mathcal{S}) = \big(g : A \longrightarrow X \mid A \in \mathbf{A} \text{ and for all } h \in T(X,\mathcal{S}), \\ \mathrm{ev} \circ ((h \circ g) \times 1) \text{ is an } \mathbf{A}\text{-morphism} \big).$$

Finally, $(X,\mathcal{S})$ is in the cartesian closed topological hull of **A** iff $IT(X,\mathcal{S}) \subset \mathcal{S}$. Structured sinks with this property are called *power-closed*. So we have shown: *The cartesian closed topological hull of* **A** *is given by the category of power-closed sinks.* ∎

3.4 The cartesian closed monotopological hull [22: 3.3.5]. With the same assumptions and notations as in 3.3, an SA-object $(X,\mathcal{S})$ is in the cartesian closed monotopological hull of **A** iff $IT(X,\mathcal{S}) \subset \mathcal{S}$ and $T(X,\mathcal{S})$ separates points, i.e. iff $\mathcal{S}$ is a *power-closed mono-induced sink.* ∎

Proceeding in the same way for the remaining two hulls, we obtain:

3.5 The extensional topological hull [12: 3.1(3)]. Assume that **A** is a construct with subspaces. Following the description of the one-point extensions in 2.6, define, for an SA-object $(X,\mathcal{S})$:

$$T(X,\mathcal{S}) = \big(h : X \longrightarrow |B| \cup \{\infty\} \mid B \in \mathrm{Ob}\mathbf{A} \text{ and for all } g \in \mathcal{S}, \\ \overline{h \circ g} : (h \circ g)^{-1}(B) \longrightarrow B \text{ is an } \mathbf{A}\text{-morphism} \big)$$

and

$$IT(X,\mathcal{S}) = \big(g : A \longrightarrow X \mid A \in \mathrm{Ob}\mathbf{A} \text{ and for all } h \in T(X,\mathcal{S}),\ \overline{h \circ g} \text{ is an } \mathbf{A}\text{-morphism} \big).$$

Then $(X,\mathcal{S})$ is in the extensional topological hull of **A** iff $IT(X,\mathcal{S}) \subset \mathcal{S}$; structured sinks with this property might be called *extension-closed sinks.* Hence *the extensional topological hull of* **A** *is given by the category of extension-closed sinks.* ∎

3.6 The topological universe hull [4: Remark 4]. Let **A** be a construct with subspaces and finite concrete products. For an SA-object $(X,\mathcal{S})$, define

$$T(X,\mathcal{S}) = \big(h : X \longrightarrow \mathrm{Hom}(B,C^{\mathbf{!}}) \mid B,C \in \mathrm{Ob}\mathbf{A} \text{ and for all } g : A \longrightarrow X \in \mathcal{S}, \\ \overline{\mathrm{ev} \circ ((h \circ g) \times 1_B)} : (\mathrm{ev} \circ ((h \circ g) \times 1_B))^{-1}(C) \longrightarrow C \in \mathrm{Mor}\mathbf{A} \big)$$

and

$$IT(X,\mathcal{S}) = \big(g : A \longrightarrow X \mid A \in \mathrm{Ob}\mathbf{A} \text{ and for all } h \in T(X,\mathcal{S}),\ \overline{\mathrm{ev}\circ((h\circ g)\times 1)} \in \mathrm{Mor}\mathbf{A}\big).$$

Then $(X,\mathcal{S})$ is in the topological universe hull of **A** iff $IT(X,\mathcal{S}) \subset \mathcal{S}$; structured sinks with this property are called *partially closed.*

It might seem like the definition of a partially closed sink is not quite in terms of **A**; however, from the description of one-point extensions in the proof of 2.6, it is easily seen that $\mathrm{Hom}(B, C^{\sharp})$ consists of those maps $k : |B| \longrightarrow |C| \cup \{\infty\}$ whose restriction $\bar{k} : k^{-1}(C) \longrightarrow C$ is an **A**-morphism (where $k^{-1}(C)$ is considered as a subspace of B). ∎

3.7 Remark. (1) To appreciate the method for finding internal descriptions of hulls presented in this paper, the reader is invited to a comparison with the original proofs, which usually start out by defining the category of structured sinks with an appropriate closure property (let's call them P-closed, for short). Then, *in addition* to our argumentation, it has to be shown that the necessary constructions—initial structures, function space structures, or one-point extension structures—are P-closed. Moreover, our approach provides the natural explanation for the definition of P-closed sinks.

(2) Of course, the internal descriptions of the P-hulls are fairly complicated; it may be quite hard to work practically with them (or even with simplified versions, like the one given in [5]). For all practical reasons, if the P-hull of a specific construct is asked for, it is always preferable to perform the constructions of Theorem 1.2 in a construct **B** which can be defined in simpler terms (and which is a category, rather than a quasi-category): e.g., all P-hulls of the category of topological spaces can be formed inside the category of pseudotopological spaces, see [15].

(3) One of the theoretical merits of the given internal descriptions of the P-hulls is that they provide conditions for the existence of these hulls: the P-hull of **A** exists if and only if the quasi-category of P-closed sinks over **A** has small fibres. The existence conditions gained from this equivalence can be formulated in terms of **A**. (However, they are not always the best available ones: the fact that every construct with subspaces has an extensional topological hull was obtained by a different method in [**14**: 4.2].)

(4) Finally, the reader might ask: Why didn't we investigate extensional monotopological (or monotopological universe) hulls? The answer is that, with the definition of extensionality via injective, initial maps [**15**: 1.3], there is nothing new to investigate: if a monotopological construct is extensional, then it is already topological [**21**: Thm. 6].

References

1. Adámek, J., H. Herrlich and G. E. Strecker, Least and largest initial completions, Comment. Math. Univ. Carolinae **20** (1979), 43–77.

2. Adámek, J., H. Herrlich and G. E. Strecker, Abstract and Concrete Categories, Wiley, New York et al. 1990.

3. Adámek, J. and V. Koubek, Cartesian closed initial completions, Topology Appl. **11** (1980), 1–16.

4. Adámek, J., J. Reiterman, and F. Schwarz, On universally topological hulls and quasitopos hulls, Houston J. Math. (to appear).

5. Adámek, J., J. Reiterman, and G. E. Strecker, Realization of cartesian closed topological hulls, Manuscripta Math. **53** (1985), 1–33.

6. Adámek, J. and G. E. Strecker, Construction of cartesian closed topological hulls, Comment. Math. Univ. Carolinae **22** (1981), 235–254.

7. Adámek, J. and G. E. Strecker, Injectivity of topological categories, Algebra Universalis **26** (1989), 284–306.

8. Alderton, I. W., Cartesian closedness and the MacNeille completion of an initially structured category, Quaest. Math. **8** (1985), 63–78.

9. Antoine, P., Étude élémentaire des catégories d'ensembles structurés, Bull. Soc. Math. Belgique **18** (1966), 142–164, 387–414.

10. Hausdorff, F., Grundzüge der Mengenlehre, Veit & Comp., Leipzig 1914.

11. Herrlich, H., Initial completions, Math. Zeitschr. **150** (1976), 101–110.

12. Herrlich, H., Topological improvements of categories of structured sets, Topology Appl. **27** (1987), 145–155.

13. Herrlich, H., Hereditary topological constructs, in: Z. Frolík (ed.), General Topology and its Relations to Modern Analysis and Algebra VI (Proc. Sixth Prague Topological Symposium 1986), Heldermann, Berlin 1988, 249–262.

14. Herrlich, H., On the representability of partial morphisms in Top and in related constructs, in: F. Borceux (ed.), Categorical Algebra and its Applications (Proc. Louvain-La-Neuve 1987), Lecture Notes Math. **1348**, Springer, Berlin et al. 1988, 143–153.

15. Herrlich, H., E. Lowen-Colebunders, and F. Schwarz, Improving Top: PrTop and PsTop, these proceedings.

16. Herrlich, H. and L. D. Nel, Cartesian closed topological hulls, Proc. Amer. Math. Soc. **62** (1977), 215–222.

17. Nel, L. D., Initially structured categories and cartesian closedness, Canadian J. Math. **27** (1975), 1361–1377.

18. Schwarz, F., Powers and exponential objects in initially structured categories and applications to categories of limit spaces, Quaest. Math. **6** (1983), 227–254.

19. Schwarz, F., Hereditary topological categories and topological universes, Quaest. Math. **10** (1986), 197–216.

20. Schwarz, F., Description of the topological universe hull, in: H. Ehrig et al. (eds.), Categorical Methods in Computer Science with Aspects from Topology (Proc. Berlin 1988), Lecture Notes Computer Science **393**, Springer, Berlin et al. 1989, 325–332.

21. Schwarz, F., Representability of partial morphisms in topological and monotopological constructs, Quaest. Math. (to appear).

22. Weck-Schwarz, S., Cartesisch abgeschlossene monotopologische Hüllen und die Rolle der T_0-Reflexion, Dissertation, Universität Hannover 1988.

23. Wyler, O., Are there topoi in topology?, in: E. Binz and H. Herrlich (eds.), Categorical Topology (Proc. Mannheim 1975), Lecture Notes Math. **540**, Springer, Berlin et al. 1976, 699–719.