Category Theory at Work
H. Herrlich, H.-E. Porst (eds.)
© Heldermann Verlag Berlin 1991
21–34

Improving Top: PrTop and PsTop

H. Herrlich, E. Lowen-Colebunders, F. Schwarz

Introduction

Many of the categories investigated in general topology have the property that initial structures exist with respect to arbitrary families of maps, but lack other desirable properties, such as cartesian closedness or extensionality. A possible approach to improve the situation is to look at extensions which have (some of) these properties; of course the originally given category should be nicely contained in such extensions.

As an example of this method of *"improvement by enlargement"*, we will, in this paper, study the case of the category **Top** of topological spaces and continuous maps.

The presentation is self-contained, as far as **Top** and its extensions are concerned. (The only prerequisite is a basic knowledge of topology and filters.) The theoretical background material on categorical topology is formulated, but without proofs. To round out the picture, we mention—also without proofs—a few generalizations of the results on **Top** that apply, in particular, to the epireflective subcategories of **Top**. A general procedure for the construction of "smallest" extensions with desirable properties, that is applicable to any concrete category over **Set**, is explained in [29].

With respect to the literature, we have tried to refer to the original papers. A comprehensive presentation of categorical topology is contained in the recent book [2].

1. The category Top of topological spaces and continuous maps

The main categorical feature, which makes the category **Top** of topological spaces and continuous maps a nice category to work in, is the existence of initial structures: Given a family $\big((X_i, t_i) \mid i \in I\big)$ of topological spaces, indexed by a class I (possibly proper or empty), and a family $\big(f_i : X \longrightarrow X_i \mid i \in I\big)$ of maps, there is a unique topology t on X such that $\big(f_i : (X, t) \longrightarrow (X_i, t_i) \mid i \in I\big)$ is an initial source, i.e. such that for any topological space (Z, u) and any function $g : Z \longrightarrow X$, we have $g : (Z, u) \longrightarrow (X, t)$ is continuous if and only if $f_i \circ g : (Z, u) \longrightarrow (X_i, t_i)$ is continuous for every $i \in I$.

This fundamental property is the basis for the following definition:

1.1 Definition [5: 2.1]. Let **A** be a *construct* (i.e. **A** may be regarded as a category of "structured sets", whose morphisms are "structure-compatible" maps, together with the usual composition of maps). Then **A** is called *topological* if it satisfies

(1) For any set X, any family $\big((X_i, \alpha_i) \mid i \in I\big)$ of **A**-objects and any family $\big(f_i : X \longrightarrow X_i \mid i \in I\big)$ of maps, there is a unique **A**-object (X, α) such that $\big(f_i : (X, \alpha) \longrightarrow (X_i, \alpha_i) \mid i \in I\big)$ is an initial source in **A**;

and **A** is called *well-fibred* if it fulfils the conditions

(2) For any set X, the class $\{(Y, \alpha) \mid (Y, \alpha) \in \mathrm{Ob}\mathbf{A} \text{ and } Y = X\}$ of **A**-objects with underlying set X is a set.

(3) For any set X with cardinality at most one, there exists exactly one **A**-object with underlying set X.

This work is an original contribution and will not appear elsewhere.

Convention. In this paper, *we will only consider constructs that are well-fibred.* To keep the terminology brief, we will speak of "constructs" and "topological constructs" instead of "well-fibred constructs" and "well-fibred topological constructs".

In this language, **Top** is a topological construct. The initial topology t on X with respect to the structured source $\left(f_i : X \longrightarrow (X_i, t_i) \mid i \in I\right)$ is, in terms of open sets, described by having

$$\bigcup_{i \in I} \{\, f_i^{-1}(U) \mid U \text{ open in } (X_i, t_i) \,\}$$

as a subbase.

As a topological construct, **Top** has also final structures: for any structured sink $\left(f_i : (X_i, t_i) \longrightarrow X \mid i \in I\right)$, there exists a unique topology t on X such that $\left(f_i : (X_i, t_i) \longrightarrow (X, t) \mid i \in I\right)$ is a final sink in **Top**. In particular, **Top** has subspaces, products, quotients and coproducts. Subspaces and products are obtained by endowing the subobjects and products in **Set** with the corresponding initial topologies. (The subspaces are the extremal subobjects in **Top**.) Quotients and coproducts are obtained by taking the respective colimits in **Set**, endowed with the corresponding final topologies. Moreover, the topologies on a fixed set X form a complete lattice if one defines

$$t_1 \leq t_2 \iff 1_X : (X, t_1) \longrightarrow (X, t_2) \text{ is continuous.}$$

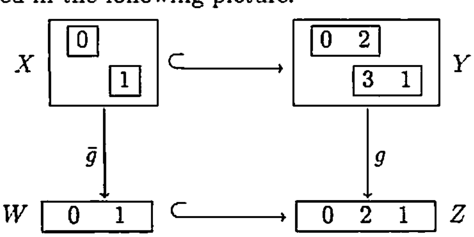
As we have seen, **Top** has many useful properties, which can analogously be formulated and proved in any topological construct. In the following we will discuss other desirable properties of topological constructs: *extensionality* and *cartesian closedness*. Roughly speaking, they characterize the compatibility between initial and final constructions. As we will see, **Top** does not possess either of these properties. How can we improve **Top**? A reasonable approach is to enlarge **Top**, i.e. to embed **Top** into bigger categories that have nicer properties, but are small enough to retain as much structure as possible.

Coproducts in **Top** commute with subspaces in the following sense: If $X = \coprod X_i$ is a coproduct in **Top**, Y is a subspace of X, and Y_i are the pre-images of Y under the natural injections, considered as subspaces of the corresponding X_i, then $Y = \coprod Y_i$. In other words, if for each index k we form the pullback

$$\begin{array}{ccc} Y_k & \hookrightarrow & X_k \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & \coprod X_i \end{array}$$

then $Y = \coprod Y_i$, in short: "Coproducts in **Top** are preserved by pullbacks along embeddings" or: "Coproducts in **Top** are hereditary". However, coproducts cannot be replaced by arbitrary final episinks in this statement. As the following example shows, quotients are not preserved by pullbacks along embeddings.

1.2 Example [14: Thm. 2]. Consider the topological spaces W, X, Y, Z whose non-empty open sets are illustrated in the following picture:



W and X are subspaces of Z and Y, respectively, the square is a pullback, and g is a quotient, but the restriction $\bar{g}$ of g to X is not a quotient.

The usefulness of the property that final episinks are hereditary has been pointed out in [19: Section 2]. A more constructive characterization was given in [16], in the following way.

1.3 Definition. Let **A** be a topological construct.

(1) A *partial morphism* from X to Y is a morphism $f : Z \longrightarrow Y$ whose domain Z is a subspace of X.

(2) *Partial morphisms to* Y *are representable* provided Y can be embedded via the addition of a single point ∞_Y into an object $Y^\sharp$ with the property that for every partial morphism $f : Z \longrightarrow Y$ from X to Y, the map $f^X : X \longrightarrow Y^\sharp$, defined by $f^X(x) = f(x)$ if $x \in Z$, $f^X(x) = \infty_Y$ if $x \in X - Z$, is a morphism. The object $Y^\sharp$ is called *one-point extension* of Y.

(3) **A** is *extensional* if partial morphisms to all **A**-objects are representable.

1.4 Theorem [16]. *For a topological construct* **A**, *the following conditions are equivalent:*

(1) **A** *is extensional.*

(2) *Quotients and coproducts in* **A** *are hereditary.*

(3) *Final episinks in* **A** *are hereditary.*

It follows from 1.2 and 1.4 that Top is not extensional. This can also be seen directly from the definition:

1.5 Example. Denote by 2 the *Sierpinski space*, i.e. the set $\{0,1\}$ endowed with the topology $\{\emptyset, \{1\}, \{0,1\}\}$. Assume the one-point extension $2^\sharp$ exists; it is customary to denote ∞ by 2 in this context. Choosing f to be the identity on 2 in 1.3(2), and letting X run through all topological spaces with underlying set $\{0,1,2\}$ which have 2 as subspace, it is seen that in all these cases, the map f^X is the identity on $\{0,1,2\}$. Consequently, $2^\sharp$ would have to carry the largest (i.e. coarsest) topology which makes 2 a subspace. But it is easy to verify that such a topology does not exist.

The considerations of 1.5 apply generally: Whenever **A** is an extensional topological construct, then the one-point extension of any **A**-object Y carries the largest structure that makes Y a subspace [25: 2.6]. Note, however, that this necessary condition for the extensionality of **A** is not sufficient [28: 3(2)], i.e. if for each Y, there exists an extension of Y by one point, carrying the largest structure for which Y is a subspace, then partial morphisms to Y need not be representable.

Any non-trivial topological construct inside Top contains all spaces where each open set is closed [14: Prop. 3(3)]. Applying this fact to 1.2 leads to the following result:

1.6 Generalization [14: Thm. 2]. The only topological constructs contained in Top that are extensional are the discrete and the indiscrete spaces.

2. The category PrTop of pretopological spaces

Since Top itself is not extensional, we weaken the axioms of a topological space in order to obtain a supercategory of Top that is extensional. This can be done by looking at topologies in terms of their convergent filters:

2.1 Definition. A *pretopology* on a set X is a function q assigning to each $x \in X$ a set of proper filters on X, the "filters converging to x", subject to the following conditions:

(1) for every $x \in X$: $\dot{x} \in q(x)$ (where $\dot{x}$ is the principal ultrafilter generated by $\{x\}$);

(2) if $\mathcal{F}$, $\mathcal{G}$ are filters on X with $\mathcal{G} \supset \mathcal{F} \in q(x)$, then $\mathcal{G} \in q(x)$;

(3) for every $x \in X$: $\bigcap q(x) \in q(x)$.

(X, q) is then called a *pretopological space*. It is customary (and more suggestive) to write $\mathcal{F} \longrightarrow x$ instead of $\mathcal{F} \in q(x)$; if necessary, the pretopology or the pretopological space is used as an index to avoid ambiguities.

A map $f : (X_1, q_1) \longrightarrow (X_2, q_2)$ between pretopological spaces is *continuous* if $\mathcal{F} \in q_1(x)$ implies $f(\mathcal{F}) \in q_2(f(x))$ (where $f(\mathcal{F})$ denotes the filter on X_2 generated by $\{ f(F) \mid F \in \mathcal{F} \}$).

The category of pretopological spaces and continuous maps is denoted by **PrTop**.

Like topologies, pretopologies can be characterized in a variety of ways; cf. [2: 5N(b)]. We mention here only one, which we will use in the sequel. Clearly, a pretopology q on a set X is completely determined by the coarsest filters converging to the points of X: For each $x \in X$, $\mathcal{V}(x) := \bigcap q(x)$ is a filter with $\mathcal{V}(x) \subset \dot{x}$; conversely, if a filter $\mathcal{V}(x)$ with $\mathcal{V}(x) \subset \dot{x}$ is assigned to each $x \in X$, then a pretopology q is defined by

$$\mathcal{F} \in q(x) \iff \mathcal{F} \supset \mathcal{V}(x).$$

In analogy to topologies, $\mathcal{V}(x)$ is called the *neighborhood filter* of x. A map $f : (X_1, q_1) \longrightarrow (X_2, q_2)$ is continuous if and only if $f(\mathcal{V}_1(x)) \supset \mathcal{V}_2(f(x))$ for every $x \in X_1$, or equivalently, if $\mathcal{V}_1(x) \supset f^{-1}(\mathcal{V}_2(f(x)))$ for every $x \in X_1$. (Here $f^{-1}(\mathcal{V}_2(f(x)))$ denotes the filter generated by $\{ f^{-1}(V) \mid V \in \mathcal{V}_2(f(x)) \}$; observe that all sets $f^{-1}(V)$ are non-empty.)

Obviously, **Top** can be regarded as a subcategory of **PrTop**.

2.2 Proposition. **PrTop** *is a topological construct.*

Proof. If $((X_i, q_i) \mid i \in I)$ is a family of pretopological spaces and $(f_i : X \longrightarrow X_i \mid i \in I)$ a family of maps, then the initial pretopology q on X can be described in the following ways:

(a) $\mathcal{F} \underset{q}{\longrightarrow} x \iff \forall i \in I\colon\ f_i(\mathcal{F}) \underset{q_i}{\longrightarrow} f_i(x)$.

(b) $\forall x \in X\colon\ \mathcal{V}_q(x) = \sup \{ f_i^{-1}(\mathcal{V}_i(f_i(x))) \mid i \in I \}$ (where the supremum is taken in the lattice of filters on X, ordered by inclusion). ∎

Since initial structures in **Top** can be described in the same way as those in **PrTop**, it follows that **Top** is closed under formation of initial sources in **PrTop**, and we have:

2.3 Proposition. **Top** *is a bireflective subcategory of* **PrTop**.

For a pretopological space (X, q), the bireflection is given by

$$1_X : (X, q) \longrightarrow (X, t)$$

where $\{ U \subset X \mid \forall x \in U\colon\ U \in \mathcal{V}_q(x) \}$ is the collection of open sets of t.

2.4 Theorem [16: p. 250]. **PrTop** *is extensional.*

Proof. We give the construction of the one-point extensions in **PrTop** and show that they fulfil the requirements of 1.3(2). For a pretopological space Y, define $Y^\sharp = Y \cup \{\infty\}$ with $\infty \notin Y$. Let $j : Y \longrightarrow Y^\sharp$ denote the inclusion map. The convergence structure of $Y^\sharp$ has to be as large as possible such that Y is a subspace of $Y^\sharp$; hence we define

$$\mathcal{G} \underset{Y^\sharp}{\longrightarrow} y \iff y = \infty \text{ or } \mathcal{G} = \dot{\infty} \text{ or } \mathcal{G}|_Y \underset{Y}{\longrightarrow} y,$$

i.e., all filters on $Y^\sharp$ converge to ∞, and $\dot{\infty}$ converges to all points of $Y^\sharp$, while in any other case, the convergence behavior of a filter $\mathcal{G}$ on $Y^\sharp$ is determined by the convergence behavior of its trace $\mathcal{G}|_Y$ on Y. (Note that $\mathcal{G}|_Y$ is a proper filter whenever $\mathcal{G} \neq \dot{\infty}$.) Observing that

$(\bigcap\{\mathcal{G}_i \mid i \in I\})|_Y = \bigcap\{\mathcal{G}_i|_Y \mid i \in I\}$, this definition makes $Y^\sharp$ a pretopological space. For any filter $\mathcal{F}$ on Y and any $y \in Y$, we have $j(\mathcal{F})|_Y = \mathcal{F}$, and consequently,

$$\mathcal{F} \xrightarrow[Y]{} y \iff j(\mathcal{F}) \xrightarrow[Y^\sharp]{} j(y),$$

which proves that $j : Y \longrightarrow Y^\sharp$ is initial. Finally, let $f : Z \longrightarrow Y$ be a partial morphism from X to Y. We have to show that $f^X : X \longrightarrow Y^\sharp$ is continuous. Let $\mathcal{F} \xrightarrow[X]{} x$. It is clear that $f^X(\mathcal{F}) \xrightarrow[Y^\sharp]{} f^X(x)$ if $x \in X - Z$ (i.e., $f^X(x) = \infty$) or if $X - Z \in \mathcal{F}$ (i.e., $f^X(\mathcal{F}) = \dot{\infty}$). In any other case, the trace $\mathcal{F}|_Z$ is a proper filter, and $\mathcal{F}|_Z \xrightarrow[Z]{} x$, since Z is a subspace of X. By the continuity of f, we obtain $f^X(\mathcal{F})|_Y = f(\mathcal{F}|_Z) \xrightarrow[Y]{} f(x) = f^X(x)$. ∎

We have seen that by a quite natural generalization of **Top**, we obtained the extensional topological supercategory **PrTop**. In fact, **PrTop** is the "smallest" extensional topological extension of **Top**. Before we can make this statement precise, we need the following definition:

2.5 Definition. A subclass **D** of a topological construct **B** is called *finally dense* (resp. *initially dense*) in **B**, if for each **B**-object X, there is a final sink $(f_i : D_i \longrightarrow X \mid i \in I)$ (resp. an initial source $(f_i : X \longrightarrow D_i \mid i \in I)$) with all D_i in **D**. If $E : \mathbf{A} \longrightarrow \mathbf{B}$ is an embedding of a construct **A** into **B**, then the *embedding* E and the *extension* **B** of **A** are said to be *finally dense* (resp. *initially dense*) if the class $\{E(X) \mid X \in \mathrm{Ob}\mathbf{A}\}$ is finally dense (resp. initially dense) in **B**. (We assume throughout this paper that subcategories are full and embeddings are full and concrete.)

Final density plays an important rôle: If we consider a finally dense subclass **D** of a topological construct **A** as a full subcategory of **A**, then the embedding functor preserves initial sources [18: Prop. 10]. Hence a finally dense extension of a category retains much of its original character.

2.6 Definition [15: 2.3]. An extensional topological construct **B** is called an *extensional topological hull* of a construct **A** if **B** is a finally dense extension of **A** with the property that any finally dense embedding of **A** into an extensional topological construct can be uniquely extended to **B**.

The extensional topological hull of a construct—if it exists—is unique up to isomorphism. (Surprisingly, the existence is already ensured if **A** has subspaces [17: 4.2].) It can be characterized in terms of one-point extensions:

2.7 Theorem [15: 2.3], [16: p. 258 ff]. *The extensional topological hull* **B** *of a construct* **A** *is characterized by the following properties:*

(1) **B** *is an extensional topological construct.*
(2) **A** *is finally dense in* **B**.
(3) $\{Y^\sharp \mid Y \in \mathrm{Ob}\mathbf{A}\}$ *is initially dense in* **B**.

A detailed proof of 2.7 can be found in [29: 1.3]. We will use this characterization theorem to show that **PrTop** is the extensional topological hull of **Top**.

2.8 Proposition (cf. [21: 1.2]). **Top** *is finally dense in* **PrTop**.

Proof. For any filter $\mathcal{F}$ on a set X and $a \in X$, we obtain a topology $t_{a,\mathcal{F}}$ on X by defining the neighborhood filters of the points of X as follows: $\mathcal{V}(a) = \mathcal{F} \cap \dot{a}$, $\mathcal{V}(x) = \dot{x}$ if $x \neq a$. Now if q is a pretopology on X, then the sink $\left(1_X : (X, t_{a,\mathcal{F}}) \longrightarrow (X, q) \mid a \in X,\ \mathcal{F} \xrightarrow[q]{} a\right)$ is a final episink. ∎

In terms of neighborhood filters, the Sierpinski space **2** is described by $\mathcal{V}(0) = \dot{0} \cap \dot{1}$, $\mathcal{V}(1) = \dot{1}$.

2.9 Definition. The one-point extension $2^\sharp$ of the Sierpinski space is denoted by 3. (Again we use 2 instead of ∞.) So 3 is the pretopological space described by $\mathcal{V}(0) = \mathcal{V}(2) = \dot{0} \cap \dot{1} \cap \dot{2}$, $\mathcal{V}(1) = \dot{1} \cap \dot{2}$.

2.10 Proposition [8: II.2]. *$\{3\}$ is initially dense in* **PrTop**.

Proof. Let X be a pretopological space. We have to show that $(f : X \longrightarrow 3 \mid f \text{ continuous})$ is an initial source, i.e. that for $a \in X$:

$$\mathcal{V}(a) = \sup \{ f^{-1}(\mathcal{V}(f(a))) \mid f : X \longrightarrow 3 \text{ continuous} \}.$$

The inclusion $\supset$ is clear. To prove the converse inclusion, define, for each $V \in \mathcal{V}(a)$, a map $f_{V,a} : X \longrightarrow 3$ by $f_{V,a}(x) = 0$ if $x \in X - V$, $f_{V,a}(a) = 1$, $f_{V,a}(x) = 2$ if $x \in V - \{a\}$. All $f_{V,a}$ are continuous, and $V = {f_{V,a}}^{-1}(\{1,2\}) \in {f_{V,a}}^{-1}(\mathcal{V}(1)) = {f_{V,a}}^{-1}(\mathcal{V}(f_{V,a}(a)))$. ∎

The preceding results are summed up in the following theorem:

2.11 Theorem [16: p. 259]. **PrTop** *is the extensional topological hull of* **Top**.

PrTop is also the extensional topological hull of T_0**Top**, the category of T_0-spaces. For many other well-known subcategories of **PrTop**, surprisingly, only one additional category occurs as their extensional topological hull: this is the case if a symmetry axiom is satisfied, as e.g. for the T_1-spaces (T_1**Top**), the completely regular spaces (**CReg**), the Tychonoff spaces (**Tych** = **CReg** $\cap$ T_1**Top**) and the zero-dimensional spaces (**ZDim**). See also the diagram at the end of this paper.

2.12 Generalization [27: Cor. of Thm. 1]. A pretopological space X is an *R_0-space* if for all points $x, y \in X$, $\dot{x} \longrightarrow y$ is equivalent to $\dot{y} \longrightarrow x$. The subcategory of **PrTop** consisting of the R_0-spaces is denoted by R_0**PrTop**. If **A** is an epireflective subcategory of **PrTop** that contains a non-indiscrete space, then the extensional topological hull of **A** is **PrTop** if $2 \in \text{Ob}\mathbf{A}$, and R_0**PrTop** if $2 \notin \text{Ob}\mathbf{A}$.

We now turn our attention to a second desirable property of (topological) categories: cartesian closedness.

2.13 Definition [11: p. 550]. A category **A** with finite products is *cartesian closed* if for each $X \in \text{Ob}\mathbf{A}$, the functor $- \times X : \mathbf{A} \longrightarrow \mathbf{A}$ has a right adjoint, denoted by $[X, -]$.

For topological constructs, cartesian closedness is characterized by the existence of *canonical function spaces*, i.e. $[X, Y]$ is given by the set $\text{Hom}(X, Y)$, endowed with a structure fulfilling condition (2) of the following theorem, and the counit of the adjunction is the usual evaluation map.

2.14 Theorem [13: p. 7 Thm.]. *For a topological construct* **A**, *the following are equivalent:*

(1) **A** *is cartesian closed.*

(2) *For each pair X, Y of* **A**-*objects, the set* $\text{Hom}(X, Y)$ *can be endowed with an* **A**-*structure α such that*

- (a) *the evaluation map* $\text{ev} : (\text{Hom}(X, Y), \alpha) \times X \longrightarrow Y$, $\text{ev}(f, x) = f(x)$, *is a morphism, and*
- (b) *for each* **A**-*object W and each morphism $h : W \times X \longrightarrow Y$, the map $h^* : W \longrightarrow (\text{Hom}(X, Y), \alpha)$ defined by $h^*(w)(x) = h(w, x)$ is a morphism.*

(3) *For every $X \in \text{Ob}\mathbf{A}$, the functor $- \times X$ preserves quotients and coproducts.*

(4) *For every $X \in \text{Ob}\mathbf{A}$, the functor $- \times X$ preserves final episinks.*

Note the analogy of this characterization theorem to 1.4.

The map h^* in 2.14(2)(b) is called the *transpose* of h. Observe that h^* is indeed a map from W to $\mathrm{Hom}(X,Y)$, since for each $w \in W$, the map $h^*(w)$ is basically the restriction of h to $\{w\} \times X$.

While coproducts in **Top** and **PrTop** are finitely productive, quotients are generally not preserved by the functors $- \times X$; a counterexample for the case **Top** is given in [12: 2.4.20]. Consequently, **Top** and **PrTop** are not cartesian closed. The following proof of this negative result uses condition (2) of 2.14 and is essentially due to Arens [6: Thm. 3].

2.15 Theorem. *Top and PrTop are not cartesian closed.*

Proof. Let $X = \mathbf{Q}$ (the rationals) and $Y = [0,1]$ (the unit interval) with their usual topologies. Suppose there is a (pre)topology q on $\mathrm{Hom}(X,Y)$ which fulfils 2.14(2). We will show that this assumption implies that $X = \mathbf{Q}$ is locally compact, a contradiction.— Let $a \in X$ be fixed. Denote by f the constant map $f : X \longrightarrow Y$ with value 0. Choose a neighborhood W of 0 in Y with $1 \notin W$. Since the evaluation map is continuous with respect to q, there exist $H \in \mathcal{V}_q(f)$ and $V \in \mathcal{V}(a)$ such that $\mathrm{ev}(H \times V) \subset W$. We can choose V to be closed, because X is regular. We will prove that V is compact. Let $\mathcal{C}$ be an open covering of V; we have to find a finite subcovering. Without restriction, we can assume that $X - V \in \mathcal{C}$, i.e., that $\mathcal{C}$ covers X. Put

$$\mathcal{A} = \{\, A \subset X \mid A \text{ closed and } A \subset C \text{ for some } C \in \mathcal{C} \,\}$$

and

$$\mathcal{S} = \{\, \langle A, U\rangle \mid A \in \mathcal{A} \text{ and } U \subset Y \text{ open } \},$$

where

$$\langle A, U\rangle = \{\, g \in \mathrm{Hom}(X,Y) \mid g(A) \subset U \,\}.$$

Since $\mathcal{S}$ covers $\mathrm{Hom}(X,Y) = \langle \emptyset, Y\rangle$, $\mathcal{S}$ is a subbase for a topology t on $\mathrm{Hom}(X,Y)$ (the so-called *$\mathcal{A}$-open topology*). By regularity of X, $\mathcal{V}(x) \cap \mathcal{A}$ is a neighborhood base of x for any $x \in X$. Consequently, the evaluation map is continuous with respect to t: Given $g \in \mathrm{Hom}(X,Y)$, $x \in X$ and an open neighborhood U of $g(x)$, there is a neighborhood $U' \in \mathcal{V}(x) \cap \mathcal{A}$ with $g(U') \subset U$ by continuity of g. It follows that $\langle U', U\rangle$ is a neighborhood of g with respect to t, and $\mathrm{ev}(\langle U', U\rangle \times U') \subset U$. Now the continuity of

$$\mathrm{ev} : (\mathrm{Hom}(X,Y), t) \times X \longrightarrow Y$$

implies the continuity of

$$\mathrm{ev}^* : (\mathrm{Hom}(X,Y), t) \longrightarrow (\mathrm{Hom}(X,Y), q).$$

Since ev^*, as a map, is the identity on $\mathrm{Hom}(X,Y)$, we have $t \leq q$. Consequently, $H \in \mathcal{V}_t(f)$, and there are $A_1, \ldots, A_n \in \mathcal{A}$ and open subsets $U_1, \ldots, U_n$ of Y with $f \in \langle A_1, U_1\rangle \cap \ldots \cap \langle A_n, U_n\rangle \subset H$. Since every A_i is contained in some $C_i \in \mathcal{C}$, it is now sufficient to show: $V \subset A_1 \cup \ldots \cup A_n$. Let $z \in V$. Assume $z \notin A_1 \cup \ldots \cup A_n$. Since $A_1 \cup \ldots \cup A_n$ is a closed subset of X and X is completely regular, there is a continuous map $h : X \longrightarrow [0,1] = Y$ with $h(z) = 1$ and $h(A_1 \cup \ldots \cup A_n) \subset \{0\}$. Then $h \in \langle A_1, U_1\rangle \cap \ldots \cap \langle A_n, U_n\rangle \subset H$, and $1 = h(z) = \mathrm{ev}(h, z) \in \mathrm{ev}(H \times V) \subset W$, a contradiction. ∎

2.16 Generalization [24: 3.3]. If **A** is an epireflective subcategory of **PrTop** containing a non-indiscrete space, then **A** fails to be cartesian closed.

3. The category PsTop of pseudotopological spaces

By weakening the axioms of convergence in a pretopological space, we will obtain a cartesian closed topological extension of **PrTop**.

3.1 Definition [10]. A *pseudotopology* on a set X is a function q assigning to each $x \in X$ a set of proper filters on X such that

(1) for every $x \in X$: $\dot{x} \in q(x)$;
(2) if $\mathcal{F}, \mathcal{G}$ are filters on X with $\mathcal{G} \supset \mathcal{F} \in q(x)$, then $\mathcal{G} \in q(x)$;
(3) $\mathcal{F} \in q(x)$ whenever $\mathcal{U} \in q(x)$ for every ultrafilter $\mathcal{U} \supset \mathcal{F}$.

Continuous maps between pseudotopological spaces are defined in the same way as for pretopological spaces. The category of pseudotopological spaces and continuous maps is denoted by **PsTop**.

Since every filter is the intersection of all finer ultrafilters, **PrTop** is a subcategory of **PsTop**.

3.2 Proposition. PsTop *is a topological construct.*

Proof. Initial structures can be described as in (a) of the proof of 2.2, i.e. the initial pseudotopology q on X with respect to $\big(f_i : X \longrightarrow (X_i, q_i) \,\big|\, i \in I\big)$ is given by:

$$\mathcal{F} \underset{q}{\longrightarrow} x \iff \forall i \in I\colon\ f_i(\mathcal{F}) \underset{q_i}{\longrightarrow} f_i(x).$$

(Use Lemma 3.5(1) to show that q fulfils condition (3) of Definition 3.1.) ∎

Observe that a pseudotopology is completely determined by the convergence behavior of the ultrafilters; conditions (2) and (3) of 3.1 can be summed up as: "$\mathcal{F} \in q(x)$ if and only if $\mathcal{U} \in q(x)$ for every ultrafilter $\mathcal{U} \supset \mathcal{F}$". As a consequence, the initial pseudotopology q in the proof of 3.2 is already determined if the equivalence $\mathcal{F} \underset{q}{\longrightarrow} x \iff \forall i \in I\colon f_i(\mathcal{F}) \underset{q_i}{\longrightarrow} f_i(x)$ holds for all *ultra*filters $\mathcal{F}$ on X.

Since **PrTop** is closed under formation of initial sources in **PsTop**, we have:

3.3 Proposition. PrTop *is a bireflective subcategory of* PsTop.

For a pseudotopological space (X, q), the bireflection is given by

$$1_X : (X, q) \longrightarrow (X, r),$$

where r is the pretopology defined by

$$\mathcal{F} \underset{r}{\longrightarrow} x \iff \mathcal{F} \supset \bigcap q(x).$$

We are now going to show that **PsTop** is cartesian closed. Given pseudotopological spaces X, Y, we have to find a pseudotopology q on $\mathrm{Hom}(X, Y)$ which fulfils (a) and (b) of 2.14(2). The greater q is (i.e. the more filters converge), the more likely is it that (b) is satisfied. In contrast, (a) is only fulfilled if q is small enough. Thus when looking for a structure fulfilling (a) and (b), we have to look for the greatest structure that satisfies (a), i.e. that makes the evaluation map continuous.

3.4 Definition [7: 1.1]. For pseudotopological spaces X, Y, the *structure of continuous convergence* c on $\mathrm{Hom}(X, Y)$ is defined by

$$\mathcal{H} \underset{c}{\longrightarrow} f \iff \forall \mathcal{F} \underset{X}{\longrightarrow} x\colon\ \mathrm{ev}(\mathcal{H} \times \mathcal{F}) \underset{Y}{\longrightarrow} f(x)$$

for $f \in \mathrm{Hom}(X, Y)$ and filters $\mathcal{H}$ on $\mathrm{Hom}(X, Y)$. (Here $\mathcal{H} \times \mathcal{F}$ is the filter generated by $\{H \times F \mid H \in \mathcal{H},\ F \in \mathcal{F}\}$.)

In order to show that the structure of continuous convergence is pseudotopological, the following observation about ultrafilters will be useful.

3.5 Lemma. (1) *Let $f : X \longrightarrow Y$ be a map. If $\mathcal{F}$ is a filter on X and $\mathcal{U}$ is an ultrafilter on Y with $f(\mathcal{F}) \subset \mathcal{U}$, then there exists an ultrafilter $\mathcal{V}$ on X with $\mathcal{F} \subset \mathcal{V}$ and $f(\mathcal{V}) \subset \mathcal{U}$.*

(2) *Let $g : X_1 \times X_2 \longrightarrow Y$ be a map. If $\mathcal{F}_1, \mathcal{F}_2$ are filters on X_1, X_2, respectively, and $\mathcal{U}$ is an ultrafilter on Y with $g(\mathcal{F}_1 \times \mathcal{F}_2) \subset \mathcal{U}$, then there exists an ultrafilter $\mathcal{W}$ on X_1 such that $\mathcal{F}_1 \subset \mathcal{W}$ and $g(\mathcal{W} \times \mathcal{F}_2) \subset \mathcal{U}$.*

Proof. (1) For all $F \in \mathcal{F}$, $U \in \mathcal{U}$, we have $f(F) \cap U \in \mathcal{U}$, and consequently, $f(F) \cap U \neq \emptyset$. Then $F \cap f^{-1}(U) \neq \emptyset$. It follows that $\mathcal{F} \sqcup f^{-1}(\mathcal{U})$ is a proper filter on X. Choose any ultrafilter $\mathcal{V}$ on X with $\mathcal{V} \supset \mathcal{F} \sqcup f^{-1}(\mathcal{U})$. Since $f(V) \cap U \neq \emptyset$ for any $V \in \mathcal{V}$, $U \in \mathcal{U}$, we obtain $f(\mathcal{V}) \subset \mathcal{U}$.

(2) By (1), there is an ultrafilter $\mathcal{V}$ on $X_1 \times X_2$ with $\mathcal{F}_1 \times \mathcal{F}_2 \subset \mathcal{V}$ and $g(\mathcal{V}) \subset \mathcal{U}$. Put $\mathcal{W} = p_1(\mathcal{V})$. Then $\mathcal{F}_1 \subset p_1(\mathcal{V}) = \mathcal{W}$, $\mathcal{F}_2 \subset p_2(\mathcal{V})$, and $g(\mathcal{W} \times \mathcal{F}_2) \subset g(p_1(\mathcal{V}) \times p_2(\mathcal{V})) \subset g(\mathcal{V}) \subset \mathcal{U}$. ∎

3.6 Proposition [21: 3.7]. *For pseudotopological spaces X, Y, the structure of continuous convergence on $\mathrm{Hom}(X, Y)$ is a pseudotopology.*

Proof. Let $f \in \mathrm{Hom}(X, Y)$.

(1) $\dot{f} \underset{c}{\longrightarrow} f$: Let $\mathcal{F} \underset{X}{\longrightarrow} x$. Since f is continuous, $\mathrm{ev}(\dot{f} \times \mathcal{F}) = f(\mathcal{F}) \underset{Y}{\longrightarrow} f(x)$.

(2) Let $\mathcal{H} \underset{c}{\longrightarrow} f$ and $\mathcal{H}' \supset \mathcal{H}$. Then for any $\mathcal{F} \underset{X}{\longrightarrow} x$, $\mathrm{ev}(\mathcal{H}' \times \mathcal{F}) \supset \mathrm{ev}(\mathcal{H} \times \mathcal{F}) \underset{Y}{\longrightarrow} f(x)$, so that $\mathrm{ev}(\mathcal{H}' \times \mathcal{F}) \underset{Y}{\longrightarrow} f(x)$.

(3) Let $\mathcal{H}$ be a filter on $\mathrm{Hom}(X, Y)$ such that every ultrafilter finer than $\mathcal{H}$ converges to f. We have to prove that $\mathcal{H} \underset{c}{\longrightarrow} f$. Let $\mathcal{F} \underset{X}{\longrightarrow} x$. We have to show that $\mathrm{ev}(\mathcal{H} \times \mathcal{F}) \underset{Y}{\longrightarrow} f(x)$, i.e. that every ultrafilter $\mathcal{U}$ finer than $\mathrm{ev}(\mathcal{H} \times \mathcal{F})$ converges to $f(x)$. By Lemma 3.5(2), there is an ultrafilter $\mathcal{W}$ on $\mathrm{Hom}(X, Y)$ with $\mathcal{W} \supset \mathcal{H}$ and $\mathrm{ev}(\mathcal{W} \times \mathcal{F}) \subset \mathcal{U}$. By assumption, $\mathcal{W} \underset{c}{\longrightarrow} f$ and $\mathcal{F} \underset{X}{\longrightarrow} x$, so that $\mathrm{ev}(\mathcal{W} \times \mathcal{F}) \underset{Y}{\longrightarrow} f(x)$ by definition of c. It follows that $\mathcal{U} \underset{Y}{\longrightarrow} f(x)$. ∎

3.7 Theorem [22: 3.2.10]. *The category* **PsTop** *is cartesian closed.*

Proof. We show that for pseudotopological spaces X, Y, the structure of continuous convergence on $\mathrm{Hom}(X, Y)$ fulfils condition 2.14(2). In view of 3.4 and 3.6, it remains to be shown that for $W \in$ **PsTop** and $h : W \times X \longrightarrow Y$ continuous, the map $h^* : W \longrightarrow (\mathrm{Hom}(X, Y), c)$ is continuous. Let $\mathcal{G} \underset{W}{\longrightarrow} w$. In order to prove $h^*(\mathcal{G}) \underset{c}{\longrightarrow} h^*(w)$, we have to show that $\mathrm{ev}(h^*(\mathcal{G}) \times \mathcal{F}) \underset{Y}{\longrightarrow} \mathrm{ev}(h^*(w), x) = h(w, x)$ whenever $\mathcal{F} \underset{X}{\longrightarrow} x$. Since $\mathrm{ev}(h^*(\mathcal{G}) \times \mathcal{F}) = h(\mathcal{G} \times \mathcal{F})$, this follows immediately from the continuity of h. ∎

We have found that **PsTop** is a cartesian closed topological extension of **Top** and **PrTop**. Is it the smallest possible? The answer will turn out to be "yes" in case of **PrTop**.

3.8 Definition [18]. A cartesian closed topological construct **B** is called a *cartesian closed topological hull* of a construct **A** if **B** is a finally dense extension of **A** with the property that any finally dense embedding of **A** into a cartesian closed topological construct can be uniquely extended to **B**.

As in the case of the extensional topological hull, the cartesian closed topological hull is unique up to isomorphism (but it does not always exist [3]). It can be characterized in terms of function spaces:

3.9 Theorem [18]. *The cartesian closed topological hull* **B** *of a construct* **A** *is characterized by the following properties:*

(1) **B** *is a cartesian closed topological construct.*

(2) **A** *is finally dense in* **B**.

(3) $\{ [X, Y] \mid X, Y \in \mathrm{Ob}\mathbf{A} \}$ *is initially dense in* **B**.

Replacing pretopologies by pseudotopologies in the proof of 2.8, we obtain:

3.10 Proposition. Top *and* PrTop *are finally dense in* PsTop.

In order to prove that PsTop is the cartesian closed topological hull of PrTop, it now remains to be shown that $\{ [X,Y] \mid X,Y \in \text{ObPrTop} \}$ is initially dense in PsTop. By comparison with the proof of the initial density of $\{ Y^{\natural} \mid Y \in \text{ObTop} \}$ in PrTop, where it was sufficient to consider the class $\{2^{\natural}\}$ (2.10), one might hope to find a single object of the form $[X,Y]$ with $X,Y \in \text{ObPrTop}$ which constitutes an initially dense subclass of PsTop. However, by [20: Cor.], PsTop does not possess an initially dense singleton subclass. Hence a more sophisticated approach is needed to prove condition (3) of 3.9. The following fact, which was observed by Bourdaud [8: II.4.3], will be very helpful; the proof given here is considerably simpler than his original proof.

3.11 Proposition. *For every pseudotopological space X, a continuous, initial map $j : X \longrightarrow [[X,3],3]$ is given by defining $j(x)(f) = f(x)$ for $x \in X$, $f \in \text{Hom}(X,3)$.*

Proof. The composition of the obvious isomorphism $X \times [X,3] \longrightarrow [X,3] \times X$ with the evaluation map is a continuous map. Consequently, its transpose, which coincides with j, is also a continuous map.

In order to show that j is initial, we have to prove that an ultrafilter $\mathcal{U}$ on X converges to $a \in X$ whenever $j(\mathcal{U}) \underset{c}{\longrightarrow} j(a)$. We use contraposition: Let $\mathcal{U}$ be an ultrafilter on X with $\mathcal{U} \not\longrightarrow a$. We have to find a filter $\mathcal{H}$ on $\text{Hom}(X,3)$ and $f \in \text{Hom}(X,3)$ such that $\mathcal{H} \underset{c}{\longrightarrow} f$ but $\text{ev}(j(\mathcal{U}) \times \mathcal{H}) \not\longrightarrow f(a)$; then clearly $j(\mathcal{U}) \underset{c}{\not\longrightarrow} j(a)$ by definition of continuous convergence on $\text{Hom}([X,3],3)$.— For any set $A \subset X$, the set $\langle X-A, \{2\}\rangle = \{ g \in \text{Hom}(X,3) \mid g(X-A) \subset \{2\} \}$ is non-empty; moreover, $\langle X-A, \{2\}\rangle \cap \langle X-B, \{2\}\rangle = \langle X-(A\cap B), \{2\}\rangle$ for $A, B \subset X$. Hence $\{ \langle X-U, \{2\}\rangle \mid U \in \mathcal{U} \}$ generates a filter $\mathcal{H}$ on $\text{Hom}(X,3)$. Define $f : X \longrightarrow 3$ by $f(a) = 1$, $f(x) = 2$ for $x \neq a$. Since $f(X) = \{1,2\}$, we have $f(\mathcal{F}) \supset \dot{1} \cap \dot{2}$ for all filters $\mathcal{F}$ on X, so that f is continuous. We will show that $\mathcal{H}$ and f satisfy the above conditions.

$\mathcal{H} \underset{c}{\longrightarrow} f$: It suffices to show that $\mathcal{F} \underset{X}{\longrightarrow} a$ implies $\text{ev}(\mathcal{H} \times \mathcal{F}) \longrightarrow f(a) = 1$. Since $\mathcal{U} \underset{X}{\not\longrightarrow} a$, we know $\mathcal{U} \not\supset \mathcal{F}$, and there exists a set $F \in \mathcal{F}$ with $F \notin \mathcal{U}$. Then $X - F \in \mathcal{U}$, because $\mathcal{U}$ is an ultrafilter. It follows that $\{2\} = \text{ev}(\langle F, \{2\}\rangle \times F) \in \text{ev}(\mathcal{H} \times \mathcal{F})$, hence $\text{ev}(\mathcal{H} \times \mathcal{F}) = \dot{2} \longrightarrow 1$.

$\text{ev}(j(\mathcal{U}) \times \mathcal{H}) \not\longrightarrow f(a)$: It suffices to show that $\text{ev}(j(\mathcal{U}) \times \mathcal{H}) \subset \dot{0}$, i.e. that $0 \in \text{ev}(j(U_1) \times \langle X - U_2, \{2\}\rangle)$ for all $U_1, U_2 \in \mathcal{U}$. Let $U_1, U_2 \in \mathcal{U}$. Put $U = U_1 \cap U_2$. Then $U \in \mathcal{U}$, $U \subset U_1$, and $\langle X-U, \{2\}\rangle \subset \langle X-U_2, \{2\}\rangle$. Define $g : X \longrightarrow 3$ by $g(x) = 0$ if $x \in U$, $g(x) = 2$ if $x \in X-U$. Since all filters on 3 converge to 0 and 2, the map g is continuous. Clearly, $g \in \langle X-U, \{2\}\rangle$. Choose some $x_0 \in U$. Then $0 = g(x_0) = \text{ev}(j(x_0), g) \in \text{ev}(j(U) \times \langle X-U, \{2\}\rangle) \subset \text{ev}(j(U_1) \times \langle X - U_2, \{2\}\rangle)$. ∎

3.12 Theorem [9: 3.2]. PsTop *is the cartesian closed topological hull of* PrTop.

Proof. We have to show that function spaces of PrTop-objects are initially dense in PsTop. Let X be a pseudotopological space. Since PrTop is finally dense in PsTop, there is a final episink $\big(f_i : Y_i \longrightarrow [X,3] \mid i \in I\big)$ where all Y_i are pretopological spaces. Application of the functor $[-,3]$ transforms this final episink into an initial source $\big((f_i,1) : [[X,3],3] \longrightarrow [Y_i,3] \mid i \in I\big)$ [18: Lemma 6]. By Proposition 3.11, the source $j : X \longrightarrow [[X,3],3]$ is also initial. Composition of these initial sources provides an initial source as required by 3.9(3). ∎

The cartesian closed topological hull of Top is *not* given by PsTop, but by the subcategory of *Antoine spaces* [9: 3.3]. However, if we are looking for topological extensions of Top that are both cartesian closed and extensional, PsTop will again be the "best" choice.

With the same construction as for PrTop (2.4), we obtain:

3.13 Proposition. PsTop *is extensional.*

Proof. Using the notation of 2.4, it is sufficient to show that $Y^{\natural}$ is pseudotopological. Let $\mathcal{G}$ be a filter on $Y^{\natural}$ such that $\mathcal{U} \underset{Y^{\natural}}{\longrightarrow} y$ for every ultrafilter $\mathcal{U} \supset \mathcal{G}$. If $y = \infty$ or $\mathcal{G} = \dot{\infty}$, we have $\mathcal{G} \underset{Y^{\natural}}{\longrightarrow} y$. Otherwise, $y \in Y$, and the trace $\mathcal{G}|_Y$ is a proper filter on Y. Since $j(\mathcal{G}|_Y) \supset \mathcal{G}$, every ultrafilter $\mathcal{V}$ on Y with $\mathcal{V} \supset \mathcal{G}|_Y$ can be represented as the trace $\mathcal{U}|_Y$ of some ultrafilter $\mathcal{U} \supset \mathcal{G}$ with $\mathcal{U} \neq \dot{\infty}$ (namely, $\mathcal{U} = j(\mathcal{V})$); from $\mathcal{U} \underset{Y^{\natural}}{\longrightarrow} y$, it follows that $\mathcal{V} = \mathcal{U}|_Y \underset{Y}{\longrightarrow} y$. Consequently, $\mathcal{G}|_Y \underset{Y}{\longrightarrow} y$, and we obtain $\mathcal{G} \underset{Y^{\natural}}{\longrightarrow} y$. ∎

Categories which are cartesian closed as well as extensional are extremely nice to work in (Wyler called them "ultra-convenient categories for topologists" [30: p. 699]).

3.14 Definition [23]. A topological construct is called a *topological universe* if it is both cartesian closed and extensional.

One of the reasons for the importance of topological universes is the following characterization theorem:

3.15 Theorem [15: 1.4]. *A topological construct is a topological universe if and only if final episinks are preserved by pullbacks along arbitrary morphisms, i.e. whenever* $(f_i : Y_i \longrightarrow Y \mid i \in I)$ *is a final episink,* $f : X \longrightarrow Y$ *is a morphism, and for each* $i \in I$, *the diagram*

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

is a pullback, then $(g_i : X_i \longrightarrow X \mid i \in I)$ *is a final episink.*

By 3.7 and 3.13, we have:

3.16 Theorem [30: 4.9]. PsTop *is a topological universe.*

Moreover, PsTop is the "smallest" topological universe extension of both Top and PrTop.

3.17 Definition (cf. [1: p. 317]). A topological universe B is called *topological universe hull* of a construct A if B is a finally dense extension of A such that any finally dense embedding of A into a topological universe can be uniquely extended to B.

Uniqueness (up to isomorphism) of the topological universe hull follows immediately from the definition, while the existence is not ensured (not even if A is a cartesian closed topological construct [4: Example 1]). In analogy to the extensional and cartesian closed topological hulls, we have the following characterization theorem (a proof of which can be found in [29: 1.3]):

3.18 Theorem [26: 3.6], [4: Remark 1(b)]. *The topological universe hull* B *of a construct* A *is characterized by the following properties:*

(1) B *is a topological universe.*
(2) A *is finally dense in* B.
(3) $\{ [X, Y^{\natural}] \mid X, Y \in \mathrm{Ob}\,\mathrm{A} \}$ *is initially dense in* B.

In condition (3) of 3.9, the description of the cartesian closed topological hull, Y can be restricted to an initially dense subclass of **A**, and X to a finally dense subclass of **B** contained in **A**; i.e. 3.9(3) can be replaced by the condition

(3′) $\{ [X,Y] \mid X \in \mathbf{C},\ Y \in \mathbf{D} \}$ is initially dense in **B**,

where **D** is initially dense in **A** and $\mathbf{C} \subset \mathbf{A}$ is finally dense in **B**. Applying 3.18, we obtain:

3.19 Theorem [26: 3.5]. *If the topological universe hull of a construct* **A** *exists, then it can be described as the cartesian closed topological hull of the extensional topological hull of* **A**.

Notice, however, that the topological universe hull of **A** generally *cannot* be obtained as the extensional topological hull of the cartesian closed topological hull of **A**; a counterexample is already given by **A** = **Top** [26: 2.1].

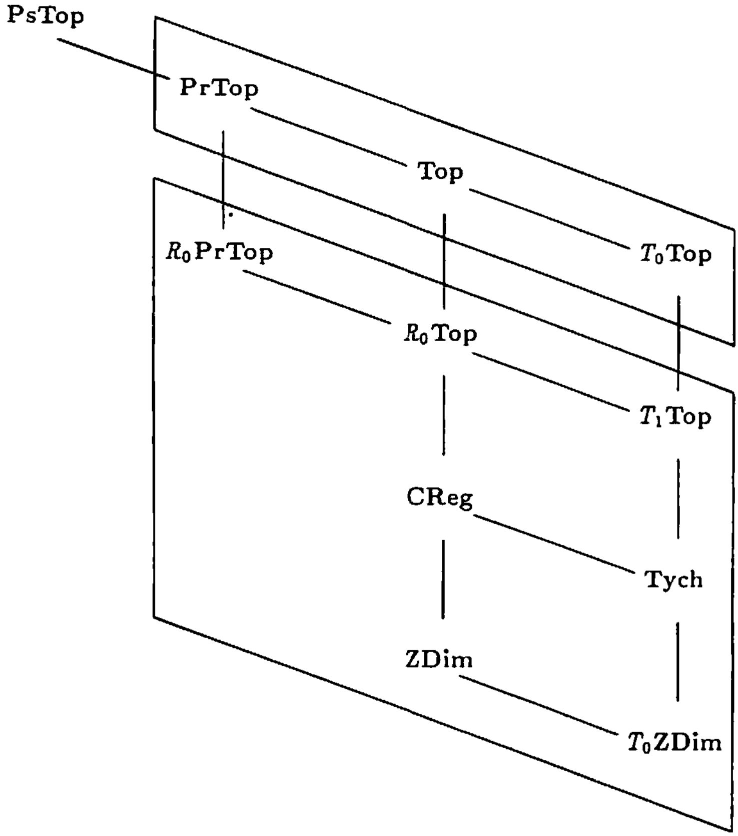
By 3.19, we obtain as an immediate consequence of 2.11 and 3.12:

3.20 Theorem [30: 4.9]. **PsTop** *is the topological universe hull of the constructs* **Top** *and* **PrTop**.

In 2.12, we mentioned that formation of the extensional topological hull of epireflective subcategories of **PrTop** produces only two distinct categories (beside the trivial one consisting of indiscrete spaces). In case of the topological universe hull, the situation is even more restrictive:

3.21 Generalization [27: Remark 3(1)]. For every epireflective subcategory **A** of **PrTop** containing a non-indiscrete space, the topological universe hull of **A** is given by **PsTop**.

Some applications of 2.12, 2.16 and 3.21 are depicted in the following diagram. Except for **PsTop**, none of the categories shown is cartesian closed, and **PsTop** is their topological universe hull. The extensional topological hull of the categories contained in either of the two boxes is the uppermost category in that box.



References

1. Adámek, J., Classification of concrete categories, Houston J. Math. 12 (1986), 305–326.
2. Adámek, J., H. Herrlich and G. E. Strecker, Abstract and Concrete Categories, Wiley, New York et al. 1990.
3. Adámek, J. and V. Koubek, Cartesian closed initial completions, Topology Appl. 11 (1980), 1–16.
4. Adámek, J., J. Reiterman, and F. Schwarz, On universally topological hulls and quasitopos hulls, Houston J. Math. (to appear).
5. Antoine,P., Étude élémentaire des catégories d'ensembles structurés, Bull. Soc. Math. Belgique 18 (1966), 142–164, 387–414.
6. Arens, R. F., A topology for spaces of transformations Ann. Math. **47** (1946), 480–495.
7. Binz, E. and H. H. Keller, Funktionenräume in der Kategorie der Limesräume, Ann. Acad. Sci. Fenn., Ser. A. I., **383** (1966), 1–21.
8. Bourdaud, G., Espaces d'Antoine et semi-espaces d'Antoine, Cahiers Topol. Géom. Diff. **16** (1975), 107–133.
9. Bourdaud, G., Some cartesian closed topological categories of convergence spaces, in: E. Binz and H. Herrlich (eds.), Categorical Topology (Proc. Mannheim 1975), Lecture Notes Math. **540**, Springer, Berlin et al. 1976, 93–108.

10. Choquet, G., Ann. Univ. Grenoble Sect. Sci. Math. Phys. (N.S.) **23** (1948), 57–112.
11. Eilenberg, S. and G. M. Kelly, Closed categories, in: S. Eilenberg et al. (eds.), Proceedings of the Conference on Categorical Algebra (La Jolla 1965), Springer, Berlin et al. 1966, 421–562.
12. Engelking, R., General Topology, Heldermann, Berlin 1989.
13. Herrlich, H., Cartesian closed topological categories, Math. Colloq. Univ. Cape Town **9** (1974), 1–16.
14. Herrlich, H., Are there convenient subcategories of Top?, Topology Appl. **15** (1983), 263–271.
15. Herrlich, H., Topological improvements of categories of structured sets, Topology Appl. **27** (1987), 145–155.
16. Herrlich, H., Hereditary topological constructs, in: Z. Frolík (ed.), General Topology and its Relations to Modern Analysis and Algebra VI (Proc. Sixth Prague Topological Symposium 1986), Heldermann, Berlin 1988, 249–262.
17. Herrlich, H., On the representability of partial morphisms in Top and in related constructs, in: F. Borceux (ed.), Categorical Algebra and its Applications (Proc. Louvain-La-Neuve 1987), Lecture Notes Math. **1348**, Springer, Berlin et al. 1988, 143–153.
18. Herrlich, H. and L. D. Nel, Cartesian closed topological hulls, Proc. Amer. Math. Soc. **62** (1977), 215–222.
19. Herrlich, H., G. Salicrup, and R. Vázquez, Light factorization structures, Quaest. Math. **3** (1979), 189–213.
20. Lowen, E. and R. Lowen, On the nonsimplicity of some convergence categories, Proc. Amer. Math. Soc. **105** (1989), 305–308.
21. Machado, A., Espaces d'Antoine et pseudo-topologies, Cahiers Topol. Géom. Diff. **14** (1973), 309–327.
22. Nel, L. D., Initially structured categories and cartesian closedness, Canadian J. Math. **27** (1975), 1361–1377.
23. Nel, L. D., Topological universes and smooth Gelfand-Naimark duality, Contemporary Math. **30** (1984), 244–276.
24. Schwarz, F., Cartesian closedness, exponentiality, and final hulls in pseudotopological spaces, Quaest. Math. **5** (1982), 289–304.
25. Schwarz, F., Hereditary topological categories and topological universes. Quaest. Math. **10** (1986), 197–216.
26. Schwarz, F., Description of the topological universe hull, in: H. Ehrig et al. (eds.), Categorical Methods in Computer Science with Aspects from Topology (Proc. Berlin 1988), Lecture Notes Computer Science **393**, Springer, Berlin et al. 1989, 325–332.
27. Schwarz, F., Extensionable topological hulls and topological universe hulls inside the category of pseudotopological spaces, Comment. Math. Univ. Carolinae **31** (1990), 123–127.
28. Schwarz, F., Representability of partial morphisms in topological and monotopological constructs, Quaest. Math. (to appear).
29. Schwarz, F. and S. Weck-Schwarz, Internal description of hulls: a unifying approach, these proceedings.
30. Wyler, O., Are there topoi in topology?, in: E. Binz and H. Herrlich (eds.), Categorical Topology (Proc. Mannheim 1975), Lecture Notes Math. **540**, Springer, Berlin et al. 1976, 699–719.