Category Theory at Work
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Improving Constructions in Topology

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Abstract: Many desirable topological statements, unfortunately, fail to be true. However, several of these concerning the stability of topological properties under topological constructions, can be made true provided the familiar constructions of subspaces, products, and quotients are replaced by more appropriate constructions. Whereas the familiar constructions are the "right" ones in the category Top of topological spaces and continuous maps, the new constructions are the "right" ones in various suitable supercategories of Top.

§1 Introduction

Generally topologists are concerned with the study of topological spaces and their relations to each other by means of continuous maps, in other words: with the study of the construct Top. Theorems in Top almost invariably involve

(I) properties of spaces or maps (e.g., compactness, Hausdorffness, connectedness, being an open map, a perfect map, etc.)

(II) categorically defined constructions (e.g., products, subspaces, etc.)

A prototypical theorem in Top is the famous Tychonoff theorem: products of compact spaces are compact. Such theorems are fundamental and have far reaching consequences for the development and usefulness of a theory. Unfortunately—many desirable theorems fail to hold; the categorically defined constructions all too often destroy useful properties of spaces or maps. In order to nevertheless obtain useful results, topologists have often forced (I) in the sense of adding supplementary and extraneous conditions or even changing the definition of a property altogether, and left (II) well alone, i.e., essentially continued working in Top.

Our aim in this paper is to provide evidence that doing precisely the opposite, i.e., leaving concepts as they are but stepping outside Top and thereby changing constructions in an appropriate way will illuminate the situation and provide a natural setting or solution for problems for which no decent solution in Top or in any reasonable subcategory of Top seems to exist. For each of the three important constructions, subspaces, products, and quotients, we provide examples of statements which are untrue in Top but which become true when the same constructions are performed in a suitable supercategory of Top.

§2 Topological Instability

For a pleasant structure theory, it is desirable that

(a) many of the especially important and useful properties of objects should remain stable under several standard constructions,

(b) certain types of standard constructions should commute.

This work is an original contribution and will not appear elsewhere.

In topology this is frequently not the case as the following examples show. All of the following "theorems" are false:

(A) **Formation of subspaces**

(1) Every subspace of a zero-dimensional[1] space is zero-dimensional.

(2) Every subspace of a paracompact space is paracompact.

(3) Every subspace of a compact space is compact.

(4) Every subspace of a Lindelöf space is Lindelöf.

(5) Every dense subspace of a connected space is connected.

(B) **Formation of products**

(6) Products of zero-dimensional[1] spaces are zero-dimensional.

(7) Products of paracompact spaces are paracompact.

(8) Products of Lindelöf spaces are Lindelöf.

(C) **Formation of quotients**

(9) Subspace-restrictions of quotient maps are quotient maps.

(10) Products of quotient maps are quotient maps.

1 EXAMPLE [(1)–(4) are false.]

Let $X = \omega^* \times \Omega^*$ where ω^* (respectively Ω^*) is the one point compactification of the space ω (respectively Ω) of all finite (respectively countable) ordinals. Then clearly X is a zero-dimensional[1] compact Hausdorff space. The subspace $Y = X \setminus \{(\omega, \Omega)\}$ however is not normal (the disjoint closed sets $\omega \times \{\Omega\}$ and $\{\omega\} \times \Omega$ have no disjoint neighborhoods), hence not zero-dimensional and not paracompact. Y is also clearly not Lindelöf ($\{\omega \times \Omega^*\} \cup \left\{ \omega^* \times [0, \alpha[\;\middle|\; \alpha < \Omega \right\}$ has no countable subcover) and thus also not compact. □

2 EXAMPLE [(5) is false.]

Let C be the Cantor set and let $p = (\frac{1}{2}, \frac{1}{2})$. We denote by X the cone over C with vertex p, but such that line segments connecting end points of deleted intervals with p contain only points with second coordinate rational and line segments connecting other points of C with p contain only points with second coordinate irrational. Then X is connected but $X \setminus \{p\}$ is totally disconnected [KK21]. □

Note that the foregoing examples prove that (1)–(5) are false in a very drastic way. In each case a property is destroyed by the removal of a single point.

3 EXAMPLE [(6)–(8) are false.]

Let X denote the Sorgenfrey line, i.e., $\mathbb{R}$ equipped with the topology generated by all half-open intervals $[a, b[$. This space is paracompact, Lindelöf, and zero-dimensional. However $X \times X$ has none of these properties. That $X \times X$ is not Lindelöf follows, e.g., from the fact that the set

$$A = \left\{ (x, y) \;\middle|\; x + y = 0 \right\}$$

[1] In the sense of the Lebesgue covering dimension.

is closed but discrete in its subspace topology. That $X \times X$ is not normal follows, e.g., from the fact that $A \cap (\mathbb{Q} \times \mathbb{Q})$ and $A \setminus (\mathbb{Q} \times \mathbb{Q})$ are closed disjoint sets which cannot be separated by open sets. Hence $X \times X$ is also not zero-dimensional and not paracompact. □

That (9) is false follows from an elegantly simple example, which is included in the paper "Improving Top: PrTop and PsTop" by Herrlich, Lowen-Colebunders, Schwarz in this volume.

4 EXAMPLE [(10) is false.]

Let $X = \mathbb{R}$, let Y be its quotient obtained by identifying all points in $\mathbb{Z}$, and let $\phi : X \to Y$ be the quotient map. Then consider the surjective map

$$\phi \times id_{\mathbb{Q}} : X \times \mathbb{Q} \to Y \times \mathbb{Q}.$$

If $\phi \times id_{\mathbb{Q}}$ were a quotient map it would map closed saturated sets to closed sets. We construct a closed and saturated set A as follows. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of irrational numbers converging to 0. For each n let $(r_{n,m})_{m=1}^{\infty}$ be a sequence of rational numbers converging to α_n. Then put

$$A = \left\{ (n + \frac{1}{m}, r_{n,m}) \;\middle|\; n, m \in \mathbb{N} \setminus \{0\} \text{ and } m > 1 \right\}.$$

A is closed and saturated in $\mathbb{R} \times \mathbb{Q}$ but $A = \phi \times id_{\mathbb{Q}}[A]$ is not closed in $Y \times \mathbb{Q}$ since $(\phi(0), 0) \notin A$ but each neighborhood of $(\phi(0), 0)$ in $Y \times \mathbb{Q}$ intersects A. □

The false theorems above are actually of two kinds. Those under (A) and (B) are specific for Top. The concepts used in them are not necessarily meaningful in constructs other than Top. Those under (C) are formulated in terms of purely categorical concepts and the question of their validity can be posed in any "sufficiently nice", for example topological, construct [AHS90]. We will show that all of the statements can be turned into true ones by appropriately "extending" or "enlarging" Top. For the statements under (A) and (B) this is achieved by embedding Top[1] in the construct Near of nearness spaces as introduced by Herrlich [H74a], and for (C) this is achieved by embedding Top in the constructs PrTop of pretopological spaces and PsTop of pseudotopological spaces, as introduced essentially by Choquet [Ch48]. More details on the above examples with respect to (1)–(8) can be found, e.g., in [H76]. Example 4 is taken from [H86; 4.4.20]. There are several other equally striking false statements which can be "corrected" by the theory displayed in §3–§5. For more information on this we refer the interested reader to [BHR76], [H74b], [H76], and [H77]. For results and terminology of a categorical nature we refer the interested reader to [AHS90].

§3 Characterizing Topological Spaces by Uniform Covers

An analysis of the above examples (1)–(8) shows that properties which are unstable under the formation of subspaces or products are frequently defined by means of open (equivalently, interior) covers. A cover of a topological space is called an interior cover provided it remains a cover even if we replace all sets in it by their interiors (equivalently, if it is refined[2] by some open cover). Therefore our

[1] More precisely, we embed not Top but rather its rather large subcategory Top_S of all symmetric topological spaces.

[2] A cover $\mathcal{U}$ of a set X is said to refine a cover $\mathcal{V}$ provided every member of $\mathcal{U}$ is a subset of some member of $\mathcal{V}$.

first step should be to try to axiomatize the concept of interior cover in such a way that an alternative definition of topological spaces results.

We have to make a very mild restriction on the type of topological spaces we consider: they should be symmetric, i.e., should satisfy the axiom of Šanin [Š43]:

$$x \in \mathrm{cl}\{y\} \iff y \in \mathrm{cl}\{x\}.$$

Symmetric spaces have also been called R_0 or essentially T_1.

As is usual in mathematics and if no confusion can result we denote the pair consisting of a set and some structure on it simply by the underlying set, e.g., we speak of the topological space X rather than (X, τ). In a topological space, int and cl refer to interior and closure as usual.

5 PROPOSITION

In a symmetric topological space X, the following are equivalent:

(1) $x \in \mathrm{int}_X A$.

(2) $\{X \setminus \{x\}, A\}$ *is an interior cover of* X.

Proof: (1)$\Longrightarrow$(2): If $x \in \mathrm{int}_X A$ then since X is symmetric it follows that $\mathrm{cl}_X\{x\} \subset \mathrm{int}_X A$. Hence, the collection $\{X \setminus \mathrm{cl}_X\{x\}, \mathrm{int}_X A\}$ is an open cover of X which refines the cover $\{X \setminus \{x\}, A\}$. (2)$\Longrightarrow$(1) is obvious. □

Note that for technical reasons (in particular, in order to guarantee that the empty set will carry exactly one structure), we do not want the empty collection to be called a cover of the empty set. So by "cover" we always mean non-empty cover.

6 PROPOSITION

If μ is the set of all interior covers of a symmetric topological space X then we have:

(C1) $\{X\} \in \mu$.

(C2) Each cover of X which is refined by a member of μ is itself a member of μ.

(C3) From $\mathcal{U} \in \mu$ and $\mathcal{V} \in \mu$ it follows that the collection

$$\mathcal{U} \wedge \mathcal{V} = \left\{ U \cap V \;\middle|\; U \in \mathcal{U} \text{ and } V \in \mathcal{V} \right\}$$

is a member of μ.

(C4) From $\mathcal{U} \in \mu$ it follows that the collection

$$\mathrm{int}_\mu \mathcal{U} = \left\{ \mathrm{int}_\mu U \;\middle|\; U \in \mathcal{U} \right\}$$

is also a member of μ, where by definition

$$\mathrm{int}_\mu U = \left\{ x \in X \;\middle|\; \{U, X \setminus \{x\}\} \in \mu \right\}.$$

(C5) If $\mathrm{int}_\mu \mathcal{U}$ is a cover of X then $\mathcal{U} \in \mu$. □

The proof of the above proposition is easy and is left to the reader.

7 PROPOSITION

If X is a set and μ is a set of covers of X which satisfies conditions (C1)–(C5) of the preceding proposition, then there exists precisely one symmetric topology τ on X so that μ is the set of all interior covers of (X,τ).

Proof: We define $A \in \tau \iff \forall x \in A \quad x \in \mathrm{int}_\mu A$. □

It follows from the two above propositions that a symmetric topological space can be defined as a pair (X,μ) where μ is a set of covers of X satisfying conditions (C1)–(C5).

8 PROPOSITION

If $f : X \to Y$ is a map between symmetric topological spaces and if μ_X (respectively, μ_Y) is the set of all interior covers of X (respectively, Y), then the following are equivalent:

(1) $f : X \to Y$ is continuous.

(2) $\mathcal{U} \in \mu_Y$ implies $f^{-1}[\mathcal{U}] = \{ f^{-1}[U] \mid U \in \mathcal{U} \} \in \mu_X$. □

It follows from the foregoing that the construct with objects all pairs (X,μ) where X is a set and μ is a set of covers of X fulfilling (C1)–(C5), and morphisms those functions fulfilling the condition (2) of Proposition 8 is actually concretely isomorphic to $\mathsf{Top_S}$ ($\mathsf{Top_S}$ denotes the category of all symmetric topological spaces). If we denote the newly constructed construct by TNear then this means the functor

$$\begin{aligned} \mathsf{Top_S} &\longrightarrow \mathsf{TNear} \\ (X,\tau) &\longrightarrow (X,\mu) \\ f &\longrightarrow f \end{aligned}$$

is full, faithful, and bijective on objects. For simplicity, we will identify $\mathsf{Top_S}$ with TNear by means of the above concrete isomorphism.

If one defines symmetric topological spaces by means of covers as above, then one is led to define new kinds of subspaces and products directly by means of covers in the following natural manner.

9 DEFINITION

Let (X,μ) be a symmetric topological space and let Y be a subset of X. For each $\mathcal{U} \in \mu$ we call

$$\mathcal{U}_Y = \left\{ U \cap Y \;\middle|\; U \in \mathcal{U} \right\}$$

the **trace** of $\mathcal{U}$ on Y and

$$\mu_Y = \left\{ \mathcal{U}_Y \;\middle|\; \mathcal{U} \in \mu \right\}$$

the **trace** of μ on Y. Then (Y,μ_Y) is called the **subspace** of (X,μ) determined by Y. □

If Y is closed in (X,μ) then (Y,μ_Y) coincides with the topological subspace of (X,μ) as it is usually defined (modulo our convention that symmetric topological spaces have their structure given by the interior covers). In general, however, (Y,μ_Y) is not a topological space: μ_Y always fulfills the conditions (C1)–(C4), but usually not (C5). In other words this means that the subspace structure we so naturally defined does not agree with the subspace structure in TNear and does not correspond to the subspace structure in $\mathsf{Top_S}$ (which is the same as in Top).

10 DEFINITION

Let $(X_i, \mu_i)_{i \in I}$ be a family of symmetric topological spaces. Let $(\pi_i : X \to X_i)_{i \in I}$ denote the Set product of the family of sets $(X_i)_{i \in I}$. We define

$$\mu = \bigotimes_{i \in I} \mu_i$$

as the set of all covers $\mathcal{W}$ of X for which there exist a family $(\mathcal{U}_j)_{j \in J}$, with J a finite subset of I and with each $\mathcal{U}_j \in \mu_j$, such that the cover

$$\bigwedge_{j \in J} \pi_j^{-1}[\mathcal{U}_j] = \left\{ \bigcap_{j \in J} \pi_j^{-1}[U_j] \;\middle|\; U_j \in \mathcal{U}_j \text{ for all } j \in J \right\}$$

refines $\mathcal{W}$. Then (X, μ) is called the **product** of the family $(X_i, \mu_i)_{i \in I}$. □

If each (X_i, μ_i) is compact, then the product defined above coincides with the topological product of $(X_i, \mu_i)_{i \in I}$ as it is usually defined (again, modulo the same convention as above). In general, however, (X, μ) again fulfills (C1)–(C4) but not necessarily (C5). Therefore this product also does not correspond to the product in Top_S (which is the same as in Top).

§4 Axiomatizing Uniform covers: Near

Since the constructions described in the preceding section leave the class of topological spaces by violating condition (C5), it is natural to broaden the concept of topological space by the renunciation of condition (C5). The result is the construct Near of nearness spaces, which not only is closed under the constructions described in Definitions 9 and 10 but in which these constructions are the categorically "right" ones.

11 DEFINITIONS

(1) A **nearness space** is a pair (X, μ) where μ is a set of covers of X satisfying conditions (C1)–(C4) of Proposition 6.

(2) A map $f : (X, \mu) \to (Y, \nu)$ between nearness spaces is called **uniformly continuous** provided that from $\mathcal{V} \in \nu$ it follows that $f^{-1}[\mathcal{V}] \in \mu$.

(3) Near denotes the construct of nearness spaces and uniformly continuous maps. □

The construct Near has subspaces (in the categorical sense as initial structures on subsets), and products (also in the categorical sense) and these are both given by the constructions described in the preceding section. Actually, much more can be said.

12 THEOREM

Near is a topological construct with Top_S *embedded as a concretely coreflective subcategory.*

Proof: The main property of being a topological construct is the existence of initial structures. That these do exist is easily seen to hold in Near. If $(X_i, \mu_i)_{i \in I}$ is a (possibly large) family of nearness spaces and $(f_i : X \to X_i)_{i \in I}$ is a source in Set, then the initial nearness structure on X is given by μ where

$\mathcal{U} \in \mu$ iff $\{X\}$ refines $\mathcal{U}$ or $\mathcal{U}$ is a cover of X for which there exist $i_1, \cdots, i_n \in I$ and $\mathcal{U}_j \in \mu_{i_j}$, $j = 1, \cdots, n$ such that

$$\forall (U_j)_{j=1}^n \in \prod_{j=1}^n \mathcal{U}_j \quad \exists U \in \mathcal{U} \quad \bigcap_{j=1}^n f_{i_j}^{-1}[U_j] \subset U,$$

or shortly, $\bigwedge_{j=1}^n f_{i_j}^{-1}\mathcal{U}_j$ refines $\mathcal{U}$. That $\mathsf{Top_S}$ is embedded as a concretely coreflective subcategory is also very easily seen. We already know it is embedded in Near from the foregoing section. Given a nearness space (X, μ) we define μ_t to be the set of all those covers $\mathcal{U}$ of X for which $X = \cup \operatorname{int}_\mu \mathcal{U}$. It is straightforward to verify that (X, μ_t) is in $\mathsf{TNear} = \mathsf{Top_S}$ and that

$$\begin{aligned} \mathsf{T} : \mathsf{Near} &\to \mathsf{Top_S} \\ (X, \mu) &\to (X, \mu_t) \end{aligned}$$

is indeed a coreflector. $\mathsf{T}(X, \mu) = (X, \mu_t)$ is called the topological coreflection of (X, μ). □

We remark that it follows from the fact that the concrete functor $\mathsf{T} : \mathsf{Near} \to \mathsf{Top_S}$ is a coreflector that it preserves subspaces and products. It follows that topological subspaces (respectively, products) can be constructed in two steps: first, one forms the subspace (respectively, the product) using the constructions in Definitions 9 and 10, i.e., in the construct Near; second, one takes the topological coreflection. In the next section, we shall see that it is the second step (i.e., the pressing back of the covering structure, obtained in the first step, to the tighter domain of topological spaces) that is responsible for the fact that the topological properties mentioned in (A) and (B) get lost.

Topological spaces exhibit a richness in that they can have their structure given by various alternative means: by prescribing, e.g., any one of the following: the open sets, the closed sets, the interior operator, the neighborhood systems, etc., or as we have suggested above, in the case of symmetric topological spaces, by prescribing the set of interior covers. Nearness spaces exhibit a similar richness in the variety of ways that the structure can be given. Thus if (X, μ) is a nearness space and $\mathcal{A}$ is a collection of subsets of X then

(1) The collection $\mathcal{A}$ is said to be a **uniform cover** of (X, μ) provided that $\mathcal{A} \in \mu$.

(2) The collection $\mathcal{A}$ is said to be **near** in (X, μ) provided that for every uniform cover $\mathcal{U}$ of (X, μ) there exists $U \in \mathcal{U}$ which meets every member of $\mathcal{A}$.

(3) The collection $\mathcal{A}$ is said to be **micromeric** in (X, μ) provided that for every uniform cover $\mathcal{U}$ of (X, μ) there exists $U \in \mathcal{U}$ which contains some member of $\mathcal{A}$.

It turns out that Near can be axiomatized by means of any one of the three notions mentioned above, and in this sense the three are equivalent. As is the case for topological spaces, sometimes one or the other of these notions is more suitable than another for a particular use. We shall not pursue the investigation of the notions of nearness or micromeric collections at this time. For further information on these matters, we refer the reader to [H88] or [Pr88].

We also mention, but without presenting any details here, that Near is also nice in that it has a homology and cohomology theory, and this theory is closely related to the notion of connectedness as one would naturally expect (connectedness for nearness spaces is defined in Definition 16 below). For details we refer the reader to [Cz75], [B83], [Pr83], and [Pr84].

§5 False Theorems become True: Products and Subspaces

"Theorems" (1)–(8) mentioned in §2 become true if subspaces and products are constructed in Near and if the notions of zero-dimensionality, paracompactness, etc., are extended to Near in a natural way. We shall restrict our attention to paracompactness. First a word on notation and terminology. Given a cover $\mathcal{U}$ and a subset A of X, the star of A with respect to $\mathcal{U}$ is defined and denoted by

$$\text{star}(A,\mathcal{U}) = \bigcup \left\{ U \in \mathcal{U} \;\middle|\; U \cap A \neq \emptyset \right\}.$$

We say that $\mathcal{U}$ **star-refines** $\mathcal{V}$ if the cover $\{\text{star}(U,\mathcal{U}) \mid U \in \mathcal{U}\}$ refines $\mathcal{V}$.

13 DEFINITION

A nearness space X is said to be **paracompact** (or **uniform**) provided that every uniform cover of X is star-refined by some uniform cover of X. □

14 PROPOSITION

A topological space is paracompact in the usual topological sense if and only if it is paracompact in the above sense.

Proof: The only difference between paracompactness in Top_S and in Near is the fact that in Near we require all interior covers, and not just the open covers, to be star-refinable. This however poses no problem since every open cover is an interior cover and every interior cover is refined by some open cover. □

For this paper, we shall denote the full subcategory of Near with objects all paracompact nearness spaces by PNear.

15 THEOREM

PNear *is a concretely reflective subcategory of* Near; *in particular, products and subspaces of paracompact spaces are paracompact.*

Proof: If (X,μ) is a nearness space we let μ_p be the set of all uniform covers $\mathcal{U}$ for which there exists a sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ of uniform covers fulfilling

(1) $\mathcal{U}_0 = \mathcal{U}$.

(2) $\forall n \in \mathbb{N}$, $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$.

It is easily seen that μ_p fulfills (C1)–(C3). If $\mathcal{U} \in \mu_p$ and $(\mathcal{U}_n)_{n=1}^{\infty}$ is a sequence as above then $\mathcal{U}_n \in \mu_p$ for every n and thus the condition in Definition 13 is fulfilled. If $\mathcal{U} \in \mu_p$ and $\mathcal{V} \in \mu_p$ are such that $\mathcal{V}$ star-refines $\mathcal{U}$ then for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ for which $\text{star}(V,\mathcal{V}) \subset U$. For $x \in V$ it then follows that $\mathcal{V}$ refines $\{X \setminus \{x\}, U\}$ and thus $x \in \text{int}_{\mu_p} U$. Consequently $\mathcal{V}$ refines $\text{int}_{\mu_p} \mathcal{U}$ and also (C4) is fulfilled. It is easily seen that $id_X : (X,\mu) \to (X,\mu_p)$ is indeed a PNear reflection of (X,μ). □

As a consequence of the above theorem, items (2) and (7) in the list of desirable theorems in §2 become true in Near. Observe that what we have called paracompact nearness spaces are (in view of the fact that (C4) follows from the condition on star-refinements defining paracompactness) nothing more or less than uniform spaces (as axiomatized by Tukey). Since our definition of uniform continuity for maps between nearness spaces coincides with the notion of the same name between uniform spaces, it is clear that the construct Unif of uniform spaces is concretely isomorphic to what we have called PNear. Thus paracompact nearness spaces are not new but rather familiar entities. In exactly the same way as uniform (= paracompact nearness) spaces can be regarded as a suitable generalization of paracompact topological spaces, nearness spaces can be regarded as a suitable generalization of symmetric topological spaces.

In the same manner as "theorems" (2) and (7) become true in Near by extending the notion of paracompactness to the setting of nearness spaces as above, all "theorems" (1)–(8) become true in Near provided we extend the concepts of zero-dimensionality, etc., in the following natural way.

16 DEFINITIONS

A nearness space X is called

(1) **zero-dimensional** provided every uniform cover is refined by some uniform partition.

(2) **compact** (respectively, **Lindelöf**) provided every uniform cover is refined by some uniform cover which is finite (respectively, countable).

(3) **connected** provided that $\{A, X \setminus A\}$ can be a uniform cover only for $A = X$ or $A = \emptyset$. □

For lack of space, we omit the demonstrations that using the above concepts causes statements (1), (3)–(6), and (8) to become true as theorems about nearness spaces. For the details, the reader is referred to [H76], [H88], and [Pu77]. Finally, we remark that the books [H88] and [Pr88] are largely devoted to nearness spaces: these works should be consulted for further information on the subject.

§6 Characterizing Topological Spaces by Convergence

As we remarked earlier the failure of the statements in §2 under (C) in Top is of a different nature than the failure of the statements under (A) and (B). In order to deal with (C) we first characterize Top in a quite different way than was done in §3. First a word on notation. Given a set X, $\mathsf{F}(X)$ stands for the set of all filters on X, $\mathsf{U}(X)$ stands for the set of all ultrafilters on X, and $\mathsf{P}(X)$ stands for the power set of X. For each point $x \in X$, we let $\dot{x}$ denote the fixed ultrafilter $\{A \subset X \mid x \in A\}$.

In a topological space (X, τ) we now consider the map

$$q_\tau : X \to \mathsf{P}(\mathsf{F}(X))$$

where $\mathcal{F} \in q_\tau(x)$ iff $\mathcal{F}$ converges to x with respect to the topology τ.

17 PROPOSITION

In a topological space (X, τ), q_τ fulfills the following properties:

(F1) $\dot{x} \in q_\tau(x)$.

(F2) $\mathcal{F} \in q_\tau(x)$ *and* $\mathcal{F} \subset \mathcal{G} \in \mathsf{F}(X)$ *implies* $\mathcal{G} \in q_\tau(x)$.

(F3) If $(\mathcal{F}_i)_{i \in I}$ *is a family of filters on* X *such that* $\mathcal{F}_i \in q_\tau(x)$ *for every* $i \in I$, *then*

$$\bigcap_{i \in I} \mathcal{F}_i \in q_\tau(x).$$

(F4) If $\mathcal{F}$ is a filter on X with $\mathcal{F} \in q_\tau(a)$ and $(\mathcal{G}_x)_{x\in X}$ is a family of filters on X such that for each $x \in X$ we have $\mathcal{G}_x \in q_\tau(x)$ then

$$\bigcup_{F\in\mathcal{F}} \bigcap_{x\in F} \mathcal{G}_x \in q_\tau(a).$$ □

The above properties are characteristic for topological spaces [Ko54]. For a proof of the above result we refer the reader to the textbook by Kowalsky [Ko65].

18 PROPOSITION

If X is a set and $q : X \to \mathsf{P}(\mathsf{F}(X))$ satisfies conditions (F1)–(F4) of the preceding proposition, then there exists precisely one topology τ on X such that $q_\tau = q$.

Proof: Simply define the neighborhood filters of τ as

$$\mathcal{V}_\tau(x) = \cap q(x).$$ □

Given a map $f : X \to Y$ and a filter $\mathcal{F}$ on X we denote by $f(\mathcal{F})$ the filter on Y generated by $\{ f[F] \mid F \in \mathcal{F} \}$, i.e.,

$$f(\mathcal{F}) = \left\{ B \subset Y \;\middle|\; f^{-1}[B] \in \mathcal{F} \right\}.$$

19 PROPOSITION

If $f : X \to Y$ is a map between topological spaces and if q_X (respectively, q_Y) stand for the associated "convergences" then the following are equivalent:

(1) $f : X \to Y$ is continuous.

(2) $\mathcal{F} \in q_X(x)$ implies $f(\mathcal{F}) \in q_Y(f(x))$. □

It follows from the foregoing that the construct with objects all pairs (X, q) where X is a set and q is a "convergence" on X fulfilling (F1)–(F4), and morphisms those maps fulfilling condition (2) of Proposition 19 is concretely isomorphic to Top.

Several relaxations of the properties (F1)–(F4) lead to interesting supercategories of Top.

§7 Axiomatizing Convergence: PrTop and PsTop

20 DEFINITIONS

(1) A **pretopology** q on a set X is a map $q : X \to \mathsf{P}(\mathsf{F}(X))$ which fulfills (F1)–(F3) of Proposition 17. The pair (X, q) is called a **pretopological space**. If no confusion can result $\mathcal{F} \in q(x)$ is often denoted $\mathcal{F} \to x$.

(2) A map $f : (X, q) \to (Y, p)$ between pretopological spaces is called **continuous** if $\mathcal{F} \in q(x)$ implies $f(\mathcal{F}) \in p(f(x))$.

(3) PrTop denotes the construct whose objects are all pretopological spaces, and whose morphisms are all continuous maps between these. □

21 THEOREM

PrTop *is a topological construct with* Top *embedded in* PrTop *as a concretely reflective subcategory.*

Proof: That Top is embedded in PrTop follows at once from Propositions 17 and 18. If $(X_i, q_i)_{i \in I}$ is a (possibly large) family of pretopological spaces and $(f_i : X \to X_i)_{i \in I}$ is a source in Set then the initial pretopology on X is given by q where

$$\mathcal{F} \in q(x) \iff \forall i \in I \quad f_i(\mathcal{F}) \in q_i(f_i(x)).$$

That Top is embedded as a concretely reflective subcategory is easily seen as follows: Given a pretopological space (X, q) we define the topology $\lambda(q)$ by declaring G to be open in $\lambda(q)$ iff

$$x \in G \text{ and } \mathcal{F} \in q(x) \implies G \in \mathcal{F}.$$

Then it is easily seen that

$$\begin{aligned} \mathsf{PrTop} &\longrightarrow \mathsf{Top} \\ (X, q) &\longrightarrow (X, \lambda(q)) \end{aligned}$$

is indeed a concrete reflector. □

22 DEFINITIONS

(1) A **pseudotopology** q on a set X is a map $q : X \to \mathsf{P}(\mathsf{F}(X))$ which fulfills (F1)–(F2) and the following relaxation of (F3) of Proposition 17.

(PsT) If $\mathcal{F} \in \mathsf{F}(X)$ satisfies the condition

$$\mathcal{F} \subset \mathcal{U} \in \mathsf{U}(X) \implies \mathcal{U} \in q(x)$$

then $\mathcal{F} \in q(x)$.

The pair (X, q) is called a **pseudotopological space**. As with pretopological spaces, $\mathcal{F} \to x$ means $\mathcal{F} \in q(x)$.

(2) A map $f(X, q) \to (Y, p)$ between pseudotopological spaces is called **continuous** if $\mathcal{F} \in q(x)$ implies $f(\mathcal{F}) \in p(f(x))$. By (PsT), it is already sufficient that $\mathcal{U} \in q(x) \cap \mathsf{U}(X)$ implies $f(\mathcal{U}) \in p(f(x))$.

(3) PsTop denotes the construct whose objects are all pseudotopological spaces and whose morphisms are all continuous maps between these. □

23 THEOREM

PsTop *is a topological construct with* PrTop, *and hence also* Top, *embedded as concretely reflective subcategories.*

Proof: We use, formally, the same construction as in PrTop to obtain initial structures (see the proof of Theorem 21). To show that PrTop is concretely reflective in PsTop, given a pseudotopological space (X, q), we define the pretopology $\psi(q)$ by stating

$$\mathcal{F} \in (\psi(q))(x) \iff \cap q(x) \subset \mathcal{F}.$$

Then one can easily verify that $\psi(q)$ is indeed a pretopology and that

$$\begin{aligned} \mathsf{PsTop} &\longrightarrow \mathsf{PrTop} \\ (X, q) &\longrightarrow (X, \psi(q)) \end{aligned}$$

is a concrete reflector. That Top is embedded as a concretely reflective subcategory follows by the composition of concrete reflectors. Alternatively, one can observe that the Top reflection of a pseudotopological space is given by the same direct description as the Top reflection of a pretopological space (see the proof of Theorem 21 above). □

The above results show that we have a hierarchy:

$$\mathsf{Top} \hookrightarrow \mathsf{PrTop} \hookrightarrow \mathsf{PsTop}$$

where each subcategory is concretely reflective in the bigger ones. Consequently, initial structures, in particular products and subspaces, are the same in all three constructs. Quotients however differ.

§8 False Theorems become True: Quotients

We begin by characterizing quotients in PrTop. To simplify the notation, if (X,q) is a pretopological space, we let $\mathcal{V}_q(x) = \cap\, q(x)$ for each $x \in X$.

24 PROPOSITION [D. C. Kent]

If (X,q) and (Y,p) are pretopological spaces then $f:(X,q) \to (Y,p)$ is a quotient map in PrTop *iff $f:(X,q) \to (Y,p)$ is a continuous surjection and*

$$(QPr) \qquad \forall V \subset Y\ \forall y \in Y\ \left(\forall x \in f^{-1}(y)\quad f^{-1}[V] \in \mathcal{V}_q(x)\right) \implies V \in \mathcal{V}_p(y).$$

Proof: Assume that $f: X \to Y$ is a surjection. For each $y \in Y$ let $\mathcal{V}(y)$ stand for the filter generated by the sets $V \subset Y$ such that for all x, $x \in f^{-1}(y)$ implies $f^{-1}[V] \in \mathcal{V}_q(x)$. (QPr) then states that for all $y \in Y$, $\mathcal{V}(y) \subset \mathcal{V}_p(y)$. The converse inclusion is equivalent to the continuity of $f:(X,q) \to (Y,p)$. Therefore the condition

$$\forall y \in Y \qquad \mathcal{V}(y) = \mathcal{V}_p(y)$$

is equivalent to p being the finest pretopology on Y that makes f continuous. The desired equivalence is now clear. □

25 DEFINITION

In topological constructs, a quotient map $f: X \to Y$ is called **hereditary** provided for each $B \subset Y$, $f|f^{-1}[B] : f^{-1}[B] \to B$ is a quotient map. □

The next theorem shows that in PrTop the false "theorem" (9) becomes true. Although we will prove this theorem here, it is worthwhile to mention that it is actually a consequence of the fact that PrTop is a so-called extensional construct (see the paper "Improving Top: PrTop and PsTop" by Herrlich, Lowen-Colebunders, and Schwarz in this volume).

26 THEOREM

In PrTop *quotients are hereditary.*

Proof: We use Proposition 24. Suppose $f:(X,q) \to (Y,p)$ is a quotient map in PrTop, $B \subset Y$, $A = f^{-1}[B]$, and $g = f|A$. Let $y \in B$ and $V \subset B$ be such that for all $x \in g^{-1}(y)$, we have

$g^{-1}[V] \in \mathcal{V}_{q|A}(x)$. $W = V \cup (Y \setminus B)$ has the property that for all $x \in f^{-1}(y)$, $f^{-1}[W] \in \mathcal{V}_q(x)$. It follows that $W \in \mathcal{V}_p(y)$ and thus that $V \in \mathcal{V}_{q|B}(y)$. Therefore, g is indeed a quotient map in PrTop. □

27 DEFINITION

Given topological spaces X and Y, a continuous surjection $f : X \to Y$ is called pseudo-open if whenever $y \in Y$ and U is a neighborhood of $f^{-1}(y)$ it follows that $y \in \mathrm{int}_Y f[U]$. □

28 PROPOSITION [1]

For maps f between topological spaces, the following are equivalent:

(1) f is a hereditary quotient map in Top.

(2) f is a quotient map in PrTop.

(3) f is pseudo-open.

Proof: The equivalence of (2) and (3) follows at once from Proposition 24. That (2) implies (1) follows from the facts that Top is concretely reflective in PrTop and from Theorem 26. So all that remains to be shown is, e.g., that (1) implies (2). Let $f : (X, q) \to (Y, p)$ be a hereditary quotient map in Top. Let $y \in Y$ and $V \subset Y$ be such that for all $x \in f^{-1}(y)$ we have $f^{-1}[V] \in \mathcal{V}_q(x)$. Let $B = (Y \setminus V) \cup \{y\}$. Then by hypothesis,

$$g = f \,|\, f^{-1}[B] : f^{-1}[B] \to B$$

is a quotient map in Top. Since $g^{-1}(y) = f^{-1}[V] \cap f^{-1}[B]$ it therefore follows that $\{y\}$ is open in B. Consequently, there exists G open in Y such that $G \cap B = \{y\}$. Hence $y \in G \subset V$ and it follows that $V \in \mathcal{V}_p(y)$. □

The fact that PrTop saves the false statement (9) is not the only reason that it is interesting. This construct exhibits a richness in that its objects can have their structures given in various alternative ways, just as is true for Top and for Near as was already pointed out earlier. A pretopological space can have its structure given by prescribing the "neighborhoods" of points.

A **neighborhood filter system** $\mathcal{V}$ on a set X is a map $\mathcal{V} : X \to \mathsf{P}(\mathsf{P}(X))$ such that the two following axioms are satisfied:

(N1) $x \in \bigcap \mathcal{V}(x)$ for each $x \in X$.

(N2) $\mathcal{V}(x)$ is a filter on X for each $x \in X$.

The convergence structure and the neighborhood filter system are related by the prescription: for a filter $\mathcal{F}$ on X and for $x \in X$, we define $\mathcal{F} \to x$ iff $\mathcal{V}(x) \subset \mathcal{F}$. (The above informal description can be developed into a concrete isomorphism of categories.) Another remark about pretopological spaces: They are also isomorphic to the category of closure spaces (in the sense of Čech, i.e. without idempotency). For a proof of this isomorphism, see Section III.14.B of [Č66].

In PrTop however the false statement (10) in our list remains just as untrue as it was in Top. Actually, the same example as the one given earlier for Top shows this also for PrTop. Quotients in Top in general differ from quotients in PrTop, but in the particular case of the example given in Example

[1] The equivalence of conditions (1) and (3) is due to A. V. Arhangel'skiĭ.

4 they are the same. Obviously $id_{\mathbf{Q}} : \mathbf{Q} \to \mathbf{Q}$ is also a quotient in PrTop, and that $\phi : X \to Y$ is a quotient in PrTop follows from straightforward verification. If then

$$\phi \times id_{\mathbf{Q}} : X \times \mathbf{Q} \to Y \times \mathbf{Q}.$$

were a quotient in PrTop it would also have to be a quotient in Top and we know that this is not the case.

Therefore, in order to save statement (10) we have to go beyond PrTop. Thus we consider next the construct PsTop of all pseudotopological spaces. These spaces have even less of the flavor of topological spaces than do the pretopological spaces, e.g., their structure is no longer given by neighborhood systems. But there is, nevertheless, an added advantage in considering the pseudotopological spaces since, as we shall see, PsTop saves not only statement (10), but also shares with PrTop the ability to save statement (9).

29 PROPOSITION [D. C. Kent]

If (X,q) and (Y,p) are pseudotopological spaces then $f : (X,q) \to (Y,p)$ is a quotient map in PsTop *iff $f : (X,q) \to (Y,p)$ is a continuous surjection and*

$$(QPs) \qquad \forall y \in Y\ \forall \mathcal{U} \in \mathrm{U}(Y) \cap p(y) \quad \exists x \in f^{-1}(y) \quad \exists \mathcal{W} \in \mathrm{U}(X) \cap q(x) \qquad f(\mathcal{W}) = \mathcal{U}.$$

Proof: Assume that $f : X \to Y$ is a surjection. For $y \in Y$, let

$$r(y) = \bigcup_{x \in f^{-1}(y)} \left\{ f(\mathcal{W}) \;\middle|\; \mathcal{W} \in q(x) \cap \mathrm{U}(X) \right\}.$$

(QPs) then states that for all $y \in Y$, $p(y) \cap \mathrm{U}(Y) \subset r(y)$. The converse inclusion is equivalent to the continuity of $f : (X,q) \to (Y,p)$. Therefore the condition

$$\forall y \in Y \qquad p(y) \cap \mathrm{U}(Y) = r(y)$$

is equivalent to p being the finest pretopology on Y that makes f continuous. The desired equivalence is now clear. □

30 THEOREM

In PsTop *quotients are hereditary.*

Proof: We use Proposition 29. Suppose $f : (X,q) \to (Y,p)$ is a quotient map in PsTop, $B \subset Y$, $A = f^{-1}[B]$, and $g = f|A$. Let $y \in B$ and $\mathcal{U} \in \mathrm{U}(B) \cap (p|B)(y)$ and let $\mathcal{V}$ be the filter on Y generated by $\mathcal{U}$. Then $\mathcal{V} \in \mathrm{U}(Y) \cap p(y)$ and thus there exist $x \in f^{-1}(y) = g^{-1}(y)$ and $\mathcal{W} \in \mathrm{U}(X) \cap q(x)$ such that $f[\mathcal{W}] = \mathcal{V}$. Since $B \in \mathcal{U}$, $A \in \mathcal{W}$ and therefore $g[\mathcal{W}|A] = \mathcal{U}$, we are done. □

Our next result shows that the false "theorem" (10) can be remedied in PsTop.

31 THEOREM

If $(f_i : X_i \to Y_i)_{i \in I}$ is a family of quotient maps in PsTop *then the product map*

$$\prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$$

is also a quotient map in PsTop.

Proof: To simplify the notation, let

$$f = \prod_{i\in I} f_i, \qquad X = \prod_{i\in I} X_i, \quad \text{and} \quad Y = \prod_{i\in I} Y_i$$

and let $\pi_i : X \to X_i$ and $\pi_i' : Y \to Y_i$ denote the canonical projection maps. We have to show that f fulfills the condition (QPs) of Proposition 29. Let $\mathcal{U}$ be an ultrafilter on Y such that $\mathcal{U} \to y$. Then for each $i \in I$ we have $\pi_i'(\mathcal{U}) \to y_i$ and consequently there exist $\mathcal{W}_i \in \mathrm{U}(X_i)$ and $x_i \in f_i^{-1}(y_i)$ such that $\mathcal{W}_i \to x_i$ and $f_i(\mathcal{W}_i) = \pi_i'(\mathcal{U})$. Let $x = (x_i)_{i\in I}$ and let $\mathcal{W}$ denote the filter on X generated by all sets of the form

$$\bigcap_{j\in J} \pi_i^{-1}[U_j]$$

where J is a finite subset of I and $U_j \in \mathcal{U}_j$ for all $j \in J$. Now suppose that for each ultrafilter $\mathcal{Z}$ on X containing $\mathcal{W}$ we have $f(\mathcal{Z}) \neq \mathcal{U}$. Then for each such ultrafilter we can find $Z \in \mathcal{Z}$ such that $f[Z] \notin \mathcal{U}$. In that case however we can also find a finite number of such Z's, say $Z_1 \in \mathcal{Z}_1, \cdots, Z_n \in \mathcal{Z}_n$ such that

$$\bigcup_{k=1}^{n} Z_k \in \mathcal{W}$$

and then it follows that

$$f(\bigcup_{k=1}^{n} Z_k) \in f(\mathcal{W}) \subset \mathcal{U}$$

which is a contradiction. Consequently there does exist such an ultrafilter $\mathcal{Z}$ on X for which $f(\mathcal{Z}) = \mathcal{U}$, and thus $f : X \to Y$ is a quotient map in PsTop. □

It is worthwhile to notice that the foregoing result for finite products is already a consequence of the fact that PsTop is a so-called cartesian closed topological construct. This result is proved, e.g., in the paper "Improving Top: PrTop and PsTop" by Herrlich, Lowen-Colebunders, and Schwarz in this volume.

32 DEFINITION

In topological constructs a quotient map $f : X \to Y$ is called

(1) **product-stable** provided for any object A, the map

$$f \times id_A : X \times A \to Y \times A$$

is a quotient map.

(2) **pullback-stable** provided for any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & Y \end{array}$$

we have that f' is a quotient map. □

33 PROPOSITION

In PsTop *quotient maps are pullback-stable.*

Proof: From Theorems 30 and 31 we know that quotients in PsTop are hereditary and product-stable. It now suffices to notice that pullbacks compose and that in topological constructs, any morphism $f : X \to Y$ decomposes into an embedding followed by a projection:

$$\begin{array}{ccccc} X & \xrightarrow{(id_X, f)} & X \times Y & \xrightarrow{\pi_Y} & Y \\ x & \mapsto & (x, f(x)) & \mapsto & f(x) \end{array}$$

□

34 DEFINITION

Given topological spaces X and Y, a continuous surjection $f : X \to Y$ is called **bi-quotient** if whenever $\mathcal{B}$ is a filter base on Y and y is an adherence point of $\mathcal{B}$, there exists an adherence point of the filter base $f^{-1}(\mathcal{B})$ in $f^{-1}(y)$. □

35 PROPOSITION [1]

For continuous maps f between topological spaces the following are equivalent:

(1) f is a pullback-stable quotient map in Top.

(2) f is a product-stable hereditary quotient map in Top.

(3) f is a quotient map in PsTop.

(4) f is a bi-quotient map.

Proof: That (1) and (2) are equivalent follows by the same arguments as in the proof of Proposition 33. That (3) and (4) are equivalent is an immediate consequence of Proposition 29. That (3) implies (2) follows at once from Propositions 30 and 31. Indeed, if, e.g., X, Y, A are topological spaces and $f : X \to Y$ is a quotient map in PsTop then by Proposition 31,

$$f \times id_A : X \times A \to Y \times A$$

is a quotient map in PsTop. Since Top is bireflective in PsTop it is then necessarily a quotient map in Top. The same reasoning holds for the hereditary part. Consequently, all that remains to be shown is, e.g., that (1) implies (4). Suppose that X and Y are topological spaces and that $f : X \to Y$ is a pullback-stable quotient map. Let $y \in Y$, $\mathcal{B}$ be a filterbase on Y, and y an adherence point of $\mathcal{B}$. Suppose that for all $x \in f^{-1}(y)$, x is not an adherence point of $f^{-1}[\mathcal{B}]$, i.e., there exist G_x open, $x \in G_x$, and $B_x \in \mathcal{B}$ such that $f^{-1}[B_x] \cap G_x = \emptyset$. Then $\{ G_x \mid x \in f^{-1}(y) \}$ is an open cover of $f^{-1}(y)$ and we let

$$H_x = (Y \setminus f[G_x]) \cup \{y\}, \qquad x \in f^{-1}(y).$$

Let $\mathcal{F}$ stand for the filter generated by

$$\mathcal{V}(y) \cup \left\{ H_x \;\middle|\; x \in f^{-1}(y) \right\},$$

define a new topology on Y by

$$\tau' = \mathcal{F} \cup \left\{ B \subset Y \;\middle|\; y \notin B \right\},$$

[1] The equivalence of (1) and (4) is due to B. J. Day and G. M. Kelly.

and let Y' stand for this new space. Then $id : Y' \to Y$ is continuous and if

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle id_0}\downarrow & & \downarrow{\scriptstyle id_1} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback square then X' has the coarsest topology making both f' and id_0 continuous. Now clearly

$$f^{-1}(y) = \bigcup_{x \in f^{-1}(y)} \left(G_x \cap f^{-1}[H_x]\right)$$

is open in X' and thus by our supposition $\{y\} \in \tau'$, i.e., $\{y\} \in \mathcal{F}$. Hence we can find $x_1, \cdots, x_n \in f^{-1}(y)$ and $V \in \mathcal{V}(y)$ such that

$$\{y\} = V \cap \left(\bigcap_{i=1}^{n} H_{x_i}\right)$$

i.e.,

$$V \subset \bigcup_{i=1}^{n} f[G_{x_i}].$$

Now let

$$B = B_{x_1} \cap \cdots \cap B_{x_n}.$$

Then

$$B \cap \left(\bigcup_{i=1}^{n} f[G_{x_i}]\right) \neq \emptyset$$

and if we choose $i \in \{1, \cdots, n\}$, $a \in G_{x_i}$ and $z \in B$ such that $z = f(a)$ then it follows that

$$a \in f^{-1}[B_{x_i}] \cap G_{x_i}$$

which is a contradiction and we are done. □

We remark that an immediate corollary to Theorem 31 and Proposition 35 is the theorem proved by Michael that in Top, products of bi-quotient maps are bi-quotient maps.

All the foregoing shows us (a) that the false theorem (10) in Top becomes a true theorem in the "extension" PsTop, and (b) that a peculiar type of map in Top becomes a canonical categorically defined map in PsTop.

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