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Sharp Estimate of the Cost of Controllability for a Degenerate Parabolic Equation with Interior Degeneracy

This work is motivated by the study of null controllability for the typical degenerate parabolic equation with interior degeneracy and one-sided control:

$$u_t - (|x|^\alpha u_x)_x = h(x,t)\chi_{(a,b)}, \quad x \in (-1,1),$$

with $0 < a < b < 1$. It was proved in a paper by P. Cannarsa, R. Ferretti, and P. Martinez [*Null controllability for parabolic operators with interior degeneracy and one-sided control*, SIAM J. Control Optimization 57/2 (2019) 900–924] that this equation is null controllable (in any positive time T) if and only if $\alpha < 1$, and that the cost of null controllability blows up as $\alpha \to 1^-$. This is related to the following property of the eigenvalues: the gap between an eigenvalue of odd order and the consecutive one goes to 0 as $\alpha \to 1^-$ (see the paper cited above).

The goal of the present work is to provide optimal upper and lower estimates of the null controllability cost, with respect to the degeneracy parameter (when $\alpha \to 1^-$) and in short time (when $T \to 0^+$). We prove that the null controllability cost behaves as $1/(1-\alpha)$ as $\alpha \to 1^-$ and as $e^{1/T}$ as $T \to 0^+$. Our analysis is based on the construction of a suitable family biorthogonal to the sequence $(e^{\lambda_n t})_n$ in $L^2(0,T)$, under some general gap conditions on the sequence $(\lambda_n)_n$, conditions that are suggested by a motivating example.

Keywords: Controllability, degenerate parabolic equation, biorthogonal family.

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