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On Group and Loop Spheres

We investigate the problem of *defining group or loop structures on spheres*, where by "sphere" we mean the level set $q(x) = c$ of a general $\mathbb{K}$-valued quadratic form q, for an invertible scalar c. When $\mathbb{K}$ is a field and q non-degenerate, then this corresponds to the classical theory of *composition algebras*; in particular, for $\mathbb{K} = \mathbb{R}$ and positive definite forms, we obtain the sequence of the four real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (quaternions), $\mathbb{O}$ (octonions). Our theory is more general, allowing that $\mathbb{K}$ is merely a commutative ring, and the form q possibly degenerate. To achieve this goal, we give a more geometric formulation, replacing the theory of binary composition algebras by *ternary algebraic structures*, thus defining categories of *group spherical* and of *Moufang spherical spaces*. In particular, we develop a theory of *ternary Moufang loops*, and show how it is related to the Albert-Cayley-Dickson construction and to generalized ternary octonion algebras. At the bottom, a starting point of the whole theory is the (elementary) result that *every* 2*-dimensional quadratic space carries a canonical structure of commutative group spherical space*.

Keywords: Composition algebra, quaternions, octonions, (binary) quadratic form, sphere, generalized dicyclic group, circle group, group spherical space, Moufang loop spherical space, torsor, ternary loop.

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