Strong Integrality of Inversion Subgroups of Kac-Moody Groups

Let $A$ be a symmetrizable generalized Cartan matrix with corresponding Kac-Moody algebra $\mathfrak{g}$ over $\mathbb{Q}$. Let $V = V^\lambda$ be an integrable highest weight $\mathfrak{g}$-module with dominant regular integral weight $\lambda$ and representation $\rho : \mathfrak{g} \to \text{End}(V)$, and let $V_\mathbb{Z} = V_\mathbb{Z}^\lambda$ be a $\mathbb{Z}$-form of $V$. Let $G_V(\mathbb{Q})$ be the associated minimal Kac-Moody group generated by the automorphisms $\exp(t\rho(e_i))$ and $\exp(t\rho(f_i))$ of $V$, where $e_i$ and $f_i$ are the Chevalley-Serre generators and $t \in \mathbb{Q}$. Let $G(\mathbb{Z})$ be the group generated by $\exp(t\rho(e_i))$ and $\exp(t\rho(f_i))$ for $t \in \mathbb{Z}$. Let $\Gamma(\mathbb{Z})$ be the Chevalley subgroup of $G_V(\mathbb{Q})$, that is, the subgroup that stabilizes the lattice $V_\mathbb{Z}$ in $V$. For a subgroup $M$ of $G_V(\mathbb{Q})$, we say that $M$ is integral if $M \cap G(\mathbb{Z}) = M \cap \Gamma(\mathbb{Z})$ and that $M$ is strongly integral if there exists $v \in V_\mathbb{Z}$ such that $g \cdot v \in V_\mathbb{Z}$ implies $g \in G(\mathbb{Z})$ for all $g \in M$. We prove strong integrality of inversion subgroups $U_{(w)}$ of $G_V(\mathbb{Q})$ for $w$ in the Weyl group, where $U_{(w)}$ is the group generated by positive real root groups that are flipped to negative root groups by $w^{-1}$. We use this to prove strong integrality of subgroups of the unipotent subgroup $U$ of $G_V(\mathbb{Q})$ that are generated by commuting real root groups. When $A$ has rank 2, this gives strong integrality of subgroups $U_1$ and $U_2$ where $U = U_1 \ast U_2$ and each $U_i$ is generated by ‘half’ the positive real roots.

**Keywords**: Kac-Moody groups, Chevalley groups, integrality.

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