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On Weakly Complete Universal Enveloping Algebras: A Poincaré-Birkhoff-Witt Theorem

The Poincaré-Birkhoff-Witt Theorem deals with the structure and universal property of the universal enveloping algebra $U(L)$ of a Lie algebra L, e.g., over $\mathbb{R}$ or $\mathbb{C}$. K. H. Hofmann and L. Kramer (HK) [*On weakly complete group algebras of Compact Groups*, J. Lie Theory 30 (2020) 407–426] recently introduced the weakly complete universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of a profinite-dimensional topological Lie algebra $\mathfrak{g}$. Here it is shown that the classical universal enveloping algebra $U(|\mathfrak{g}|)$ of the abstract Lie algebra underlying $\mathfrak{g}$ is a dense subalgebra of $\mathbf{U}(\mathfrak{g})$, algebraically generated by $\mathfrak{g} \subseteq \mathbf{U}(\mathfrak{g})$. It is further shown that, inspite of $\mathbf{U}$ being a left adjoint functor, it nevertheless preserves projective limits in the form $\mathbf{U}(\lim_{\mathfrak{i}} \mathfrak{g}/\mathfrak{i}) \cong \lim_{\mathfrak{i}} \mathbf{U}(\mathfrak{g}/\mathfrak{i})$, for profinite-dimensional Lie algebras $\mathfrak{g}$ represented as projective limits of their finite-dimensional quotients. The required theory is presented in an appendix which is of independent interest.

In a natural way, a weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ is a weakly complete symmetric Hopf algebra with a Lie subalgebra $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ of *primitive* elements containing $\mathfrak{g}$ (indeed properly if $\mathfrak{g} \neq \{0\}$), and with a nontrivial multiplicative pro-Lie group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ of *grouplike* units, having $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ as its Lie algebra – in contrast with the classical Poincaré-Birhoff-Witt environment of $U(L)$, thus providing a new aspect of Lie's Third Fundamental Theorem: Indeed a canonical pro-Lie subgroup $\Gamma^*(\mathfrak{g})$ of $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ is identified whose Lie algebra is naturally isomorphic to $\mathfrak{g}$. The structure of $\mathbf{U}(\mathfrak{g})$ is described in detail for $\dim \mathfrak{g} = 1$. The primitive and grouplike components and their mutual relationship are evaluated precisely.

In (HK), cited above, and in the work of R. Dahmen and K. H. Hofmann [*The pro-Lie group aspect of weakly complete algebras and weakly complete group Hopf algebras*, J. Lie Theory 29 (2019) 413–455] the real weakly complete group Hopf algebra $\mathbb{R}[G]$ of a compact group G was described. In particular, the set $\mathbb{P}(\mathbb{R}[G]))$ of primitive elements of $\mathbb{R}[G]$ was identified as the Lie algebra $\mathfrak{g}$ of G. It is now shown that for any compact group G with Lie algebra $\mathfrak{g}$ there is

a natural morphism of weakly complete symmetric Hopf algebras $\omega_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g}) \to \mathbb{R}[G]$, implementing the identity on $\mathfrak{g}$ and inducing a morphism of pro-Lie groups $\Gamma^*(G) \to \mathbb{G}(\mathbb{R}[G]) \cong G$: yet another aspect of Sophus Lie's Third Fundamental Theorem!

Keywords: Associative algebra, Lie algebra, universal enveloping algebra, weakly complete vector space, projective limit, pro-Lie group, profinite-dimensional Lie algebra, power series algebra, symmetric Hopf algebra, primitive element, grouplike element, Poincaré-Witt theorem.

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