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A Calabi-Yau Algebra with E_6 Symmetry and the Clebsch-Gordan Series of $sl(3)$

Building on classical invariant theory, it is observed that the polarised traces generate the centraliser $Z_L(sl(N))$ of the diagonal embedding of $U(sl(N))$ in $U(sl(N))^{\otimes L}$. The paper then focuses on $sl(3)$ and the case $L = 2$. A Calabi-Yau algebra $\mathcal{A}$ with three generators is introduced and explicitly shown to possess a PBW basis and a certain central element. It is seen that $Z_2(sl(3))$ is isomorphic to a quotient of the algebra $\mathcal{A}$ by a single explicit relation fixing the value of the central element. Upon concentrating on three highest weight representations occurring in the Clebsch-Gordan series of $U(sl(3))$, a specialisation of $\mathcal{A}$ arises, involving the pairs of numbers characterising the three highest weights. In this realisation in $U(sl(3)) \otimes U(sl(3))$, the coefficients in the defining relations and the value of the central element have degrees that correspond to the fundamental degrees of the Weyl group of type E_6. With the correct association between the six parameters of the representations and some roots of E_6, the symmetry under the full Weyl group of type E_6 is made manifest. The coefficients of the relations and the value of the central element in the realisation in $U(sl(3)) \otimes U(sl(3))$ are expressed in terms of the fundamental invariant polynomials associated to E_6. It is also shown that the relations of the algebra $\mathcal{A}$ can be realised with Heun type operators in the Racah or Hahn algebra.

Keywords: Calabi-Yau algebra, polarised traces, centraliser algebra, Clebsch-Gordan series, Heun operators, Weyl group of type E_6.

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