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Affine-Quadratic Problems on Lie Groups: Tops and Integrable Systems

This paper focuses on the relevance of a certain class of left-invariant Hamiltonians (affine-quadratic) on a reductive semi-simple Lie algebra $\mathfrak{g}$ for the theory of integrable systems and the equations of applied mathematics. Any semi-simple Lie group G that contains a closed subgroup K is reductive, in the sense, that the orthogonal complement $\mathfrak{p}$ in $\mathfrak{g}$ of the Lie algebra $\mathfrak{k}$ of K, relative to the Killing form, satisfies $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$. This implies that K acts (by adjoint action) on $\mathfrak{p}$ and therefore induces the semi-direct product $\mathfrak{p} \rtimes K$. Consequently, $\mathfrak{g}$, as a vector space, carries two Lie algebra structures: semi-simple, and semi-direct. Hence, the dual $\mathfrak{g}^*$ carries two Poisson structures as well. Any affine-quadratic function H on $\mathfrak{g}$ can be simultaneously viewed as a Hamiltonian for either Poisson structure.

We will show that certain coadjoint orbits relative to the semi-direct action are the cotangent bundles of $SO(n)$. This implies that the equations of an n-dimensional top can be represented on such coadjoint orbits. In this situation there is a canonical affine-quadratic Hamiltonian whose Hamiltonian equations on these coadjoint orbits coincide with the equations of the top. This implies that the integrable cases of the top correspond to the the integrable cases of the overseeing affine Hamiltonian. More generally, we will identify a subclass of affine-quadratic Hamiltonians, called isospectral, that provides new insights into the theory of integrable systems based on the contributions of S. V. Manakov, A. T. Fomenko, A. S. Mischenko and O. Bogoyavlensky listed in the references.

Keywords: Lie groups, symplectic manifolds, Poisson manifolds, control systems, Hamiltonian systems, maximum principle.

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