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On Weakly Complete Group Algebras of Compact Groups

A topological vector space over the real or complex field $\mathbb{K}$ is *weakly complete* if it is isomorphic to a power $\mathbb{K}^J$. For each topological group G there is a *weakly complete topological group Hopf algebra* $\mathbb{K}[G]$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, for which three insights are contributed:

Firstly, *there is a comprehensive structure theorem saying that the topological algebra* $\mathbb{K}[G]$ *is the cartesian product of its finite dimensional minimal ideals whose structure is clarified.*

Secondly, *for a compact* abelian *group* G *and its character group* $\hat{G}$, *the weakly complete* complex *Hopf algebra* $\mathbb{C}[G]$ *is the product algebra* $\mathbb{C}^{\hat{G}}$ *with the comultiplication* $c\colon \mathbb{C}^{\hat{G}} \to \mathbb{C}^{\hat{G}\times\hat{G}} \cong \mathbb{C}^{\hat{G}} \otimes \mathbb{C}^{\hat{G}}$, $c(F)(\chi_1,\chi_2) = F(\chi_1+\chi_2)$ *for* $F\colon \hat{G} \to \mathbb{C}$ *in* $\mathbb{C}^{\hat{G}}$. *The subgroup* $\Gamma(\mathbb{C}^{\hat{G}})$ *of grouplike elements of the group of units of the algebra* $\mathbb{C}^{\hat{G}}$ *is* $\mathrm{Hom}(\hat{G}, (\mathbb{C}\setminus\{0\}, .))$ *while the vector subspace of primitive elements is* $\mathrm{Hom}(\hat{G}, (\mathbb{C}, +))$. This forces the group $\Gamma(\mathbb{R}[G]) \subseteq \Gamma(\mathbb{C}[G])$ to be $\mathrm{Hom}(\hat{G}, \mathbb{S}^1) \cong \hat{\hat{G}} \cong G$ with the complex circle group $\mathbb{S}^1$. While the relation $\Gamma(\mathbb{R}[G]) \cong G$ remains true for *any* compact group, $\Gamma(\mathbb{C}[G]) \cong G$ holds for a compact abelian group G if and only if it is profinite.

Thirdly, for each pro-Lie algebra L a weakly complete universal enveloping Hopf algebra $\mathbf{U}_{\mathbb{K}}(L)$ over $\mathbb{K}$ exists such that *for each* connected *compact group* G *the weakly complete real group Hopf algebra* $\mathbb{R}[G]$ *is a quotient Hopf algebra of* $\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))$ *with the (pro-)Lie algebra* $\mathfrak{L}(G)$ *of* G. *The group* $\Gamma(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ *of grouplike elements of the weakly complete enveloping algebra of* $\mathfrak{L}(G)$ *maps onto* $\Gamma(\mathbb{R}[G]) \cong G$ *and is therefore nontrivial* in contrast to the case of the discrete classical enveloping Hopf algebra of an abstract Lie algebra.

Keywords: Weakly complete vector space, weakly complete algebra, group algebra, Hopf algebra, compact group, Lie algebra, universal enveloping algebra.

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