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Classical Invariant Theory for Free Metabelian Lie Algebras

Let $W_d = K^d$ be the d-dimensional vector space over a field K of characteristic 0 with the canonical action of the general linear group $GL_d(K)$ and let KX_d be the vector space of the linear functions on W_d. One of the main topics of classical invariant theory is the study of the algebra of invariants $K[X_d]^{SL_2(K)}$ of the special linear group $SL_2(K)$, when KX_d is a direct sum of $SL_2(K)$-modules of binary forms. Noncommutative invariant theory deals with the algebra of invariants $F_d(\mathfrak{V})^G$ of a group $G < GL_d(K)$ acting on the relatively free algebra $F_d(\mathfrak{V})$ of a variety of K-algebras $\mathfrak{V}$. Due to the noncommutativity it is more convenient to assume that $F_d(\mathfrak{V})$ is generated by W_d instead of by KX_d, with the corresponding action of $GL_d(K)$.
In this paper we consider the free metabelian Lie algebra $F_d(\mathfrak{A}^2)$ which is the relatively free algebra in the variety $\mathfrak{A}^2$ of metabelian (solvable of class 2) Lie algebras. We study the algebra $F_d(\mathfrak{A}^2)^{SL_2(K)}$ and describe the cases when it is finitely generated. This happens if and only if as an $SL_2(K)$-module $W_d \cong K^2 \oplus K \oplus \cdots \oplus K$ or $W_d \cong S^2(K^2)$ (and in the trivial case $KW_d \cong K \oplus \cdots \oplus K$). Here $SL_2(K)$ acts canonically on K^2, trivially on K, and $S^2(K^2)$ is the symmetric square of K^2. For small d we give a list of generators even when $F_d(\mathfrak{A}^2)^{SL_2(K)}$ is not finitely generated. The methods for establishing that the algebra $F_d(\mathfrak{A}^2)^{SL_2(K)}$ is not finitely generated work also for other relatively free algebras $F_d(\mathfrak{V})$ and for other groups G.

Keywords: Free metabelian Lie algebras, classical invariant theory, noncommutative invariant theory.

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