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Totally Geodesic Subalgebras of Nilpotent Lie Algebras

A metric Lie algebra $\mathfrak{g}$ is a Lie algebra equipped with an inner product. A subalgebra $\mathfrak{h}$ of a metric Lie algebra $\mathfrak{g}$ is said to be totally geodesic if the Lie subgroup corresponding to $\mathfrak{h}$ is a totally geodesic submanifold relative to the left-invariant Riemannian metric defined by the inner product, on the simply connected Lie group associated to $\mathfrak{g}$. A nonzero element of $\mathfrak{g}$ is called a geodesic if it spans a one-dimensional totally geodesic subalgebra. We give a new proof of Kaĭzer's theorem that every metric Lie algebra possesses a geodesic. For nilpotent Lie algebras, we give several results on the possible dimensions of totally geodesic subalgebras. We give an example of a codimension two totally geodesic subalgebra of the standard filiform nilpotent Lie algebra, equipped with a certain inner product. We prove that no other filiform Lie algebra possesses such a subalgebra. We show that in filiform nilpotent Lie algebras, totally geodesic subalgebras that leave invariant their orthogonal complements have dimension at most half the dimension of the algebra. We give an example of a 6-dimensional filiform nilpotent Lie algebra that has no totally geodesic subalgebra of dimension > 2, for any choice of inner product.

Keywords: Lie algebra, nilpotent, filiform, totally geodesic foliation.

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