Journal of Lie Theory 20 (2010) 709–737

H. Gündoğan
Fachbereich Mathematik, Technische Universität, Schlossgartenstr. 7, 64289 Darmstadt, Germany
guendogan@mathematik.tu-darmstadt.de

The Component Group of the Automorphism Group of a Simple Lie Algebra and the Splitting of the Corresponding Short Exact Sequence

Let $\mathfrak{g}$ be a simple Lie algebra of finite dimension over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathrm{Aut}(\mathfrak{g})$ the finite-dimensional Lie group of its automorphisms. We will calculate the component group $\pi_0(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})_0$ and the number of its conjugacy classes, and we will show that the corresponding short exact sequence

$$\mathbf{1} \to \mathrm{Aut}(\mathfrak{g})_0 \to \mathrm{Aut}(\mathfrak{g}) \to \pi_0(\mathrm{Aut}(\mathfrak{g})) \to \mathbf{1}$$

is split or, equivalently, there is an isomorphism $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$. Indeed, since $\mathrm{Aut}(\mathfrak{g})_0$ is open in $\mathrm{Aut}(\mathfrak{g})$, the quotient group $\pi_0(\mathrm{Aut}(\mathfrak{g}))$ is discrete. Hence a section $\pi_0(\mathrm{Aut}(\mathfrak{g})) \to \mathrm{Aut}(\mathfrak{g})$ is automatically continuous, giving rise to an isomorphism of Lie groups $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})_0 \rtimes \pi_0(\mathrm{Aut}(\mathfrak{g}))$.

Keywords: Automorphism group, simple, semisimple, Lie algebras splitting, semidirect product.

MSC: 17B20, 22E15