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Pairs of Lie Algebras and their Self-Normalizing Reductive Subalgebras

We consider a class $\mathcal{P}$ of pairs $(\mathfrak{g}, \mathfrak{g}_1)$ of **K**-Lie algebras $\mathfrak{g}_1 \subset \mathfrak{g}$ satisfying certain "rigidity conditions"; here **K** is a field of characteristic 0, $\mathfrak{g}$ is semisimple, and $\mathfrak{g}_1$ is reductive. We provide some further evidence that $\mathcal{P}$ contains a number of nonsymmetric pairs that are worth studying; e.g., in some branching problems, and for the purposes of the geometry of orbits. In particular, for an infinite series $(\mathfrak{g}, \mathfrak{g}_1) = (\mathfrak{sl}(n+1), \mathfrak{sl}(2))$ we show that it is in $\mathcal{P}$, and precisely describe a $\mathfrak{g}_1$-module structure of the Killing-orthogonal $\mathfrak{p}(n)$ of $\mathfrak{g}_1$ in $\mathfrak{g}$. Using this and the Kostant's philosophy concerning the exponents for (complex) Lie algebras, we obtain two more results. First; suppose **K** is algebraically closed, $\mathfrak{g}$ is semisimple all of whose factors are classical, and $\mathfrak{s}$ is a principal TDS. Then $(\mathfrak{g}, \mathfrak{s})$ belongs to $\mathcal{P}$. Second; suppose $(\mathfrak{g}, \mathfrak{g}_1)$ is a pair satisfying certain technical condition **C**, and there exists a semisimple $\mathfrak{s} \subseteq \mathfrak{g}_1$ such that $(\mathfrak{g}, \mathfrak{s})$ is from $\mathcal{P}$ (e.g., $\mathfrak{s}$ is a principal TDS). Then $(\mathfrak{g}, \mathfrak{g}_1)$ is from $\mathcal{P}$ as well. Finally, given a pair $(\mathfrak{g}, \mathfrak{g}_1)$, we have some useful observations concerning the relationship between the coadjoint orbits corresponding to $\mathfrak{g}$ and $\mathfrak{g}_1$, respectively.

Keywords: Pair of Lie algebras, semisimple Lie algebra, reductive subalgebra, self-normalizing subalgebra, principal nilpotent element, principal TDS, trivial extension.

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