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V. Cortés
Department Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany
cortes@math.uni-hamburg.de

L. Schäfer
Institut für Differentialgeometrie, Leibniz Universität, Welfengarten 1, 30167 Hannover, Germany
schaefer@math.uni-hannover.de

Geometric Structures on Lie Groups with Flat Bi-Invariant Metric

Let $L \subset V = \mathbb{R}^{k,l}$ be a maximally isotropic subspace. It is shown that any simply connected Lie group with a bi-invariant flat pseudo-Riemannian metric of signature (k,l) is 2-step nilpotent and is defined by an element $\eta \in \Lambda^3 L \subset \Lambda^3 V$. If η is of type $(3,0)+(0,3)$ with respect to a skew-symmetric endomorphism J with $J^2 = \varepsilon\,\mathrm{Id}$, then the Lie group $\mathcal{L}(\eta)$ is endowed with a left-invariant nearly Kähler structure if $\varepsilon = -1$ and with a left-invariant nearly para-Kähler structure if $\varepsilon = +1$. This construction exhausts all complete simply connected flat nearly (para-)Kähler manifolds. If $\eta \neq 0$ has rational coefficients with respect to some basis, then $\mathcal{L}(\eta)$ admits a lattice Γ, and the quotient $\Gamma \setminus \mathcal{L}(\eta)$ is a compact inhomogeneous nearly (para-)Kähler manifold. The first non-trivial example occurs in six dimensions.

Keywords: Flat Lie-groups, bi-invariant metrics, nearly para-Kähler manifolds, flat almost para-Hermitian manifolds, almost para-complex structures.

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