Journal of Lie Theory 16 (2006) 067–082

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# Kazhdan and Haagerup Properties in Algebraic Groups over Local Fields

Given a Lie algebra $\mathfrak{s}$, we call Lie $\mathfrak{s}$-algebra a Lie algebra endowed with a reductive action of $\mathfrak{s}$. We characterize the minimal $\mathfrak{s}$-Lie algebras with a nontrivial action of $\mathfrak{s}$, in terms of irreducible representations of $\mathfrak{s}$ and invariant alternating forms.
As a first application, we show that if $\mathfrak{g}$ is a Lie algebra over a field of characteristic zero whose amenable radical is not a direct factor, then $\mathfrak{g}$ contains a subalgebra which is isomorphic to the semidirect product of $\mathfrak{sl}_2$ by either a nontrivial irreducible representation or a Heisenberg group (this was essentially due to Cowling, Dorofaeff, Seeger, and Wright). As a corollary, if $G$ is an algebraic group over a local field $\mathbf{K}$ of characteristic zero, and if its amenable radical is not, up to isogeny, a direct factor, then $G(\mathbf{K})$ has Property (T) relative to a noncompact subgroup. In particular, $G(\mathbf{K})$ does not have Haagerup's property. This extends a similar result of Cherix, Cowling and Valette for connected Lie groups, to which our method also applies.
We give some other applications. We provide a characterization of connected Lie groups all of whose countable subgroups have Haagerup's property. We give an example of an arithmetic lattice in a connected Lie group which does not have Haagerup's property, but has no infinite subgroup with relative Property (T). We also give a continuous family of pairwise non-isomorphic connected Lie groups with Property (T), with pairwise non-isomorphic (resp. isomorphic) Lie algebras.

**Keywords**: Kazhdan's Property (T), Haagerup Property, a-T-menability.

**MSC**: 22E50; 22D10, 20G25, 17B05