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The Exponential Map and Differential Equations on Real Lie Groups

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, $\exp_{\mathrm{G}} : \mathfrak{g} \longrightarrow G$ the exponential map and $E(G)$ its range. $E^n(G)$ will denote the set of all n-fold products of elements of $E(G)$. G is called *exponential* if $E(G) = E^1(G) = G$. Since most real (or complex) connected Lie groups are not exponential, it is of interest to know that the weaker conclusion $E^2(G) = G$ is always true (Theorem 5.6). This result will be applied to prove Theorem 6.4, a generalized version of Floquet-Lyapunov theory for Lie groups. It will then be seen the property that a Lie group is exponential is equivalent to the existence of a special form of Floquet-Lyapunov theory for it (Corollary 6.3). Theorem **??**, generalizes the well-known fact that connected nilpotent Lie groups are exponential. Our methods also provide alternative proofs of some known results by arguments which seem simpler and more natural than the usual ones. Among these is part of the classical Dixmier-Saito result, Theorem 5.8.
The method employed here stems from the earliest techniques of Lie theory. It exploits connections between the exponential map and differential equations on G, starting from the observation that the one parameter subgroup $g(t) = \exp_{\mathrm{G}}(t\gamma)$ corresponding to an element $\gamma \in \mathfrak{g}$ satisfies the differential equations

$$g\,'(t) = \mathrm{dL}_{g(t)}\gamma \quad \text{and} \quad g\,'(t) = \mathrm{dR}_{g(t)}\gamma$$

on G with the initial condition $g(0) = \mathrm{e_G}$. More generally here it will be necessary to consider differential equations corresponding to certain time dependent vector fields on G, or equivalently, certain time dependent cross-sections of the tangent bundle of G.