Journal for Geometry and Graphics 18 (2014) 183–197

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Null Polarities as Generators of the Projective Group

It is well-known that the group of regular projective transformations of $\mathbb{P}^3(\mathbb{R})$ is isomorphic to the group of projective automorphisms of Klein's quadric $M_2^4 \subset \mathbb{P}^5(\mathbb{R})$. We introduce the Clifford algebra $Cl_{(3,3)}$ constructed over the quadratic space $\mathbb{R}^{(3,3)}$ and describe how points on Klein's quadric are embedded as null vectors, i.e., grade-1 elements squaring to zero. Furthermore, we discuss how geometric entities from Klein's model can be transferred to this homogeneous Clifford algebra model. Automorphic collineations of Klein's quadric can be described by the action of the so called sandwich operator applied to vectors $v \in \bigwedge^1 V$. Vectors correspond to null polarities in $\mathbb{P}^3(\mathbb{R})$. We introduce a factorization algorithm. With the help of this algorithm we are able to factorize an arbitrary versor $g \in Cl_{(3,3)}$ into a set of non-commuting vectors $v_i \in \bigwedge^1 V$, $i = 1, \dots, k$, $1 \leq k \leq 6$ corresponding to null polarities with $g = v_1 \dots v_k$. Thus, we present a new method to factorize every collineation in $\mathbb{P}^5(\mathbb{R})$ that is induced by a projective transformation acting on $\mathbb{P}^3(\mathbb{R})$ into a set of at most six involutoric automorphic collineations of Klein's quadric corresponding to null polarities respectively skew-symmetric 4×4 matrices. Moreover, we give an outlook for Lie's sphere geometry, i.e., the homogeneous Clifford algebra model constructed with the quadratic form corresponding to Lie's quadric $L_1^{n+1} \subset \mathbb{P}^{n+2}(\mathbb{R})$.

Keywords: Clifford algebra, line geometry, Klein's quadric, null polarity, factorization.

MSC: 51N15; 11E88, 51M30, 51B25