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A Generalization of a Classical Geometric Extremum Problem

Let $\partial\mathcal{C}$ be the boundary of a compact convex body $\mathcal{C}$ in $\mathbb{R}^n$, $n \geq 2$, and O be an interior point of $\mathcal{C}$. Every straight line l containing O cuts from $\mathcal{C}$ a segment $[AB]$ with end-points on $\partial\mathcal{C}$. It is shown that if $[AB]$ is the shortest such segment, then $\partial\mathcal{C}$ is smooth at the points A and B (i.e. at both of them there is only one supporting hyperplane for $\mathcal{C}$) and, something more, the normals to the unique supporting hyperplanes at the points A and B intersect at a point belonging to the hyperplane through O which is orthogonal to $[AB]$.

If $\mathcal{C}$ is a smooth compact convex body in $\mathbb{R}^n$, $n \geq 2$, the above property holds also when $[AB]$ is the longest such segment. Similar results are also valid when O is outside the set $\mathcal{C}$. The "local versions" of these results (when the length $|AB|$ of the segment $[AB]$ is locally maximal or locally minimal) are valid as well. More specific results are obtained in the particular case when $\mathcal{C}$ is a convex polytope.

Keywords: Convex set, extremal problem, Philo's line, critical point, convex polytope, concurrent lines.

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