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On Densely Complete Metric Spaces and Extensions of Uniformly Continuous Functions in ZF

A metric space $\mathbf{X}$ is called densely complete if there exists a dense set D in $\mathbf{X}$ such that every Cauchy sequence of points of D converges in $\mathbf{X}$. One of the main aims of this work is to prove that the countable axiom of choice, **CAC** for abbreviation, is equivalent to the following statements:

(1) Every densely complete (connected) metric space $\mathbf{X}$ is complete.

(2) For every pair of metric spaces $\mathbf{X}$ and $\mathbf{Y}$, if $\mathbf{Y}$ is complete and $\mathbf{S}$ is a dense subspace of $\mathbf{X}$, while $f\colon \mathbf{S} \to \mathbf{Y}$ is a uniformly continuous function, then there exists a uniformly continuous extension $F\colon \mathbf{X} \to \mathbf{Y}$ of f.

(3) Complete subspaces of metric spaces have complete closures.

(4) Complete subspaces of metric spaces are closed.

It is also shown that the restriction of (i) to subsets of the real line is equivalent to the restriction $\mathbf{CAC}(\mathbb{R})$ of **CAC** to subsets of $\mathbb{R}$. However, the restriction of (ii) to subsets of $\mathbb{R}$ is strictly weaker than $\mathbf{CAC}(\mathbb{R})$ because it is equivalent to the statement that $\mathbb{R}$ is sequential. Moreover, among other relevant results, it is proved that, for every positive integer n, the space $\mathbb{R}^n$ is sequential if and only if $\mathbb{R}$ is sequential. It is also shown that $\mathbb{R} \times \mathbb{Q}$ is not densely complete if and only if $\mathbf{CAC}(\mathbb{R})$ holds.

Keywords: Countable axiom of choice, complete metric spaces, connected metric spaces, sequential spaces.

MSC: 03E25, 54E35, 54E50, 54C20, 54D55.