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W. B. Moors
Department of Mathematics, The University of Auckland, Auckland 1142, New Zealand
w.moors@auckland.ac.nz

N. O. Tan
Department of Mathematics, The University of Auckland, Auckland 1142, New Zealand
neset.tan@auckland.ac.nz

An Abstract Variational Theorem

Let $(X, \|\cdot\|)$ be a Banach space and $f\colon X \to \mathbb{R} \cup \{\infty\}$ be a proper function. Then the *Fenchel conjugate of* f is the function $f^*\colon X^* \to \mathbb{R} \cup \{\infty\}$ defined by,

$$f^*(x^*) := \sup\{(x^* - f)(x) : x \in X\}.$$

In this article we will prove a theorem more general than the following.

Theorem: Let $f\colon X \to \mathbb{R} \cup \{\infty\}$ be a proper function on a Banach space $(X, \|\cdot\|)$. If there is a nonempty open subset A of $\mathrm{Dom}(f^*)$ such that $\mathrm{argmax}(x^* - f) \neq \varnothing$ for each $x^* \in A$, then there is a dense and G_δ subset R of A such that $(x^* - f)\colon X \to \mathbb{R} \cup \{-\infty\}$ has a strong maximum for each $x^* \in R$. In addition, if $0 \in A$ and $0 < \varepsilon$ then there is an $x^* \in X^*$ with $\|x^*\| < \varepsilon$ such that $(x^* - f)\colon X \to \mathbb{R} \cup \{-\infty\}$ has a strong maximum.

Keywords: Variational theorem, James' weak compactness theorem, convex analysis.

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