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Fenchel-Moreau Conjugation Inequalities with Three Couplings and Application to the Stochastic Bellman Equation

Given two couplings between "primal" and "dual" sets, we prove a general implication that relates an inequality involving "primal" sets to a reverse inequality involving the "dual" sets. More precisely, let be given two "primal" sets $\mathbb{X}$, $\mathbb{Y}$ and two "dual" sets $\mathbb{X}^\sharp$, $\mathbb{Y}^\sharp$, together with two coupling functions $\mathbb{X} \overset{c}{\leftrightarrow} \mathbb{X}^\sharp$ and $\mathbb{Y} \overset{d}{\leftrightarrow} \mathbb{Y}^\sharp$. We define a new coupling $c \dotplus d$ between the "primal" product set $\mathbb{X} \times \mathbb{Y}$ and the "dual" product set $\mathbb{X}^\sharp \times \mathbb{Y}^\sharp$.
Then, we consider any bivariate function $\mathcal{K} \colon \mathbb{X} \times \mathbb{Y} \to [-\infty, +\infty]$ and univariate functions $f \colon \mathbb{X} \to [-\infty, +\infty]$ and $g \colon \mathbb{Y} \to [-\infty, +\infty]$, all defined on the "primal" sets. We prove that

$$f(x) \geq \inf_{y \in \mathbb{Y}} \big(\mathcal{K}(x,y) \dotplus g(y) \big) \;\Rightarrow\; f^{c}(x^\sharp) \leq \inf_{y^\sharp \in \mathbb{Y}^\sharp} \big(\mathcal{K}^{c \dotplus d}(x^\sharp, y^\sharp) \dotplus g^{-d}(y^\sharp) \big),$$

where we stress that the Fenchel-Moreau conjugates f^c and g^{-d} are not necessarily taken with the same coupling.
We study the equality case. We display several applications. We provide a new formula for the Fenchel-Moreau conjugate of a generalized inf-convolution. We obtain formulas with partial Fenchel-Moreau conjugates. Finally, we consider the Bellman equation in stochastic dynamic programming and we provide a "Bellman-like" inequation for the Fenchel conjugates of the value functions.

Keywords: Fenchel-Moreau conjugacy, coupling, envelope function, generalized inf-convolution, Bellman equation.

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