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New Variational Principles of Symmetric Boundary Value Problems

The objective of this paper is to establish new variational principles for symmetric boundary value problems. Let V be a Banach space and V^* its topological dual. We shall consider problems of the type $\Lambda u = D\Phi(u)$ where $\Lambda : V \to V^*$ is a linear operator and $\Phi : V \to \mathbb{R}$ is a Gâteaux differentiable convex function whose derivative is denoted by $D\Phi$. It is established that solutions of the latter equation are associated with critical points of functions of the type

$$I_{\lambda,\mu}(u) := \mu\Phi^*(\Lambda u) - \lambda\Phi(u) - \frac{\mu - \lambda}{2}\langle \Lambda u, u\rangle,$$

where λ, μ are two real numbers, Φ^* is the Fenchel dual of the function Φ and $\langle ., .\rangle$ is the duality pairing between V and V^*. By assigning different values to λ and μ one obtains variety of new and classical variational principles associated to the equation $\Lambda u = D\Phi(u)$. Namely, Euler-Lagrange principle (for $\mu = 0$, $\lambda = 1$ and symmetric Λ), Clarke-Ekeland least action principle (for $\mu = 1$, $\lambda = 0$ and symmetric Λ), Brezis-Ekeland variational principle ($\mu = 1$, $\lambda = -1$) and of course many new variational principles such as

$$I_{1,1}(u) = \Phi^*(\Lambda u) - \Phi(u),$$

which corresponds to $\lambda = 1$ and $\mu = 1$. These new potential functions are quite flexible, and can be adapted to easily deal with both nonlinear and homogeneous boundary value problems.