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Free Convex Sets Defined by Rational Expressions Have LMI Representations

Suppose p is a symmetric matrix whose entries are polynomials in freely noncommutative variables and $p(0)$ is positive definite. Let $\mathcal{D}_p$ denote the component of zero of the set of those g-tuples $X = (X_1, \ldots, X_g)$ of symmetric matrices (of the same size) such that $p(X)$ is positive definite. In another paper of the authors [*Every free convex basic semi-algebraic set has an LMI representation*, Annals of Mathematics, to appear] it was shown that if $\mathcal{D}_p$ is convex and bounded, then $\mathcal{D}_p$ can be described as the set of solutions of a linear matrix inequality (LMI). This article extends that result from matrices of polynomials to matrices of rational functions in free variables.
As a refinement of a theorem of Kaliuzhnyi-Verbovetskyi and Vinnikov, it is also shown that a minimal symmetric descriptor realization r for a symmetric free matrix-valued rational function $\mathfrak{r}$ in g freely noncommuting variables $x = (x_1, \ldots, x_g)$ precisely encodes the singularities of the rational function. This singularities result is an important ingredient in the proof of the LMI representation theorem stated above.

Keywords: Matrix convexity, free convexity, linear matrix inequality, noncommutative rational function, free rational function.

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