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J. Kąkol
Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznan, Poland
kakol@amu.edu.pl

A. Kubzdela
Institute of Civil Engineering, University of Technology, 61-138 Poznan, Poland
albert.kubzdela@put.poznan.pl

Non-Archimedean Quantitative Grothendieck and Krein's Theorems

We show that the non-archimedean version of Grothendieck's theorem about weakly compact sets for $C(X,\mathbb{K})$, the space of continuous maps on X with values in a locally compact non-trivially valued non-archimedean field $\mathbb{K}$, fails in general. Indeed, we prove that if X is an infinite zero-dimensional compact space, then there exists a relatively compact set $H := \{g_n : n \in \mathbb{N}\} \subset C(X,\mathbb{K})$ in the pointwise topology τ_p of $C(X,\mathbb{K})$ which is not $w-$relatively compact, i.e. compact in the weak topology of $C(X,\mathbb{K})$, such that all $\|g_n\| = 1$ and $\gamma(H) := \sup\{|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| : (f_m)_m \subset B, (x_n)_n \subset H\} > 0$, where B is the closed unit ball in the dual $C(X,\mathbb{K})^*$ and the involved limits exist. The latter condition $\gamma(H) > 0$ shows in fact that a quantitative version of Grothendieck's theorem for real spaces (due to Angosto and Cascales) fails in the non-archimedean setting. The classical Krein and Grothendieck's theorems ensure that for any compact space X every uniformly bounded set H in a real (or complex) space $C(X)$ is τ_p-relatively compact if and only if the absolutely convex hull $acoH$ of H is τ_p-relatively compact. In contrast, we show that for an infinite zero-dimensional compact space X the absolutely convex hull $acoH$ of a τ_p-relatively compact and uniformly bounded set H in $C(X,\mathbb{K})$ needs not be τ_p-relatively compact for a locally compact non-archimedean $\mathbb{K}$. Nevertheless, our main result states that if $H \subset C(X,\mathbb{K})$ is uniformly bounded, then $acoH$ is τ_p-relatively compact if and only if H is w-relatively compact.

Keywords: Grothendieck's theorem, Krein's theorem, locally compact non-archimedean field, compactness, space of continuous functions.

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