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Convex Integrals on Sobolev Spaces

Let $j_0, j_1 : \mathbb{R} \mapsto [0, \infty)$ denote convex functions vanishing at the origin, and let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary Γ. This paper is devoted to the study of the convex functional

$$J(u) = \int_\Omega j_0(u)d\Omega + \int_\Gamma j_1(\gamma u)d\Gamma$$

on the Sobolev space $H^1(\Omega)$. We describe the convex conjugate J^* and the subdifferential ∂J. It is shown that the action of ∂J coincides pointwise a.e. in Ω with $\partial j_0(u(x))$, and a.e on Γ with $\partial j_1(u(x))$. These conclusions are non-trivial because, although they have been known for the subdifferentials of the functionals $J_0(u) = \int_\Omega j_0(u)d\Omega$ and $J_1(u) = \int_\Gamma j_1(\gamma u)d\Gamma$, the lack of any growth restrictions on j_0 and j_1 makes the sufficient *domain condition* for the sum of two maximal monotone operators ∂J_0 and ∂J_1 infeasible to verify directly. The presented theorems extend the results of H. Brézis [Intégrales convexes dans les espaces de Sobolev, Proc. Int. Symp. Partial Diff. Equations and the Geometry of Normed Linear Spaces, Jerusalem 1972, vol. 13 (1972) 9–23 (1973); MR 0341077 (49#5827)] and fundamentally complement the emerging research literature addressing supercritical damping and sources in hyperbolic PDE's. These findings rigorously confirm that a *combination* of supercritical interior and boundary damping feedbacks can be modeled by the subdifferential of a suitable convex functional on the state space.