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On the Lower Semicontinuous Quasiconvex Envelope for Unbounded Integrands (II): Representation by Generalized Controls

[For the first part of this paper see ESAIM, Control, Optimisation and Calculus of Variations.]

Motivated by the study of multidimensional control problems of Dieudonné-Rashevsky type, e.g. nonconvex correspondence problems from image processing, we raise the question how to understand to notion of quasiconvexity for a continuous function f with a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ instead of the whole space $\mathbb{R}^{nm}$ as the range of definition. Extending f by $(+\infty)$ to the complement $\mathbb{R}^{nm} \setminus \mathrm{K}$, the appropriate quasiconvex envelope turns out to be

$$f^{(qc)}(w) = \sup \big\{ g(w) \,\big|\, g\colon \mathbb{R}^{nm} \to \mathbb{R} \cup \{(+\infty)\} \text{ quasiconvex and lower semicontinuous, } g(v) \le f(v)\ \forall v \in \mathbb{R}^{nm} \big\}.$$

In the present paper, we prove that $f^{(qc)}$ admits a representation as

$$f^{(qc)}(w) = \mathrm{Min}\, \big\{ \textstyle\int_{\mathrm{K}} f(v)\, d\nu(v) \,\big|\, \nu \in \mathrm{S}^{(qc)}(w) \big\} \quad \forall w \in \mathrm{K}$$

where the sets $\mathrm{S}^{(qc)}(w)$ are nonempty, convex, weak*-sequentially compact subsets of probability measures. This theorem, forming a natural counterpart to the author's previous results about the representation of $f^{(qc)}$ in terms of Jacobi matrices, has been proven indispensable for the derivation of Jensens' integral inequality as well as of differentiability theorems for the envelope $f^{(qc)}$. The paper is mainly concerned with a detailed analysis of the set-valued map $\mathrm{S}^{(qc)}$, which will be explicitely described in terms of averages of generalized controls.

Keywords: Unbounded function, quasiconvex envelope, probability measure, generalized control, mean value theorem, set-valued map, representation theorem.

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