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Lagrangian Manifolds, Viscosity Solutions and Maslov Index

Let M be a Lagrangian manifold, let the 1-form pdx be globally exact on M and let $S(x,p)$ be defined by $dS = pdx$ on M. Let $H(x,p)$ be convex in p for all x and vanish on M. Let $V(x) = \inf\{S(x,p) : p \text{ such that } (x,p) \in M\}$. Recent work in the literature has shown that (i) V is a viscosity solution of $H(x, \partial V/\partial x) = 0$ provided V is locally Lipschitz, and (ii) V is locally Lipschitz outside the set of caustic points for M. It is well known that this construction gives a viscosity solution for finite time variational problems – the Lipschitz continuity of V follows from that of the initial condition for the variational problem. However, this construction also applies to infinite time variational problems and stationary Hamilton-Jacobi-Bellman equations where the regularity of V is not obvious. We show that for $\dim M \leq 5$, the local Lipschitz property follows from some geometrical assumptions on M – in particular that the Maslov index vanishes on closed curves on M. We obtain a local Lipschitz constant for V which is some uniform power of a local bound on M, the power being determined by $\dim M$. This analysis uses Arnold's classification of Lagrangian singularities.