IX

Subobjects, Quotient Objects, and Factorizations

He thought he saw a Garden-Door
That opened with a key;
He looked again, and found it was
A Double Rule of Three.
"And all its mystery," he said,
"Is clear as day to me!"
——Lewis Carroll†

In §17 we have already seen that if a category $\mathscr{C}$ has "sufficiently nice" smallness and completeness properties (e.g., if it is well-powered, has intersections, and is finitely complete), then it is (extremal epi, mono)-factorizable. In this chapter we will show that such a category is actually *uniquely* (extremal epi, mono)-factorizable and, moreover, that it is also uniquely (epi, extremal mono)-factorizable. Putting these results together will show that each morphism in such a category has an essentially unique three-fold (extremal epi, bi, extremal mono)-factorization. It should be noted that the (extremal epi, mono)-factorization is the one that is usually considered in algebraic categories, but is considered to be of only limited interest in categories such as **Top** or **POS**. In the latter categories, the interesting factorizations are the (epi, extremal mono)-factorizations.

In order to study these two kinds of distinguished factorizations that exist in each "reasonable" category, we will begin with a study of general $(\mathscr{E}, \mathscr{M})$-factorizations.

§33 $(\mathscr{E}, \mathscr{M})$ CATEGORIES

Throughout this section let $\mathscr{E}$ be a class of epimorphisms which is closed under composition with isomorphisms and let $\mathscr{M}$ be a class of monomorphisms that is closed under composition with isomorphisms.

† From *Alice in Wonderland.*

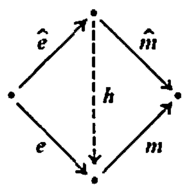
Recall that an $(\mathcal{E}, \mathcal{M})$-factorization of a morphism f is a factorization

$$\bullet \xrightarrow{f} \bullet = \bullet \xrightarrow{e} \bullet \xrightarrow{m} \bullet$$

where $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Such a factorization is called *unique* provided that whenever

$$\bullet \xrightarrow{f} \bullet = \bullet \xrightarrow{\hat{e}} \bullet \xrightarrow{\hat{m}} \bullet$$

is also an $(\mathcal{E}, \mathcal{M})$-factorization of f, there is an isomorphism h such that the diagram



commutes. Recall also that a category $\mathcal{C}$ is called (uniquely) $(\mathcal{E}, \mathcal{M})$-factorizable provided that each of its morphisms has a (unique) $(\mathcal{E}, \mathcal{M})$-factorization.

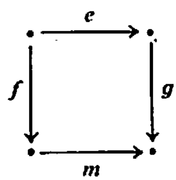
33.1 DEFINITION

A category $\mathcal{C}$ is called an **$(\mathcal{E}, \mathcal{M})$ category** provided that it is uniquely $(\mathcal{E}, \mathcal{M})$-factorizable and both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

The following property will turn out to be crucial in the study of factorizations.

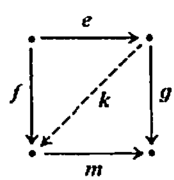
33.2 DEFINITION

A category $\mathcal{C}$ is said to have the **$(\mathcal{E}, \mathcal{M})$-diagonalization property** provided that for every commutative square in $\mathcal{C}$



with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,

there exists a morphism k that makes the diagram



commute.

33.3 THEOREM

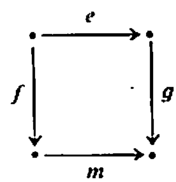
For any category $\mathcal{C}$, the following are equivalent:

(1) *$\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category.*

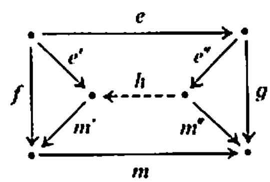
(2) *$\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-factorizable and has the $(\mathcal{E}, \mathcal{M})$-diagonalization property.*

Proof:

(*1*) ⇒ (*2*). Let



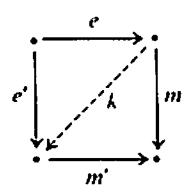
be a commutative square with $e \in \mathscr{E}$ and $m \in \mathscr{M}$. Let $f = m' \circ e'$ and $g = m'' \circ e''$ be $(\mathscr{E}, \mathscr{M})$-factorizations. Then $m'' \circ (e'' \circ e)$ and $(m \circ m') \circ e'$ are $(\mathscr{E}, \mathscr{M})$-factorizations of $g \circ e$. Thus, by uniqueness, there exists an isomorphism h such that the diagram



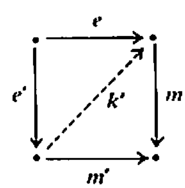
commutes.

Hence, $m' \circ h \circ e''$ is the desired diagonal morphism.

(*2*) ⇒ (*1*). If $f = m \circ e = m' \circ e'$ are $(\mathscr{E}, \mathscr{M})$-factorizations of f, then there exist morphisms k and k' such that the diagrams

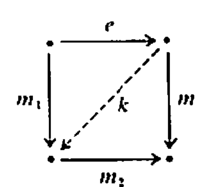


and



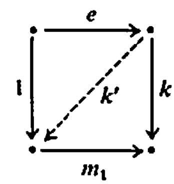
commute.

Thus $(k' \circ k) \circ e = k' \circ e' = e = 1 \circ e$, so that since e is an epimorphism, $k' \circ k = 1$; i.e., k' is a retraction. However, k' (being the first factor of a monomorphism) is also a monomorphism. Hence, k' is an isomorphism. Thus $\mathscr{C}$ is uniquely $(\mathscr{E}, \mathscr{M})$-factorizable. To show that $\mathscr{M}$ is closed under compositions, suppose that m_1 and m_2 belong to $\mathscr{M}$. If $m \circ e = m_2 \circ m_1$ is an $(\mathscr{E}, \mathscr{M})$-factorization of $m_2 \circ m_1$, then there exists a morphism k such that the diagram



commutes.

Likewise there exists a morphism k' such that the diagram



commutes.

Thus $k' \circ e = 1$, so that e is a section and an epimorphism, hence an isomorphism. Consequently, since $\mathscr{M}$ is closed under composition with isomorphisms, $m_2 \circ m_1 \in \mathscr{M}$. That $\mathscr{E}$ is closed under composition follows dually. □

33.4 PROPOSITION

For any category $\mathscr{C}$, the following are equivalent:

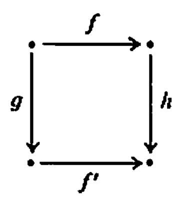
(1) *$\mathscr{C}$ is a (regular epi, mono) category.*

(2) *$\mathscr{C}$ is (regular epi, mono)-factorizable.*

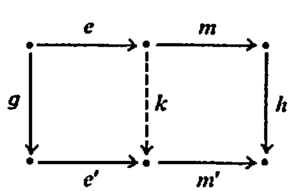
Proof: This follows immediately from the above theorem (33.3) and the fact that every category has the (regular epi, mono)-diagonalization property (17.17). □

33.5 PROPOSITION

If $\mathscr{C}$ is an $(\mathscr{E}, \mathscr{M})$ category, then the $(\mathscr{E}, \mathscr{M})$-factorizations are functorial in the following sense:
If



is a commutative square and $f = m \circ e$ and $f' = m' \circ e'$ are $(\mathscr{E}, \mathscr{M})$-factorizations of f and f', then there exists a unique morphism k such that the diagram



commutes.

Proof: By the above theorem (33.3), $\mathscr{C}$ has the $(\mathscr{E}, \mathscr{M})$-diagonalization property. □

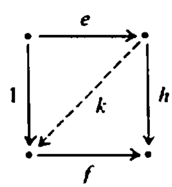
33.6 THEOREM

If $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category, then $\mathcal{E}$ and $\mathcal{M}$ uniquely determine each other; in particular,

(1) $\mathcal{M} = \{f \in Mor\,\mathcal{C} \mid$ *if* $f = h \circ e$ *and* $e \in \mathcal{E}$, *then* e *is an isomorphism*$\}$,
and

(2) $\mathcal{E} = \{f \in Mor\,\mathcal{C} \mid$ *if* $f = m \circ e$ *and* $m \in \mathcal{M}$, *then* m *is an isomorphism*$\}$.

Proof: By duality we need only to prove (1). Suppose that $f \in \mathcal{M}$ and $f = h \circ e$ where $e \in \mathcal{E}$. Since $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category, it has the $(\mathcal{E}, \mathcal{M})$-diagonalization property (33.3). Thus, there exists a morphism k such that the diagram



commutes.

Consequently, e is a section and an epimorphism; hence an isomorphism. On the other hand, suppose that f has the property that whenever $f = h \circ e$, with $e \in \mathcal{E}$, then e must be an isomorphism. Then for the $(\mathcal{E}, \mathcal{M})$-factorization $f = m \circ e$ of f, e is an isomorphism. Thus, since $\mathcal{M}$ is closed under composition with isomorphisms, f is in $\mathcal{M}$. □

33.7 COROLLARY

Let $\mathcal{C}$ be an $(\mathcal{E}, \mathcal{M})$ category.

(1) *If $\mathcal{E}$ is the class of all epimorphisms in $\mathcal{C}$, then $\mathcal{M}$ is the class of all extremal monomorphisms in $\mathcal{C}$, and*

(2) *If $\mathcal{M}$ is the class of all monomorphisms in $\mathcal{C}$, then $\mathcal{E}$ is the class of all extremal epimorphisms in $\mathcal{C}$.* □

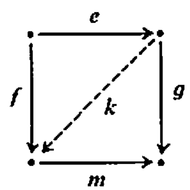
In this section we have studied $(\mathcal{E}, \mathcal{M})$ categories in general. In the next section (§34), we will show that each "reasonable" category is simultaneously an (extremal epi, mono) category and an (epi, extremal mono) category. In the last chapter (§39) we will show that a pointed category is a (normal epi, normal mono) category if and only if it is "exact". In exact categories the techniques involving exact sequences that are available in categories such as R-**Mod** will be at our disposal.

EXERCISES

33A. Show that each of **Set**, **Grp**, and **Top** is an (epi, extremal mono) category, an (epi, regular mono) category, an (extremal epi, mono) category, and a (regular epi, mono) category.

33B. Show that none of the categories **SGrp**, **Rng**, **POS**, **Top**, or $\mathbf{Top}_2$ is an (epi, mono) category.

33C. In the definition of $(\mathscr{E}, \mathscr{M})$-diagonalization property (33.2), prove that the morphism k in the diagram



must be unique, and also that commutativity of the upper triangle is sufficient to guarantee commutativity of the lower triangle, and vice versa.

33D. Prove that if $\mathscr{C}$ has the unique (epi, $\mathscr{M}$)-factorization property, then $\mathscr{M}$ is precisely the class of extremal monomorphisms in $\mathscr{C}$.

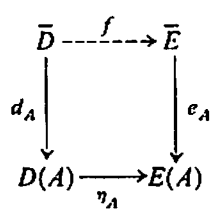
33E. Show that the fact that for an $(\mathscr{E}, \mathscr{M})$ category, $\mathscr{C}$, the $(\mathscr{E}, \mathscr{M})$-factorizations are functorial means that the factorizations may be interpreted as a functor $F: \mathscr{C}^2 \to \mathscr{C}^3$.

33F. Let $\mathscr{C}$ be an $(\mathscr{E}, \mathscr{M})$ category.

(a) Prove that $\mathscr{E} \cap \mathscr{M}$ is precisely the class of all isomorphisms in $\mathscr{C}$.

(b) Prove that if $f \circ g \in \mathscr{M}$, then $g \in \mathscr{M}$.

(c) Let $\mathscr{A} \overset{D}{\underset{E}{\rightrightarrows}} \mathscr{C}$ be functors with limits $(\bar{D}, d_A)$ and $(\bar{E}, e_A)$ respectively, let $\eta = (\eta_A): D \to E$ be a natural transformation, and let f be the unique morphism that makes the diagram



commute.

Prove that if each η_A is in $\mathscr{M}$, then f is also in $\mathscr{M}$.

(d) Prove that $\mathscr{M}$ is closed under the formation of products, pullbacks, and intersections of $\mathscr{M}$-subobjects [see 34.2].

(e) Prove that $\mathscr{M}$ contains the class of all extremal monomorphisms in $\mathscr{C}$.

(f) Provide the statements corresponding to (b), (c), (d), and (e) for the class $\mathscr{E}$.

33G. Among the morphisms in the category $\mathbf{Top}_2$ of Hausdorff spaces and continuous maps, let

$\mathscr{E}_1 = \{f \mid f \text{ is dense}\}$ [$= \{f \mid f$ is an epimorphism$\}$ (6.10(4))].

$\mathscr{M}_1 = \{f \mid f \text{ is a closed embedding}\}$ [$= \{f \mid f$ is an extremal monomorphism$\}$ (17.10(3))].

$\mathscr{E}_2 = \{f \mid f \text{ is surjective}\}$.

$\mathscr{M}_2 = \{f \mid f \text{ is an embedding}\}$.

$\mathscr{E}_3 = \{f \mid f \text{ is a quotient map}\}$ [$= \{f \mid f$ is an extremal epimorphism$\}$ (17.10(3))].

$\mathscr{M}_3 = \{f \mid f \text{ is injective}\}$ [$= \{f \mid f$ is a monomorphism$\}$ (6.3(2))].

Show that for $i = 1, 2, 3$, $\mathbf{Top}_2$ is an $(\mathscr{E}_i, \mathscr{M}_i)$ category.

33H. In $\mathbf{CRegT}_2$ let $\hat{\mathscr{E}}$ be the class of all dense, compact-extendable functions (see Definition 37.8) and let $\hat{\mathscr{M}}$ be the class of all perfect maps. Show that $\mathbf{CRegT}_2$ satisfies all of the conditions for being an $(\hat{\mathscr{E}}, \hat{\mathscr{M}})$ category except that $\hat{\mathscr{M}}$ does not consist of monomorphisms alone.

33I. Determine which of the results of this section (§33) remain valid if one drops the condition that $\mathscr{E}$ (resp. $\mathscr{M}$) consists only of epimorphisms (resp. monomorphisms).

33J. Prove that any category that has pushouts has the (epi, extremal mono)-diagonalization property.

33K. Show that each algebraic category is a (regular epi, mono) category.

33L. Let $\mathscr{C}$ be a category that has pullbacks and coequalizers. Prove that the following are equivalent:

(a) The class of regular epimorphisms in $\mathscr{C}$ is closed under composition.
(b) $\mathscr{C}$ is a (regular epi, mono) category [see Exercise 21O].

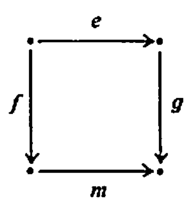
§34 (EPI, EXTREMAL MONO) AND (EXTREMAL EPI, MONO) CATEGORIES

In §17 we have shown that a well-powered category $\mathscr{C}$ that has intersections and equalizers is (extremal epi, mono)-factorizable. Next we will show that if $\mathscr{C}$ also has pullbacks, it is even an (extremal epi, mono) category.

34.1 THEOREM

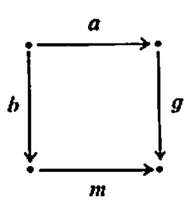
If $\mathscr{C}$ is well-powered, finitely complete, and has intersections, then $\mathscr{C}$ is an (extremal epi, mono) category.

Proof: Using earlier results (17.16 and 33.3), we need only show that $\mathscr{C}$ has the (extremal epi, mono)-diagonalization property. Let

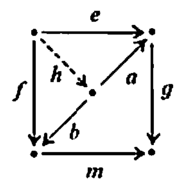


be a commutative square, where e is an extremal epimorphism and m is a monomorphism.

Let



be the pullback of m and g. Now a is a monomorphism since m is (21.13). Since the square is a pullback, there exists a unique morphism h such that the diagram



commutes.

Since e is an extremal epimorphism, a must be an isomorphism. Thus $b \circ a^{-1}$ is the desired diagonal morphism. □

Our attention now focuses on showing that each category satisfying the hypotheses of the above theorem is also an (epi, extremal mono) category. The next proposition indicates the crucial role of the (epi, extremal mono)-diagonalization property. The conclusions of the proposition should be compared with the analogous results for monomorphisms—6.4, 17.3, 21.13, and 18.16.

34.2 PROPOSITION

If $\mathscr{C}$ has the (epi, extremal mono)-diagonalization property, then in $\mathscr{C}$:

(1) *The composition of extremal monomorphisms is an extremal monomorphism.*

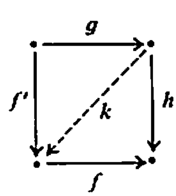
(2) *The intersection of extremal subobjects is an extremal subobject.*

(3) *The inverse image (pullback) of an extremal monomorphism is an extremal monomorphism.*

(4) *The product of extremal monomorphisms is an extremal monomorphism.*

Proof: Since monomorphisms are closed under composition, intersections, inverse images, and products, in each case we need only verify that the extremal condition (17.9(1)(ii) dual) is satisfied.

(1). If f and f' are extremal monomorphisms and $f \circ f' = h \circ g$ where g is an epimorphism, then by the (epi, extremal mono)-diagonalization property, there exists a morphism k such that the diagram

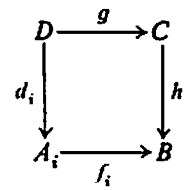


commutes.

Thus $f' = k \circ g$, where g is an epimorphism, so that g is an isomorphism.

(2). Let $(A_i, f_i)_I$ be a family of extremal subobjects of B and let (D, d) be their intersection, where for each i, $d = f_i \circ d_i$. If $d = h \circ g$, where g is an

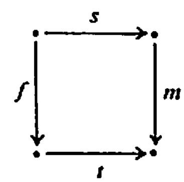
epimorphism, then for each i, the diagram



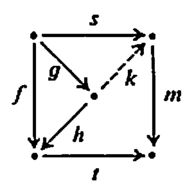
commutes.

By the diagonalization property, for each i there exists a morphism $k_i: C \to A_i$ such that $d_i = k_i \circ g$. Hence since g is an epimorphism and since the intersection, being a limit, is an (extremal mono)-source (20.4), g must be an isomorphism.

(3). Let



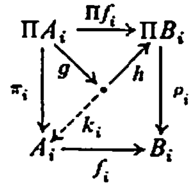
be a pullback square, where m is an extremal monomorphism. We wish to show that f is also an extremal monomorphism. If $f = h \circ g$, where g is an epimorphism, then by the diagonalization property, there exists a morphism k such that the diagram



commutes.

Thus, since pullbacks, being limits, are (extremal mono)-sources, g must be an isomorphism.

(4). If $(A_i \xrightarrow{f_i} B_i)$ is a family of extremal monomorphisms, $\Pi A_i \xrightarrow{\Pi f_i} \Pi B_i$ is their product, and $\Pi f_i = h \circ g$ where g is an epimorphism, then by the diagonalization property, for each i there is a morphism k_i such that the diagram



commutes.

Since products, being limits, are (extremal mono)-sources, g must be an isomorphism. ☐

34.3 FACTORIZATION LEMMA

Let $\mathcal{C}$ be a well-powered category that has intersections and equalizers. Let $X \xrightarrow{f} Y$ be a $\mathcal{C}$-morphism, and let $\mathcal{M}$ be a class of $\mathcal{C}$-monomorphisms with the following properties:

(1) *$\mathcal{M}$ is closed under intersections; i.e., if (D, d) is the intersection of a non-empty family $(A_i, m_i)_I$ of $\mathcal{M}$-subobjects, then $d \in \mathcal{M}$.*

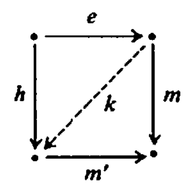
(2) *If $f = m \circ q \circ h$, where $m \in \mathcal{M}$ and q is a regular monomorphism, then $m \circ q \in \mathcal{M}$.*

(3) $1_Y \in \mathcal{M}$.

Then there exist morphisms m and e such that $m \in \mathcal{M}$, e is an epimorphism, and

(*i*) $f = m \circ e$.

(*ii*) *if $f = m' \circ h$, where $m' \in \mathcal{M}$, then there exists a morphism k such that the diagram*



commutes.

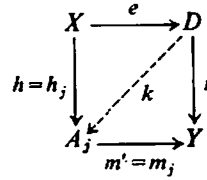
(*iii*) *if $e = \overline{m} \circ g$, where $m \circ \overline{m} \in \mathcal{M}$, then $\overline{m}$ is an isomorphism.*

Proof: Since $\mathcal{C}$ is well-powered, there exists a set-indexed family

$$\{X \xrightarrow{h_i} A_i \xrightarrow{m_i} Y\}_I$$

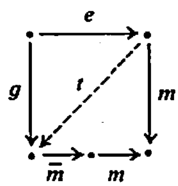
of factorizations of f, where each $m_i \in \mathcal{M}$ and $(A_i, m_i)_I$ is a representative class of all $\mathcal{M}$-subobjects of Y through which f factors. Let (D, m) be the intersection of $(A_i, m_i)_I$ (which by (3) is non-empty). By (1), $m \in \mathcal{M}$. By the definition of intersection, there exists a unique morphism e such that $f = m \circ e$.

If $f = m' \circ h$, where $m' \in \mathcal{M}$, then without loss of generality we can assume that there is some $j \in I$ with $m' = m_j$ and $h = h_j$. Since (D, m) is the intersection of $(A_i, m_i)_I$, there exists a morphism k such that $m = m_j \circ k$. Hence the diagram



commutes, so that (ii) holds.

If $e = \bar{m} \circ g$ where $m \circ \bar{m} \in \mathcal{M}$, then by (ii) there exists a morphism t such that



commutes.

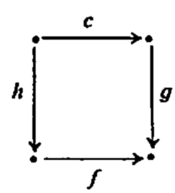
Hence $m \circ \bar{m} \circ t = m = m \circ 1$, so that since m is a monomorphism, $\bar{m} \circ t = 1$. Thus $\bar{m}$ is a retraction, and since it is the first factor of a monomorphism, it is a monomorphism. Consequently $\bar{m}$ is an isomorphism, so that (iii) is established.

To complete the proof, we need only show that e is an epimorphism. Suppose that $r \circ e = s \circ e$. Let (Q, q) be the equalizer of r and s. By the definition of equalizer, there exists a morphism h such that $e = q \circ h$. Hence $f = m \circ q \circ h$. By (2) $m \circ q \in \mathcal{M}$, so that by (iii) q is an isomorphism. Thus $r = s$. □

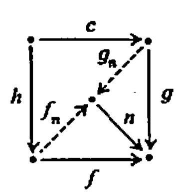
34.4 PROPOSITION

Every well-powered category $\mathscr{C}$ that has intersections and equalizers also has the (epi, extremal mono)-diagonalization property.

Proof: Let

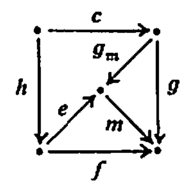


be a commutative diagram, where c is an epimorphism and f is an extremal monomorphism. Let $\mathcal{M}$ be the class of all $\mathscr{C}$-monomorphisms n with the property that there exist morphisms f_n and g_n such that the diagram



commutes.

It is a straightforward exercise to show that $\mathcal{M}$ satisfies conditions (1), (2), and (3) of the Factorization Lemma (34.3), so that there exists an (epi, $\mathcal{M}$)-factorization $f = m \circ e$ of f. Since m is a monomorphism, $f_m = e$. Hence the diagram



commutes.

Now f is an extremal monomorphism, so that since e is an epimorphism, e must be an isomorphism. Thus $e^{-1} \circ g_m$ is the desired diagonal morphism. □

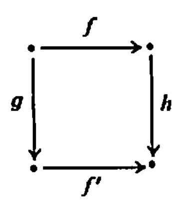
34.5 THEOREM

Every well-powered category $\mathscr{C}$ that has intersections and equalizers is an (epi, extremal mono) category.

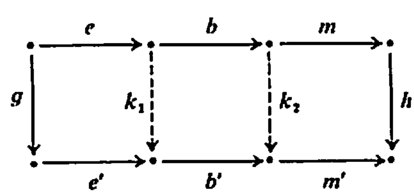
Proof: By the above proposition (34.4) $\mathscr{C}$ has the (epi, extremal mono)-diagonalization property. Thus by parts (1) and (2) of Proposition 34.2 and the fact that each regular monomorphism is an extremal monomorphism (17.11 dual), it follows that the class of all extremal monomorphisms of $\mathscr{C}$ satisfies the three hypotheses of the Factorization Lemma (34.3). Hence, by that lemma, $\mathscr{C}$ is (epi, extremal mono)-factorizable; and so it is an (epi, extremal mono) category (33.3). □

34.6 PROPOSITION

If $\mathscr{C}$ is a well-powered, finitely complete category that has intersections, then each $\mathscr{C}$-morphism f has a factorization (which is unique up to isomorphisms) of the form $f = m \circ b \circ e$, where e is an extremal epimorphism, b is a bimorphism, and m is an extremal monomorphism. Furthermore, this three-fold factorization is functorial in the sense that if



is a commutative square and $f = m \circ b \circ e$ and $f' = m' \circ b' \circ e'$ are the three-fold factorizations of f and f', then there exist unique morphisms k_1 and k_2 such that the diagram



commutes.

Proof: The existence of the factorization follows immediately from the two theorems of this section (34.1 and 34.5). (First take the (epi, extremal mono)-factorization of f, $f = m \circ e$; and then take the (extremal epi, mono)-factorization of e.) The functoriality is immediate from Proposition 33.5. □

34.7 PROPOSITION

If $\mathscr{C}$ is a well-powered category that has intersections and equalizers, then the class $\mathscr{M}$ of all extremal monomorphisms of $\mathscr{C}$ is the smallest class of $\mathscr{C}$-monomorphisms that contains all the regular monomorphisms of $\mathscr{C}$ and is closed under composition and the formation of intersections.

Proof: We already know that $\mathscr{M}$ contains the class of regular monomorphisms of $\mathscr{C}$ (17.11 dual) and that it is closed under composition and the formation of intersections (34.2). Let $\mathscr{M}'$ be another class of monomorphisms with these properties. Then $\mathscr{M}'$ satisfies the conditions (1), (2), and (3) of the Factorization Lemma (34.3) so that every $\mathscr{C}$-morphism f has an (epi, $\mathscr{M}'$)-factorization. Let $f \in \mathscr{M}$ and let $f = m \circ e$ be an (epi, $\mathscr{M}'$)-factorization of f. Since f is an extremal monomorphism and e is an epimorphism, e must be an isomorphism, so that $m \circ e \in \mathscr{M}'$. Thus $\mathscr{M} \subset \mathscr{M}'$. □

The above proposition indicates that for "sufficiently nice" categories, the class of extremal monomorphisms is the smallest sensible class of morphisms that could be called "embeddings of substructures". The largest such class is obviously the class of all monomorphisms. Consequently, in those categories in which all monomorphisms are extremal (i.e., in all balanced categories), one immediately has natural categorical concepts corresponding to "embeddings of substructures" and "images".

34.8 DEFINITION

If $\mathscr{C}$ is an $(\mathscr{E}, \mathscr{M})$ category and $X \xrightarrow{f} Y = X \xrightarrow{e} Z \xrightarrow{m} Y$ is the $(\mathscr{E}, \mathscr{M})$-factorization of a morphism f, then (Z, m) [or sometimes just m] is called the **$\mathscr{M}$-image of** f. In the case where $\mathscr{M}$ is the class of all $\mathscr{C}$-monomorphisms, we say **image** rather than $\mathscr{M}$-image and write $(Z, m) \approx Im(f)$ [or (loosely) $m \approx Im(f)$]. Dually, (e, Z) (or sometimes just e) is called the **$\mathscr{E}$-coimage of** f; abbreviated to **coimage of** f if $\mathscr{E}$ is the class of all $\mathscr{C}$-epimorphisms, in which case we write $(e, Z) \approx Coim(f)$.

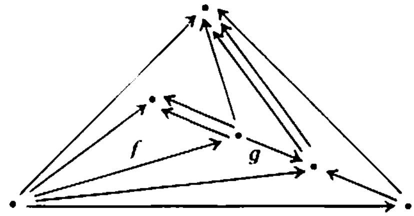
EXERCISES

34A. Let $\mathscr{C}$ be a well-powered, complete category. Prove that:

(a) The class of all extremal monomorphisms of $\mathscr{C}$ is the only class $\mathscr{M}$ for which $\mathscr{C}$ is an (epi, $\mathscr{M}$) category.

(b) The class of all extremal epimorphisms of $\mathscr{C}$ is the only class $\mathscr{E}$ for which $\mathscr{C}$ is an ($\mathscr{E}$, mono) category.

34B. Show that in an arbitrary category the composition of extremal monomorphisms is not necessarily extremal. [To do this, consider the following diagram:



(a) Prove that this uniquely depicts a category $\mathscr{B}$.
(b) Prove that, in $\mathscr{B}$, f and g are extremal monomorphisms.
(c) Prove that $g \circ f$ is not an extremal monomorphism.]

34C. By constructions similar to those described in 34B above, prove that in general categories:

(a) The intersection of extremal subobjects is not necessarily extremal.
(b) The inverse image of an extremal monomorphism is not necessarily extremal.
(c) The product of extremal monomorphisms is not necessarily extremal.

34D. For any well-powered complete category, prove that the following are equivalent:

(a) Every extremal monomorphism in $\mathscr{C}$ is regular.
(b) In $\mathscr{C}$ the composition of regular monomorphisms is regular.
(c) $\mathscr{C}$ is (epi, regular mono)-factorizable.
(d) $\mathscr{C}$ is an (epi, regular mono) category.

34E. Let $\mathscr{C}$ be an (extremal epi, mono) category, let X be a $\mathscr{C}$-object, let $(A_i, m_i)_I$ be a family of subobjects of X, and let $d: D \to X$ be a $\mathscr{C}$-monomorphism. Prove that (D, d) is an intersection of the family $(A_i, m_i)_I$ of subobjects of X if and only if (D, d) is the greatest lower bound of the family $(A_i, m_i)_I$; i.e., if and only if

(a) for each i, $(D, d) \leq (A_i, m_i)$, and
(b) if (G, g) is a subobject of X such that for each i, $(G, g) \leq (A_i, m_i)$, then $(G, g) \leq (D, d)$. (Cf. 17A.)

34F. *Factorizations in Special Categories*

(a) Let $X \xrightarrow{f} Y$ be a morphism in one of the categories **Set**, **Grp**, R-**Mod**, **SGrp**, **Rng**, **Top**, or **POS**. Let $f[X]$ denote the set-theoretic image of X under f, let $f[X] \overset{i}{\hookrightarrow} Y$ denote the inclusion function, and let $X \xrightarrow{\hat{f}} f[X]$ denote the function defined by $\hat{f}(x) = f(x)$ for all $x \in X$. Assuming now that in each instance the set $f[X]$ is equipped with the appropriate substructure inherited from Y, prove that the factorization

$$X \xrightarrow{f} Y = X \xrightarrow{\hat{f}} f[X] \xrightarrow{i} Y$$

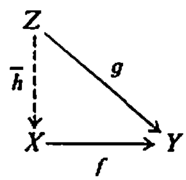
is

(i) the (epi, extremal mono)-factorization in **Set**, **Grp**, R-**Mod**, **Top**, and **POS**.
(ii) the (extremal epi, mono)-factorization in **Set**, **Grp**, R-**Mod**, **SGrp**, and **Rng**.

(b) Describe the three-fold (extremal epi, bi, extremal mono)-factorizations in **POS**, **Top**, $\mathbf{Top}_2$, and **Cat**.

34G. *Embeddings*

A morphism $X \xrightarrow{f} Y$ in a concrete category $(\mathscr{C}, U)$ will be called an **embedding** provided that for each $\mathscr{C}$-morphism $g: Z \to Y$ and each function $h: U(Z) \to U(X)$ such that $U(g) = U(f) \circ h$ there exists a unique morphism $\bar{h}: Z \to X$ such that the triangle



commutes.

Prove that:

(a) Every embedding is a monomorphism.

(b) For algebraic categories the embeddings are precisely the monomorphisms.

(c) In **Top** (considered as a concrete category) the embeddings are precisely the extremal monomorphisms (= topological embeddings).

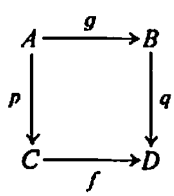
(d) In the concrete category $\mathbf{Top}_2$, the embeddings are precisely the topological embeddings. (These lie properly between the monomorphisms and the extremal monomorphisms.)

(e) Every regular monomorphism is an embedding.

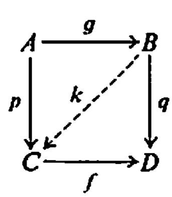
(f) The class of embeddings is closed under the formation of intersections, pullbacks (= inverse images), and products.

(g) If $\mathscr{C}$ is complete and well-powered, then every extremal monomorphism in $\mathscr{C}$ is an embedding.

(h) If the square



commutes, where $U(g)$ is surjective and f is an embedding, then there exists a unique morphism $k: B \to C$ such that the diagram



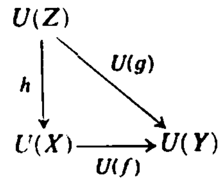
commutes.

Prove the following assuming that U preserves monomorphisms:

(i) $f: X \to Y$ is an embedding if and only if

(i) f is a monomorphism, and

(ii) for each $\mathscr{C}$-morphism $g: Z \to Y$ and each function $h: U(Z) \to U(X)$ such that the triangle



commutes, there exists a $\mathscr{C}$-morphism $\bar{h}: Z \to X$ such that $U(\bar{h}) = h$.

(j) The composition of embeddings is an embedding.

(k) If $g \circ f$ is an embedding, then f is an embedding.

34H. *Dominions*

Throughout this exercise, $\mathscr{C}$ will denote a well-powered complete category, and $X \xrightarrow{f} Y$ will denote a $\mathscr{C}$-morphism. Prove that:

(a) $X \xrightarrow{f} Y$ has a factorization $X \xrightarrow{g} D \xrightarrow{d} Y$, where d is a regular monomorphism that is characterized uniquely by any of the following equivalent conditions:

(i) for all morphisms r and s, $r \circ f = s \circ f$ implies that $r \circ d = s \circ d$.

(ii) (D, d) is the smallest regular subobject of Y through which f can be factored; i.e., if

$$X \xrightarrow{f} Y = X \xrightarrow{h} E \xrightarrow{e} Y$$

where e is a regular monomorphism, then $(D, d) \leq (E, e)$.

(iii) d is the intersection of all regular monomorphisms with codomain Y through which f can be factored.

(D, d) is called the **dominion of** f (denoted by $Dom(f)$) and $f = d \circ g$ is called the **dominion factorization of** f.

DUAL NOTION: **codominion.**

Prove that:

(b) In $\mathscr{C}$ the extremal monomorphisms are precisely the regular monomorphisms if and only if the (epi, extremal mono)-factorization of any morphism is the same as its dominion factorization.

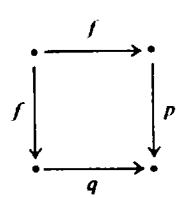
(c) $f: X \to Y$ is an epimorphism if and only if $Dom(f) \approx (Y, 1_Y)$.

(d) f is a regular monomorphism if and only if $Dom(f) \approx (X, f)$.

(e) If $f = r \circ s$, where s is an epimorphism, then $Dom(f) \approx Dom(r)$.

(f) The dominion factorization is functorial; i.e., it can be interpreted as a functor $F: \mathscr{C}^2 \to \mathscr{C}^3$.

(g) Consider the pushout square



Then (D, d) is the dominion of f if and only if it is the equalizer of p and q.

34I. *Five-Fold Factorizations*

Throughout this exercise, $\mathscr{C}$ will denote a well-powered, co-(well-powered), complete, cocomplete category, and $X \xrightarrow{f} Y$ will denote a $\mathscr{C}$-morphism.

(a) Prove that f has a unique five-fold factorization: $f = a \circ b \circ c \circ d \circ e$ where $(e, cod(e))$ is the codominion of f, d is an extremal epimorphism, c is a bimorphism, b is an extremal monomorphism and $(dom(a), a)$ is the dominion of f.

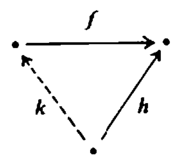
(b) Express the unique (epi, extremal mono), (extremal epi, mono), and (extremal epi, bi, extremal mono)-factorizations in terms of the five-fold factorization.

(c) Prove that the five-fold factorization is functorial.

(d) Prove that if $\mathscr{C}$ has the properties that every monomorphism is regular and every epimorphism is regular, then the middle three terms in the five-fold factorization of a $\mathscr{C}$-morphism are isomorphisms; so that every morphism in $\mathscr{C}$ has a natural unique (epi, mono)-factorization.

34J. *Strict Monomorphisms*

A morphism f is called a **strict monomorphism** provided that: whenever h is a morphism with the property that for all morphisms r and s, $r \circ f = s \circ f$ implies that $r \circ h = s \circ h$; then there exists a unique morphism k such that the diagram



commutes.

(a) Prove that every regular monomorphism is a strict monomorphism and that every strict monomorphism is an extremal monomorphism.

(b) Construct categories to show that not every extremal monomorphism is strict and not every strict monomorphism is regular.

(c) Prove that the intersection of strict subobjects is a strict subobject and that the inverse image of a strict monomorphism is strict.

(d) Prove that the product of strict monomorphisms is a strict monomorphism.

(e) Construct a category to show that the composition of strict monomorphisms is not necessarily a strict monomorphism.

(f) Prove that in well-powered categories that have equalizers, a morphism is a strict monomorphism if and only if it is the intersection of regular monomorphisms.

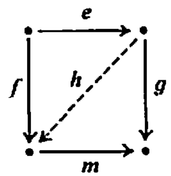
(g) Prove that in well-powered complete categories a morphism is a strict monomorphism if and only if it is regular.

(h) Prove that in a category with pushouts, a monomorphism is strict if and only if it is regular.

(i) Prove that every category has the (epi, strict mono)-diagonalization property and use this to show that for any category the composition of strict monomorphisms is an extremal monomorphism.

34K. *Strong Monomorphisms*

A morphism m is called a **strong monomorphism** provided that it is a monomorphism and whenever $m \circ f = g \circ e$ with e an epimorphism, there exists a morphism h such that the diagram



commutes.

Prove that:

(a) each strict monomorphism is strong, and each strong monomorphism is extremal.
(b) extremal monomorphisms need not be strong, and strong monomorphisms need not be strict.
(c) the class of strong monomorphisms is closed under composition, intersection, inverse images, products, and left-cancellation.
(d) in any category that has pushouts, a monomorphism is strong if and only if it is extremal.

34L. *Regular Monomorphisms in the Category of Semigroups*
Let $C = \{0, a, b, c, d, e\}$ and consider the following multiplication table:

	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	b	c
b	0	0	0	0	c	0
c	0	0	0	0	0	0
d	0	b	c	0	e	0
e	0	c	0	0	0	0

(a) Prove that C with the above multiplication is a semigroup.
(b) $A = \{0, a, b\}$ and $B = \{0, a, b, c\}$ are subsemigroups of $\mathscr{C}$. Let $f\colon A \hookrightarrow B$ and $g\colon B \hookrightarrow C$ be the inclusion homomorphisms. Prove that f and g are regular monomorphisms in **SGrp**.
(c) Prove that if $C \overset{h}{\underset{k}{\rightrightarrows}} D$ are morphisms in **SGrp** that coincide on A, then h and k coincide on B. [*Hint:* Use the equalities $b = da$ and $c = bd$.] Conclude that $g \circ f$ is *not* a regular monomorphism in **SGrp**.
(d) Let $\hat{B}$ be the free semigroup on three generators $\{\hat{a}, \hat{b}, \hat{c}\}$; let $\hat{A}$ be the subsemigroup of $\hat{B}$ generated by $\{\hat{a}, \hat{b}\hat{a}, \hat{c}\hat{a}\}$; let $\hat{C}$ be the quotient semigroup obtained by identifying the words $\hat{a}$ and $\hat{b}\hat{a}\hat{c}$; and let $i\colon \hat{A} \hookrightarrow \hat{B}$ and $\natural\colon \hat{B} \to \hat{C}$ be the inclusion map and natural map, respectively.

(i) Construct a semigroup $\hat{D}$ and homomorphisms $r, s: \hat{C} \to \hat{D}$ such that $(\hat{A}, \natural \circ i) \approx Equ(r, s)$.

(ii) Show that i is not a regular monomorphism in **SGrp**.

(e) Conclude that for the complete, well-powered category **SGrp**, the following hold:

(i) The class of regular monomorphisms coincides with the class of strict monomorphisms (34J).

(ii) There exist extremal monomorphisms that are not regular.

(iii) The class of regular monomorphisms is not closed under composition.

(iv) **SGrp** is not (epi, regular mono)-factorizable.

(v) The first factor of a regular monomorphism is not necessarily regular.

§35 (GENERATING, EXTREMAL MONO) AND (EXTREMAL GENERATING, MONO)-FACTORIZATIONS

In this section we broaden our outlook by considering factorizations of pairs (f, A) where A is an $\mathscr{A}$-object, $G: \mathscr{A} \to \mathscr{B}$ is a functor, and $f: B \to G(A)$ is a $\mathscr{B}$-morphism. In doing so we will generalize several earlier factorization results.

Recall that

$$B \xrightarrow{f} G(A) = B \xrightarrow{g} G(\hat{A}) \xrightarrow{G(m)} G(A)$$

is called

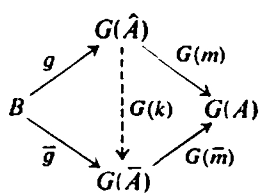
(1) An (extremal G-generating, mono)-factorization of (f, A) provided that g extremally G-generates $\hat{A}$ and $m: \hat{A} \to A$ is a monomorphism.

(2) A (G-generating, extremal mono)-factorization of (f, A) provided that g G-generates $\hat{A}$ and $m: \hat{A} \to A$ is an extremal monomorphism.

In either case, the factorization is said to be *unique* if whenever

$$B \xrightarrow{f} G(A) = B \xrightarrow{\bar{g}} G(\bar{A}) \xrightarrow{G(\bar{m})} G(A)$$

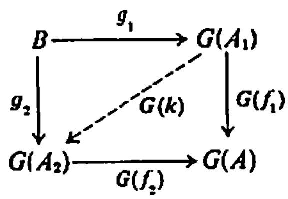
is another such factorization, there exists a unique $\mathscr{A}$-isomorphism $k: \hat{A} \to \bar{A}$ such that the diagram



commutes.

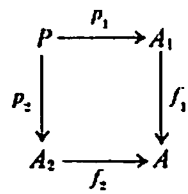
35.1 DIAGONALIZATION THEOREM 1

Suppose that $\mathscr{A}$ has pullbacks, $G: \mathscr{A} \to \mathscr{B}$ preserves pullbacks, and for $i = 1, 2, f_i: A_i \to A$ are $\mathscr{A}$-morphisms and $g_i: B \to G(A_i)$ are $\mathscr{B}$-morphisms such *that $G(f_1) \circ g_1 = G(f_2) \circ g_2$. If g_1 extremally G-generates A_1 and f_2 is a monomorphism, then there exists a unique $\mathscr{A}$-morphism $k: A_1 \to A_2$ such that the diagram*

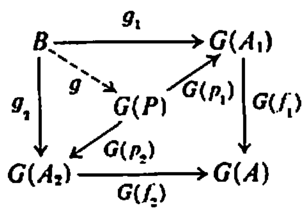


commutes.

Proof: Let



be a pullback square. Since f_2 is a monomorphism, so is p_1 (21.13). Since G preserves pullbacks, there exists a $\mathscr{B}$-morphism $g: B \to G(P)$ such that the diagram

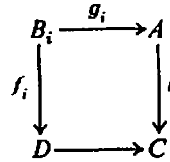


commutes.

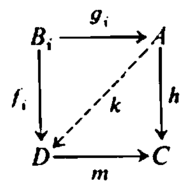
Since g_1 extremally G-generates A_1, the monomorphism p_1 must be an isomorphism. Thus $k = p_2 \circ p_1^{-1}$ is the desired diagonal morphism. Uniqueness is immediate since g_1 G-generates A_1. □

35.2 COROLLARY

Let $\mathscr{A}$ be a category that has pullbacks, let $((g_i)_I, A)$ and $((f_i)_I, D)$ be sinks in $\mathscr{A}$, and let m and h be $\mathscr{A}$-morphisms such that for each $i \in I$ the square



commutes. If (g_i, A) is an (extremal epi)-sink and m is a monomorphism, then there exists a unique morphism $k: A \to D$ such that for each $i \in I$ the diagram

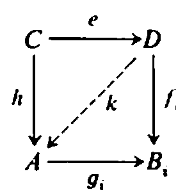


commutes.

Proof: Let $G: \mathscr{A} \to \mathscr{A}^I$ be the "constant functor" functor (15.8), where I is the discrete category determined by the index set for the sinks. Apply the theorem. □

35.3 COROLLARY

Let $\mathscr{A}$ be a category that has pushouts, let $(A, (g_i)_I)$ and $(D, (f_i)_I)$ be sources in $\mathscr{A}$, and let e and h be $\mathscr{A}$-morphisms such that for each $i \in I$, $f_i \circ e = g_i \circ h$. If $(A, (g_i)_I)$ is an (extremal mono)-source and e is an epimorphism, then there is a unique morphism $k: D \to A$ such that for each $i \in I$ the diagram



commutes.

Proof: Dualize the above corollary (35.2). □

35.4 COROLLARY

Each category that has pullbacks has the (extremal epi, mono)-diagonalization property and each category that has pushouts has the (epi, extremal mono)-diagonalization property (33J). □

35.5 THEOREM

If $\mathscr{A}$ is well-powered, finitely complete and has intersections and if $G: \mathscr{A} \to \mathscr{B}$ is a functor that preserves limits, then for any $\mathscr{A}$-object A and any $\mathscr{B}$-morphism $f: B \to G(A)$ there exists a unique (extremal G-generating, mono)-factorization of (f, A).

Proof: We have already established the existence of such a factorization (28.6). To show the uniqueness, apply the above diagonalization theorem (35.1). □

35.6 COROLLARY

If $\mathscr{A}$ is well-powered, finitely complete, and has intersections, then every sink in $\mathscr{C}$ has a unique [(extremal epi)-sink, mono]-factorization. (cf. 19.14) □

35.7 COROLLARY

Every well-powered, finitely complete category which has intersections is an (extremal epi, mono) category. (cf. 34.1) □

35.8 PROPOSITION

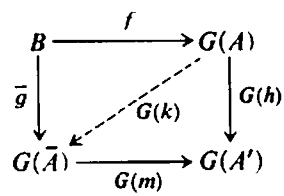
Let $G: \mathcal{A} \to \mathcal{B}$ be a functor and $f: B \to G(A)$ be a $\mathcal{B}$-morphism that extremally G-generates A. Then the following hold:

(1) *If $\mathcal{A}$ has pullbacks, G preserves pullbacks, and $h: A \to A'$ is an extremal epimorphism, then $G(h) \circ f$ extremally G-generates A'.*

(2) *If $\mathcal{B}$ has pullbacks, G preserves monomorphisms, and $g: B' \to B$ is an extremal epimorphism, then $f \circ g$ extremally G-generates A.*

Proof:

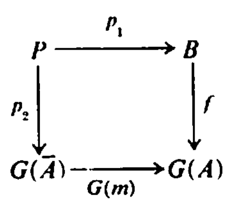
(*1*). Since h is an epimorphism, it is clear that $G(h) \circ f$ G-generates A'. To verify the extremal condition let $G(h) \circ f = G(m) \circ \bar{g}$, where $m: \bar{A} \to A$ is an $\mathcal{A}$-monomorphism. According to the preceding diagonalization theorem (35.1), there is an $\mathcal{A}$-morphism $k: A \to \bar{A}$ such that the diagram



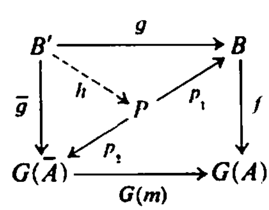
commutes.

Since f G-generates A, $h = m \circ k$. Hence since h is an extremal epimorphism, m must be an isomorphism.

(*2*). Since g is an epimorphism, it is clear that $f \circ g$ G-generates A. To verify the extremal condition, let $f \circ g = G(m) \circ \bar{g}$ be a factorization of $f \circ g$, where $m: \bar{A} \to A$ is an $\mathcal{A}$-monomorphism. Let



be a pullback square. Then there exists a morphism $h: B' \to P$ such that the diagram



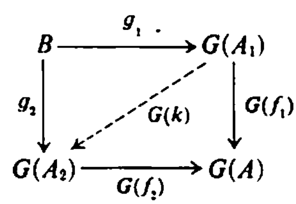
commutes.

Since G preserves monomorphisms, $G(m)$ is a monomorphism, so that p_1 is a monomorphism (21.13). Thus since g is extremal, p_1 is an isomorphism. Since f extremally G-generates A, $f = G(m) \circ (p_2 \circ p_1^{-1})$ implies that m is an isomorphism. □

We now turn our attention to the consideration of (generating, extremal mono)-factorizations.

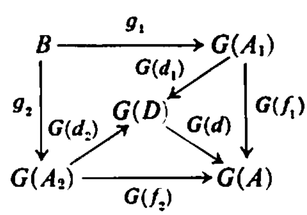
35.9 DIAGONALIZATION THEOREM II

Suppose that $\mathscr{A}$ is well-powered, has intersections and equalizers, $G: \mathscr{A} \to \mathscr{B}$ preserves monomorphisms, and for $i = 1, 2$, $f_i: A_i \to A$ are $\mathscr{A}$-morphisms and $g_i: B \to G(A_i)$ are $\mathscr{B}$-morphisms such that $G(f_1) \circ g_1 = G(f_2) \circ g_2$. If g_1 G-generates A_1 and f_2 is an extremal monomorphism, then there exists a unique $\mathscr{A}$-morphism $k: A_1 \to A_2$ such that the diagram



commutes.

Proof: Let $(X_i, m_i)_I$ be the class of all subobjects of A for which there exist $\mathscr{A}$-morphisms h_i^1, h_i^2 with $f_1 = m_i \circ h_i^1$ and $f_2 = m_i \circ h_i^2$. Let (D, d) be the intersection of $(X_i, m_i)_I$ (17.7). Then there exist $\mathscr{A}$-morphisms d_1, d_2 with $f_1 = d \circ d_1$ and $f_2 = d \circ d_2$. Since G preserves monomorphisms, $G(d)$ is a monomorphism, so that the diagram



commutes.

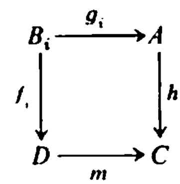
We wish to show that d_2 is an $\mathscr{A}$-epimorphism. Assume that r and s are morphisms such that $r \circ d_2 = s \circ d_2$ and let $(E, e) \approx Equ(r, s)$. Then there exists an $\mathscr{A}$-morphism e_2 such that $d_2 = e \circ e_2$. On the other hand

$$G(r \circ d_1) \circ g_1 = G(r \circ d_2) \circ g_2 = G(s \circ d_2) \circ g_2 = G(s \circ d_1) \circ g_1$$

and the fact that g_1 G-generates A_1, implies that $r \circ d_1 = s \circ d_1$. Hence, there is an $\mathscr{A}$-morphism e_1 with $d_1 = e \circ e_1$. Consequently, $f_1 = (d \circ e) \circ e_1$ and $f_2 = (d \circ e) \circ e_2$ implies that $(E, d \circ e)$ belongs to $(X_i, m_i)_I$. It follows that e is a retraction. Thus since it is also a monomorphism, it must be an isomorphism. Hence $r = s$ (16.7); so that d_2 is an epimorphism. But since f_2 is extremal, this implies that d_2 is an isomorphism. Thus $k = d_2^{-1} \circ d_1$ is the desired diagonal morphism. Uniqueness follows from the fact that g_1 G-generates A_1. □

35.10 COROLLARY

Let $\mathscr{A}$ be a well-powered category that has intersections and equalizers, let $((g_i)_I, A)$ and $((f_i)_I, D)$ be sinks in $\mathscr{A}$, and let m and h be $\mathscr{A}$-morphisms such that for each $i \in I$ the square



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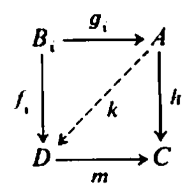
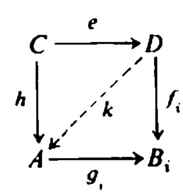
If $((g_i)_I, A)$ is an epi-sink and m is an extremal monomorphism, then there exists a unique morphism $k: A \to D$ such that for each $i \in I$, the diagram



commutes. □

35.11 COROLLARY

Let $\mathscr{A}$ be a co-(well-powered) category that has cointersections and co-equalizers, let $(A, (g_i)_I)$ and $(D, (f_i)_I)$ be sources in $\mathscr{A}$, and let e and h be $\mathscr{A}$-morphisms such that for each $i \in I$, $f_i \circ e = g_i \circ h$. If $(A, (g_i)_I)$ is a mono-source and e is an extremal epimorphism, then there is a unique morphism $k: D \to A$ such that for each $i \in I$ the diagram



commutes. □

35.12 THEOREM

If $\mathscr{A}$ is well-powered, has intersections and equalizers, and if $G: \mathscr{A} \to \mathscr{B}$ is a functor that preserves intersections and equalizers, then for any $\mathscr{A}$-object A and *any $\mathscr{B}$-morphism $f: B \to G(A)$, there exists a unique (G-generating, extremal mono)-factorization of (f, A).*

Proof: Let $(X_i, m_i)_I$ be the family of all extremal subobjects of A for which there is some $\mathscr{B}$-morphism g_i such that $f = G(m_i) \circ g_i$. Let (D, d) be the intersection of $(X_i, m_i)_I$ (17.7). Since $\mathscr{A}$ has the (epi, extremal mono)-diagonalization property (34.4), (D, d) must be an extremal subobject of A (34.2(2)). Since G preserves intersections, $(G(D), G(d))$ is the intersection of the family $(G(X_i), G(m_i))_I$. Hence there exists a $\mathscr{B}$-morphism $g: B \to G(D)$ such that

$$B \xrightarrow{f} G(A) = B \xrightarrow{g} G(D) \xrightarrow{G(d)} G(A).$$

To show that g G-generates D, let $D \underset{s}{\overset{r}{\rightrightarrows}} \hat{A}$ be a pair of $\mathscr{A}$-morphisms where $G(r) \circ g = G(s) \circ g$. Let $(E, e) \approx Equ(r, s)$. Since G preserves equalizers, there is a $\mathscr{B}$-morphism $\bar{g}$ such that $g = G(e) \circ \bar{g}$. Since $d \circ e$ is an extremal monomorphism in $\mathscr{A}$ (34.2(1)) with $f = G(d \circ e) \circ \bar{g}$, $(E, d \circ e)$ must belong to $(X_i, m_i)_I$. It follows that e is a retraction, so that since it is also a monomorphism, it must be an isomorphism. Thus $r = s$. Consequently, g generates D. Uniqueness follows immediately from the above diagonalization theorem (35.9). □

35.13 COROLLARY

If $\mathscr{A}$ is well-powered and has intersections and equalizers, then each sink in $\mathscr{A}$ has a unique (epi-sink, extremal mono)-factorization. □

35.14 COROLLARY

If $\mathscr{A}$ is co-(well-powered) and has cointersections and coequalizers, then each source in $\mathscr{A}$ has a unique (extremal epi, mono-source)-factorization. □

35.15 COROLLARY

Each category that is well-powered and has intersections and equalizers is an (epi, extremal mono) category (34.5). □

EXERCISES

35A. Let $G: \mathscr{A} \to \mathscr{B}$ be a functor, let $h: A \to A'$ be an $\mathscr{A}$-morphism, and let $g: B' \to B$ and $f: B \to G(A)$ be $\mathscr{B}$-morphisms. Prove the following:

(a) If $G(h) \circ f$ G-generates A', then h is an epimorphism.
(b) If $G(h) \circ f$ extremally G-generates A', then h is an extremal epimorphism.
(c) If $f \circ g$ G-generates A, then f G-generates A.
(d) If $f \circ g$ extremally G-generates A, then f extremally G-generates A.
(e) If g is an epimorphism and f G-generates A, then $f \circ g$ G-generates A.
(f) If h is an epimorphism and f G-generates A, then $G(h) \circ f$ G-generates A'. (cf. 35.8)

35B. Prove Theorem 35.12 after replacing the hypothesis that $\mathscr{A}$ is well-powered by the hypothesis that it is extremally well-powered.

35C. Prove that if $\mathscr{A}$ is well-powered and complete, and $G: \mathscr{A} \to \mathscr{B}$ is a limit-preserving functor, then for any $\mathscr{A}$-object A and any $\mathscr{B}$-morphism $f: B \to G(A)$ there exists a unique factorization

$$B \xrightarrow{f} G(A) = B \xrightarrow{g} G(A') \xrightarrow{G(k)} G(A'') \xrightarrow{G(m)} G(A)$$

where g extremally G-generates A', $k: A' \to A''$ is a bimorphism, and $m: A'' \to A$ is an extremal monomorphism.

35D. Prove that if $\mathscr{C}$ is co-(well-powered) and cocomplete and S is an extremal separator for $\mathscr{C}$, then $hom(S, \_\_)$ reflects limits.