VIII

Set-Valued Functors

A harmless necessary cat.

—W. Shakespeare†

Set-valued functors are of special interest for several reasons. First of all, most of the motivating examples for category theory are actually concrete categories, i.e., pairs $(\mathscr{A}, U)$ where $\mathscr{A}$ is a category and U is a faithful set-valued functor with domain $\mathscr{A}$. Secondly, due to many of the well-known nice properties of Set, set-valued functors are usually easier to handle than arbitrary functors. For example, in this chapter we will see that if $G: \mathscr{A} \to \text{Set}$ is a functor, and if there is a G-universal map for at least one non-empty set, then G preserves limits. Also if $\mathscr{A}$ is cocomplete and there is a G-universal map for a singleton set, then there is a G-universal map for each set—so that G has a left adjoint. A third reason for the importance of set-valued functors is the fact that together with any category $\mathscr{A}$, there is associated a whole class of set-valued functors—namely the *hom*-functors $hom(A, \_)$: $\mathscr{A} \to \text{Set}$. We shall see that an arbitrary functor $G: \mathscr{A} \to \mathscr{B}$ has a left adjoint provided that for each $\mathscr{B}$-object B, the set-valued functor $hom(B, \_) \circ G$ is naturally isomorphic to some *hom*-functor with domain $\mathscr{A}$. This then provides yet another approach to adjoint situations.

§29 HOM-FUNCTORS

One of the reasons for the usefulness of *hom*-functors is that they not only preserve commutative diagrams (as do all functors) but also, as we shall see in this section, acting in concert they "detect" commutativity of diagrams. Moreover,

† From *The Merchant of Venice.*

they preserve limits (and thus monomorphisms) and—as a whole—"detect" them as well.

29.1 PROPOSITION

Let $C \overset{f}{\underset{g}{\rightrightarrows}} D$ *be a pair of* $\mathscr{A}$*-morphisms. Then the following are equivalent:*

(1) $f = g$.

(2) *For each* $\mathscr{A}$*-object* A, $hom(A, \_)(f) = hom(A, \_)(g)$.

Proof: Clearly (1) implies (2). If (2) holds, then

$$f = f \circ 1_C = hom(C, f)(1_C) = hom(C, g)(1_C) = g \circ 1_C = g. \quad \square$$

The above proposition can be rephrased as follows:

A triangle (or diagram) in $\mathscr{A}$ *commutes if and only if for each* $\mathscr{A}$*-object* A*, the image triangle (or diagram) under* $hom(A, \_)$ *commutes.*

29.2 COROLLARY

Let $\mathscr{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathscr{C}$ *be a pair of functors and let* $\eta = (\eta_A\colon F(A) \to G(A))$ *be a family of* $\mathscr{C}$*-morphisms indexed by* $Ob(\mathscr{A})$*. Then the following are equivalent:*

(1) $\eta = (\eta_A)\colon F \to G$ *is a natural transformation.*

(2) *For each* $\mathscr{C}$*-object* C,

$$hom(C, \_) * \eta = (hom(C, \eta_A))\colon hom(C, \_) \circ F \longrightarrow hom(C, \_) \circ G$$

is a natural transformation. $\square$

29.3 THEOREM

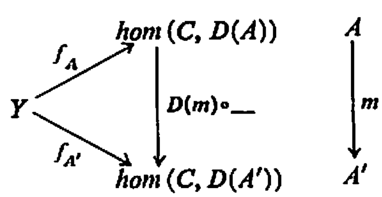
Let $D\colon \mathscr{A} \to \mathscr{C}$ *be a functor and let* $(L, L \xrightarrow{l_A} D(A))$ *be a source in* $\mathscr{C}$*. Then the following are equivalent:*

(1) $(L, (l_A))$ *is a limit of* D.

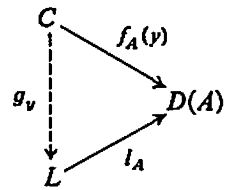
(2) $(hom(C, L), (hom(C, l_A)))$ *is a limit of* $hom(C, \_) \circ D$*, for each* $C \in Ob(\mathscr{C})$.

Proof:

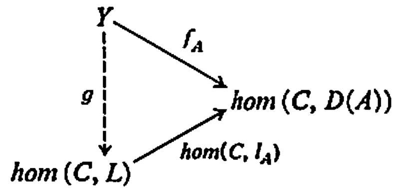
(1) $\Rightarrow$ *(2).* Since functors preserve commutative triangles, it is clear that for each $C \in Ob(\mathscr{C})$, $(hom(C, L), (hom(C, l_A)))$ is a natural source for $hom(C, \_) \circ D$. If $(Y, (f_A))$ is also a natural source for $hom(C, \_) \circ D$, then for each $y \in Y$, $f_A(y) \in hom(C, D(A))$ and commutativity of the triangle



guarantees that $D(m) \circ f_A(y) = f_{A'}(y)$. Thus $(C, (f_A(y)))$ is a natural source for D, so that there exists a unique morphism g_y: $C \to L$ such that for each A the triangle



commutes. Now for each $y \in Y$ let $g(y) = g_y$. Then g: $Y \to hom(C, L)$, and the triangle

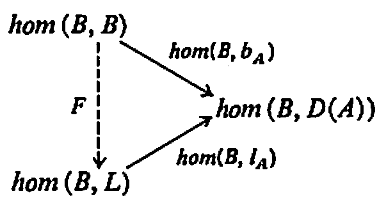


commutes, since it does for each $y \in Y$. Uniqueness follows from the uniqueness of each g_y. Hence $(hom(C, L), (hom(C, l_A)))$ is a limit of $hom(C, \_\_) \circ D$.

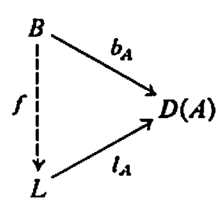
(2) $\Rightarrow$ (1). Since *hom*-functors "detect" commutative triangles, it is clear that $(L, (l_A))$ is a natural source for D. Now suppose that $(B, (b_A))$ is also a natural source for D. Then since functors preserve commutativity,

$$(hom(B, B), (hom(B, b_A)))$$

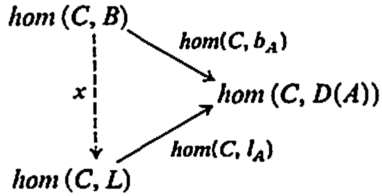
is a natural source for $hom(B, \_\_) \circ D$. Thus by the definition of limit, there is a unique function F: $hom(B, B) \to hom(B, L)$ such that for each A the triangle



commutes. Hence $(hom(B, l_A) \circ F)(1_B) = hom(B, b_A)(1_B)$; i.e., $l_A \circ F(1_B) = b_A$, so that $f = F(1_B)$ makes the triangle



commute. To show uniqueness, suppose that $f'\colon B \to L$ also makes the triangle commute. But then for each $\mathscr{C}$-object C, each of $hom(C, f)$ and $hom(C, f')$ is a function x such that the triangle



commutes. Thus by the uniqueness condition in the definition of limit, $hom(C, f) = hom(C, f')$ for each $C \in Ob(\mathscr{C})$. Consequently $f = f'$ (29.1). □

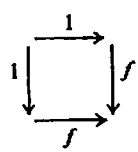
29.4 COROLLARY

(1) *The following are equivalent:*
  (*i*) *f is a $\mathscr{C}$-monomorphism*
  (*ii*) *$hom(C, f)$ is an injective function for each $C \in Ob(\mathscr{C})$.*

(2) *The following are equivalent:*
  (*i*) *T is a $\mathscr{C}$-terminal object*
  (*ii*) *$hom(C, T)$ is a singleton set for each $C \in Ob(\mathscr{C})$.*

Proof:

(1) f is a monomorphism if and only if



is a pullback (21.12) and in Set monomorphisms are precisely the injective functions.

(2) T is a terminal object if and only if it is a limit of the empty functor, and in Set terminal objects are the singleton sets. □

At this point the question naturally arises as to when *hom*-functors reflect limits. The next proposition yields a partial answer.

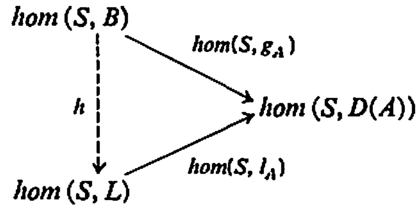
29.5 PROPOSITION

If S is a retract-separator in $\mathscr{C}$ (19.7), *then $hom(S, \_)$ reflects $\mathscr{A}$-limits for each category $\mathscr{A}$.*

Proof: Let $D\colon \mathscr{A} \to \mathscr{C}$ be a functor and let $(L, L \xrightarrow{l_A} D(A))$ be a source in $\mathscr{A}$ such that $(hom(S, L), (hom(S, l_A)))$ is a limit of $hom(S, \_) \circ D\colon \mathscr{A} \to \mathbf{Set}$. Since S is a retract-separator, it is a separator, so that $hom(S, \_)$ is faithful. Thus $(L, (l_A))$ is a natural source for D.

Let $(B, (g_A))$ also be a natural source for D. Then $(hom(S, B), (hom(S, g_A)))$ is a natural source for $hom(S, \_) \circ D$, so that there exists a unique function h

such that the triangle



commutes for each A; i.e., for each $x: S \to B$

$$g_A \circ x = l_A \circ h(x).$$

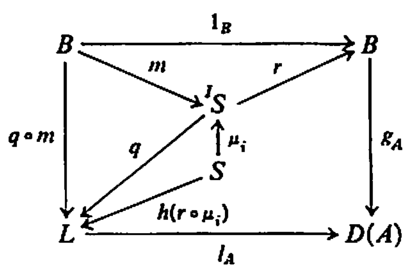
Now since S is a retract-separator, there is some copower $((\mu_i), {}^IS)$ of S and morphisms m and r such that

$$B \xrightarrow{m} {}^IS \xrightarrow{r} B = B \xrightarrow{1_B} B.$$

By the above, for each $i \in I$, and each $A \in Ob(\mathscr{A})$

$$g_A \circ (r \circ \mu_i) = l_A \circ h(r \circ \mu_i).$$

Hence by the definition of coproduct there exists a unique morphism $q: {}^IS \to L$ such that for each i, $q \circ \mu_i = h(r \circ \mu_i)$. Since $((\mu_i), {}^IS)$ is an epi-sink, the diagram



commutes for each $A \in Ob(\mathscr{A})$. Since h is unique and $hom(S, \_\_)$ is faithful, $q \circ m$ is the unique morphism that makes the outer square commute for each A. Consequently $(L, (l_A))$ must be a limit of D. □

29.6 COROLLARY

If B is a non-empty set, then $hom(B, \_\_)$: Set $\to$ Set *preserves and reflects limits.* □

EXERCISE

29A. Prove that if $\mathscr{A}$ is a complete, co-(well-powered) category and A is an extremal separator in $\mathscr{A}$, then $hom(A, \_\_): \mathscr{A} \to$ Set preserves and reflects limits.

§30 REPRESENTABLE FUNCTORS

Since in category theory any two isomorphic entities are regarded as essentially the same, it is only natural to consider, together with the *hom*-functors, those functors that are naturally isomorphic to them. These are called representable functors. With them we can obtain yet another way of investigating adjoint situations. The link between representability and adjoints is through universal maps and is provided by the seemingly technical, yet nevertheless extremely useful, Yoneda Lemma.

30.1 DEFINITION

Let $G: \mathscr{A} \to \mathbf{Set}$ be a functor.

(1) A **representation of** G is a pair (A, δ) where A is an $\mathscr{A}$-object and $\delta: hom(A, \_) \to G$ is a natural isomorphism.

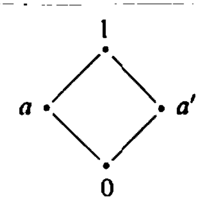
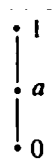
(2) G is said to be **representable** provided that there exists a representation (A, δ) of G. In this case we also say that A **represents** G.

30.2 PROPOSITION

Representable functors preserve limits.

Proof: Immediate from the fact that *hom*-functors preserve limits (29.3) and naturally isomorphic functors have identical limit-preservation properties (24.10). □

30.3 EXAMPLES

$\mathscr{A}$	The "usual" forgetful functor $U: \mathscr{A} \to \mathbf{Set}$ is representable and is represented by:
Set	any singleton set
SGrp	$(\mathbf{N} - \{0\}, +)$
Mon	$(\mathbf{N}, +)$
Grp	$(\mathbf{Z}, +)$
Ab	$(\mathbf{Z}, +)$
R-**Mod**	R
Rng	$\mathbf{Z}[x]$
BooAlg	
Top	any singleton space
POS	any singleton partially-ordered set
Lat	any singleton lattice
complete lattices	

30.4 UNIQUENESS OF REPRESENTATION

If a functor $G: \mathscr{A} \to \mathbf{Set}$ is represented by each of the objects A and B, then A and B must be isomorphic. More explicitly, if

$$(\delta_C): hom(A, \_) \longrightarrow hom(B, \_)$$

is a natural isomorphism, then $\delta_A(1_A) \in hom(B, A)$ is an isomorphism. Moreover the assignment $\delta \mapsto \delta_A(1_A)$ provides a bijective function from the set of all natural isomorphisms from $hom(A, \_)$ to $hom(B, \_)$ onto the set of all isomorphisms from B to A. This will be proved in that which follows. Nevertheless, it should be instructive for the reader to prove these statements directly before continuing on.

30.5 NOTATION

Throughout the remainder of this section, whenever F and G are functors with common domain $\mathscr{A}$ and common codomain $\mathscr{B}$, then $[F, G]$ will denote the conglomerate of natural transformations from F to G; i.e., it will denote $hom_{[\mathscr{A},\mathscr{B}]}(F, G)$.

30.6 YONEDA LEMMA

If $G: \mathscr{A} \to$ **Set** *and* A *is an* $\mathscr{A}$*-object, then there is a bijective function*

$$Y : [hom(A, \_), G] \longrightarrow G(A)$$

defined by

$$\delta \longmapsto \delta_A(1_A)$$

whose inverse

$$Y': G(A) \to [hom(A, \_), G]$$

is defined by

$$x \mapsto \xi = (\xi_B),$$

where

$$\xi_B(f) = G(f)(x)$$

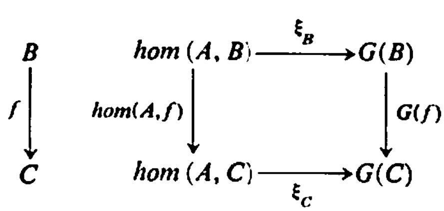
for all $f \in hom(A, B)$.

[Y and Y' are called the **Yoneda functions for** G **and** A.]

Proof: Clearly Y is a function. We first wish to show that Y' is also a function; i.e., that $Y'(x) = \xi$ is a natural transformation for each $x \in G(A)$. To see this, let $f: B \to C$ and let $g \in hom(A, B)$. Then

$$\begin{aligned}(\xi_C \circ hom(A, f))(g) &= \xi_C(f \circ g) = G(f \circ g)(x) \\ &= (G(f) \circ G(g))(x) = (G(f) \circ \xi_B)(g).\end{aligned}$$

Since this is true for each $g \in hom(A, B)$, we have commutativity of the diagram



Thus ξ is a natural transformation from $hom(A, \_)$ to G. Now by the definition of Y and Y', for each $x \in G(A)$

$$(Y \circ Y')(x) = Y(\xi) = \xi_A(1_A) = G(1_A)(x) = 1_{G(A)}(x).$$

Hence $Y \circ Y' = 1_{G(A)}$.

To show that $Y' \circ Y$ is the identity, let δ: $hom(A, \_) \to G$. Then

$$Y(\delta) = \delta_A(1_A).$$

Let ξ be $Y'(\delta_A(1_A))$. Then by definition

$$\xi_A(1_A) = G(1_A)(\delta_A(1_A)) = \delta_A(1_A).$$

Now let B be any $\mathscr{A}$-object and let f be any morphism from A to B. Then

$$\xi_B(f) = \xi_B(f \circ 1_A) = (\xi_B \circ hom(A, f))(1_A).$$

By the naturality of ξ, this is $(G(f) \circ \xi_A)(1_A)$; which by the above is

$$(G(f) \circ \delta_A)(1_A).$$

By the naturality of δ this is

$$(\delta_B \circ hom(A, f))(1_A) = \delta_B(f \circ 1_A) = \delta_B(f).$$

Hence $\delta = \xi$, so that $Y' \circ Y = 1_{[hom(A,\_),G]}$. Consequently Y is bijective and Y' is its inverse. □

30.7 COROLLARY

Let (A, B) be a pair of $\mathscr{A}$-objects. Then the Yoneda mapping

$$\tilde{Y}: hom_{\mathscr{A}}(B, A) \longrightarrow [hom(A, \_), hom(B, \_)]$$

that associates with each f: $B \to A$, the natural transformation $\xi = (\xi_B)$ defined by:

$$\xi_B(g) = g \circ f; \quad \textit{for each} \quad g \in hom(A, C),$$

is a bijective function. □

In order to show that f: $B \to A$ is an isomorphism if and only if $\tilde{Y}(f)$ is a natural isomorphism, we next prove that the functions

$$\tilde{Y}: hom_{\mathscr{A}}(B, A) \longrightarrow [hom(A, \_), hom(B, \_)]$$

defined above, are the restrictions

$$E \,|_{hom_{\mathscr{A}^{op}}(A,B)}^{[E(A),E(B)]}$$

of a functor $E: \mathscr{A}^{op} \to [\mathscr{A}, \mathbf{Set}]$.

30.8 FULL EMBEDDING THEOREM

If $\mathscr{A}$ is any category, then $E: \mathscr{A}^{op} \to [\mathscr{A}, \mathbf{Set}]$ defined by:

$$E(A) = hom(A, \_) \quad \textit{for each} \quad \mathscr{A}\textit{-object } A$$

$$E(f)_C(g) = g \circ f \quad \textit{for } \mathscr{A}\textit{-morphisms} \quad B \xrightarrow{f} A \quad \textit{and} \quad A \xrightarrow{g} C,$$

is a full embedding.

Proof: By the way E is defined, it is clear that it preserves identities and compositions. Hence it is a functor. Since in any category, morphism sets are pairwise disjoint, E must be injective on objects. By the above corollary (30.7) E must be full and faithful. Thus E is a full embedding. □

30.9 COROLLARY

Every category (resp. small category) can be fully embedded in a complete and cocomplete quasicategory (resp. category).

Proof: Since **Set** is complete and cocomplete, so is $[\mathscr{A}, \mathbf{Set}]$, for any category $\mathscr{A}$ (25.7). □

30.10 COROLLARY

Let $\tilde{Y}: hom_{\mathscr{A}}(B, A) \longrightarrow [hom(A, \_), hom(B, \_)]$ *be the Yoneda function defined in* 30.7. *Then an* $\mathscr{A}$*-morphism* f *is an* $\mathscr{A}$*-isomorphism if and only if* $\tilde{Y}(f)$ *is a natural isomorphism.*

Proof: Every functor preserves isomorphisms, and full embeddings reflect them (12.9). □

30.11 COROLLARY (UNIQUENESS OF REPRESENTATIONS)

If each of (A, δ) *and* (B, ξ) *is a representation of the functor* $G: \mathscr{A} \to \mathbf{Set}$, *then there exists an isomorphism* $f: A \to B$ *with* $\delta \circ \tilde{Y}(f) = \xi$. □

If $G: \mathscr{A} \to \mathbf{Set}$, A is an $\mathscr{A}$-object, and $Y': G(A) \to [hom(A, \_), G]$ is the Yoneda function, then by 30.10 in the case that $G = hom(B, \_)$, an element of $G(A) = hom(B, A)$ is mapped by Y' onto a natural isomorphism if and only if it is an $\mathscr{A}$-isomorphism. The question naturally arises as to what happens in the case of other functors G; i.e., can we characterize those elements of $G(A)$ whose values under Y' are natural isomorphisms? The reader who is interested in a general answer should see Exercise 30D (Universal Points). We now focus our attention on the case where $G = hom(B, \_) \circ F$, for some $F: \mathscr{A} \to \mathscr{B}$ and some $\mathscr{B}$-object, B. (Actually this can be considered as either a generalization or as a specialization depending upon one's point of view.) The solution of this problem will yield a fundamental relationship between representable functors and universal maps; and consequently between representable functors and adjoint situations.

30.12 THEOREM

Let $G: \mathscr{A} \to \mathscr{B}$, B *be a* $\mathscr{B}$*-object*, $\delta: hom(A, \_) \longrightarrow hom(B, \_) \circ G$ *be a natural transformation, and*

$$Y: [hom(A, \_), hom(B, \_) \circ G] \longrightarrow hom(B, G(A))$$

be the corresponding Yoneda function. Then the following are equivalent:

(1) δ *is a natural isomorphism.*

(2) (A, δ) *is a representation of* $hom(B, \_) \circ G$.

(3) $(Y(\delta), A)$ *is a G-universal map for* B.

Proof: Clearly, by the definition of representation, (1) implies (2). To show that (2) implies (3), suppose that (A, δ) is a representation of $hom(B, \_\_) \circ G$. Clearly $Y(\delta) = \delta_A(1_A)\colon B \to G(A)$. If $f\colon B \to G(A')$, then since $\delta_{A'}$ is a bijective function, there exists a unique morphism $\bar{f}\colon A \to A'$ such that $\delta_{A'}(\bar{f}) = f$. Also since δ is a natural transformation, the square

$$\begin{array}{ccc} hom(A, A) & \xrightarrow{\ \delta_A\ } & hom(B, G(A)) \\ {\scriptstyle hom(A, \bar{f})}\downarrow & & \downarrow{\scriptstyle hom(B, G(\bar{f}))} \\ hom(A, A') & \xrightarrow[\ \delta_{A'}\ ]{} & hom(B, G(A')) \end{array}$$

commutes. Applying this commutativity to the element $1_A \in hom(A, A)$, we have

$$(G(\bar{f}) \circ \delta_A)(1_A) = \delta_{A'}(\bar{f}) = f.$$

Hence $\bar{f}$ is the unique morphism from A to A' for which the triangle

$$\begin{array}{cccc} B & \xrightarrow{\ Y(\delta) = \delta_A(1_A)\ } & G(A) & \quad A \\ & {\scriptstyle f}\searrow & \vdots\,{\scriptstyle G(\bar{f})} & \quad \vdots\,{\scriptstyle \bar{f}} \\ & & G(A') & \quad A' \end{array}$$

commutes. Thus $(Y(\delta), A)$ is a G-universal map for B. To show that (3) implies (1), we must show that for each $\mathscr{A}$-object A',

$$\delta_{A'}\colon hom(A, A') \to hom(B, G(A'))$$

is a bijective function. This is easily established since if $f \in hom(B, G(A'))$, then because $(Y(\delta), A)$ is a G-universal map for B, there exists one and only one $\bar{f} \in hom(A, A')$ such that the above triangle commutes; i.e., such that

$$G(\bar{f}) \circ \delta_A(1_A) = f.$$

But, again, since δ is a natural transformation,

$$G(\bar{f}) \circ \delta_A(1_A) = \delta_{A'}(\bar{f}). \quad \square$$

30.13 THEOREM

Let $G\colon \mathscr{A} \to \mathscr{B}$ be a functor. Then

(1) *a $\mathscr{B}$-object B has a G-universal map if and only if $hom(B, \_\_) \circ G$ is representable.*

(2) *G has a left adjoint if and only if for each $\mathscr{B}$-object B, the functor $hom(B, \_\_) \circ G$ is representable.*

Proof: Immediate from the preceding theorem and the connection between adjoint situations and universal maps (27.3). $\square$

30.14 THEOREM

*If $G: \mathscr{A} \to$ **Set** is a functor, then an $\mathscr{A}$-object A represents G if and only if A is a G-universal object for a singleton set P; i.e., provided that there is a singleton set P and a morphism $u: P \to G(A)$ such that (u, A) is a G-universal map for P.*

Proof: It is easily seen that there exists a natural isomorphism between the functors $hom(P, \_)$: **Set** $\to$ **Set**, and $1_{\mathbf{Set}}$: **Set** $\to$ **Set**, and that this induces a bijective function

$$B: [hom(A, \_), G] \longrightarrow [hom(A, \_), hom(P, \_) \circ G]$$

that preserves and reflects isomorphisms. Let

$$Y: [hom(A, \_), hom(P, \_) \circ G] \longrightarrow hom(P, G(A))$$

be the corresponding Yoneda bijection. Then (A, δ) is a representation of G if and only if $(A, B(\delta))$ is a representation of $hom(P, \_) \circ G$, and this is the case if and only if $(Y(B(\delta)), A)$ is a G-universal map for P (30.12). □

30.15 COROLLARY

A set-valued functor G is representable if and only if the singleton sets have G-universal maps. □

30.16 COROLLARY

If a set-valued functor G has a G-universal map for a singleton set, then G preserves limits. □

This corollary can be considerably strengthened, as the following theorem shows.

30.17 THEOREM

*If a set-valued functor $G: \mathscr{A} \to$ **Set** has a G-universal map for at least one non-empty set, B, then G preserves limits.*

Proof: By Theorem 30.13, $hom(B, \_) \circ G$ is representable. Hence it preserves limits. Since B is non-empty, $hom(B, \_)$ reflects limits (29.6). Hence G must preserve limits. □

30.18 COROLLARY

Let $\mathscr{A}$ be a full subcategory of **Set**, *containing at least one non-empty set. Then the embedding functor $E: \mathscr{A} \to$ **Set** preserves limits.* □

Note that an analogue of Theorem 30.17 is not valid for functors that are not set-valued. In particular, if G: **Set** $\to$ **Top** is the functor that sends each set to the discrete space on that set, then each discrete space has a G-universal map, but G does not preserve limits.

30.19 PROPOSITION

*A functor $G: \mathscr{A} \to$ **Set** has a left adjoint if and only if it is represented by an $\mathscr{A}$-object A for which there exist arbitrary copowers ${}^{I}A$.*

Proof: G has a left adjoint if and only if each set I has a G-universal map (27.3). Since each set I is isomorphic to the copower ${}^{I}P$ of the singleton set P, it follows from Lemma 26.9 that each set I has a G-universal map if and only if P has a G-universal map (u, A) and A has arbitrary copowers ${}^{I}A$. The characterization thus follows from Theorem 30.14. □

30.20 COROLLARY

*If $\mathscr{A}$ is cocomplete, then $G: \mathscr{A} \to$ **Set** has a left adjoint if and only if G is representable.* □

The above corollary has a partial converse (see Exercise 31A).

The following theorem ties together several of the results of this section and of the previous sections (for the case of set-valued functors).

30.21 THEOREM

*Let $G: \mathscr{A} \to$ **Set** be a functor. If $\mathscr{A}$ satisfies either of the following conditions (i) or (ii), then the conditions (1) through (6) are equivalent:*

(i) $\mathscr{A}$ is complete, well-powered, and has a coseparator.
(ii) $\mathscr{A}$ is complete, well-powered, extremally co-(well-powered), and each set extremally G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects.

(1) *G preserves limits.*
(2) *G has a left adjoint.*
(3) *There is a G-universal map for each set.*
(4) *There is a G-universal map for at least one non-empty set.*
(5) *There is a G-universal map for the singleton set.*
(6) *G is representable.* □

30.22 EXAMPLES

(1) The category $\mathscr{A}$ of complete lattices is complete and well-powered, and the obvious forgetful functor $U: \mathscr{A} \to$ **Set** is representable. However, no set having more than two elements has a U-universal map.

(2) The category $\mathscr{A}$ of complete boolean algebras is complete, well-powered and extremally co-(well-powered), and the obvious forgetful functor $U: \mathscr{A} \to$ **Set** is representable. However, no infinite set has a U-universal map (cf. Exercise 30G).

(3) There is a concrete category $(\mathscr{A}, U)$ that is complete, cocomplete, well-powered, and co-(well-powered) such that $U: \mathscr{A} \to$ **Set** preserves limits, but is not representable. (See Exercise 28D.)

EXERCISES

30A. For each of the examples in 30.3, find a natural isomorphism $\delta: hom(A, \_\_) \to U$ (where in each case A is the object specified).

30B. Show that for each category $\mathscr{A}$, the Yoneda lemma (30.6) actually establishes a natural isomorphism between the evaluation functor (15.7)

$$E: \mathbf{Set}^{\mathscr{A}} \times \mathscr{A} \to \mathbf{Set}$$

and the functor

$$N: \mathbf{Set}^{\mathscr{A}} \times \mathscr{A} \to \mathbf{Set}$$

defined by:

$$N(G, A) = [hom(A, \_\_), G]$$

$$N(\eta, f)(\delta)(A)(g) = \eta_A(\delta_A(g \circ f)).$$

30C. Consider the full embedding $E: \mathscr{A}^{op} \to [\mathscr{A}, \mathbf{Set}]$ of Theorem 30.8.

(a) Show that if $\mathscr{R}$ is the full subcategory of $[\mathscr{A}, \mathbf{Set}]$ consisting of all representable functors from $\mathscr{A}$ to **Set**, then $[\mathscr{A}, \mathbf{Set}]$ is the "colimit hull" of $\mathscr{R}$ in the sense that for each functor $F \in Ob[\mathscr{A}, \mathbf{Set}]$ there exists some functor $D: \mathscr{C} \to [\mathscr{A}, \mathbf{Set}]$ and some sink $((k_C), F)$ such that for each $C \in Ob(\mathscr{C})$, $D(C) \in Ob(\mathscr{R})$ and $((k_C), F) \approx Colim\ D$.
(b) Prove that E is a limit preserving functor, but in general is not colimit preserving.
(c) Show that the full subcategory $\bar{\mathscr{C}}$ of $[\mathscr{A}, \mathbf{Set}]$ consisting of all limit preserving functors, is both complete and cocomplete.
(d) Show that the embedding $E: \mathscr{A}^{op} \to \bar{\mathscr{C}}$ is both limit and colimit preserving and that the "colimit hull" of its image is all of $\bar{\mathscr{C}}$.

30D. *Universal Points*

Let $G: \mathscr{A} \to \mathbf{Set}$ be a functor. A pair (a, A) is called a **universal point of** G provided that A is an $\mathscr{A}$-object, $a \in G(A)$, and for any such pair (x, X) there is a unique $\mathscr{A}$-morphism $f: A \to X$ such that $G(f)(a) = x$.

(a) Show that if $G: \mathscr{A} \to \mathbf{Set}$, $A \in Ob(\mathscr{A})$, and $Y': G(A) \to [hom(A, \_\_), G]$ is the Yoneda function, then the following are equivalent:

(α) (a, A) is a universal point of G.

(β) $(A, Y'(a))$ is a representation of G.

(b) Show that if $G: \mathscr{A} \to \mathscr{B}$, $A \in Ob(\mathscr{A})$, $B \in Ob(\mathscr{B})$, and $u: B \to G(A)$, then the following are equivalent:

(α) (u, A) is a G-universal map for B.

(β) (u, A) is a universal point of $hom(B, \_\_) \circ G$.

(c) Let $G: \mathscr{A} \to \mathbf{Set}$ and let $F: \mathbf{1} \to \mathbf{Set}$ be a functor whose value at the single object $*$ is the singleton set $\{p\}$. Show that the following are equivalent:

(α) (a, A) is a universal point of G.

(β) $(*, f, A)$ is an initial object of the comma category (F, G), where

$$f: (p) \to G(A)$$

is the function defined by $f(p) = a$ (see 20D and 26H).

30E. Let $(\mathscr{A}, U)$ be a complete, well-powered, concrete category with the property that U preserves limits. Show that if (u, A) is a U-universal map for X and $Y \subset X$, then there exists a U-universal map (v, B) for Y, and a morphism $m: B \to A$ such that (B, m) is a subobject of A.

30F. Suppose that $G: \mathscr{A} \to \mathbf{Set}$ has a left adjoint F.

(a) Show that for each set I, there is a bijective function from the set $G \circ F(I)$ to the set $[hom(I, \_\_) \circ G, G]$.
(b) Illustrate the above result for the case $\mathscr{A} = \mathbf{Grp}$, G is the forgetful functor, and I is a two-element set.

30G. *Complete Boolean Algebras*
Let $(\mathscr{A}, U)$ be the concrete category of complete boolean algebras and complete homomorphisms.
(a) Prove that $\mathscr{A}$ is complete, well-powered, and extremally co-(well-powered).
(b) Prove that there exists a U-universal map for each finite set.
(c) Let X be a topological space. A subset A of X is called **regular open** provided that $int(clA) = A$ (where "*int*" designates "interior" and "*cl*" designates "closure"). Show that the set $R(X)$ of all regular open subsets of X is a complete boolean algebra with respect to the following operations:

$$\vee M = int(cl(\cup M)), \quad \text{for} \quad M \subset R(X)$$

$$\wedge M = int(\cap M), \quad \text{for} \quad \varnothing \neq M \subset R(X)$$

$$A' = int(X - A), \quad \text{for} \quad A \in R(X).$$

(d) (Solovay) Let K be an infinite cardinal. Let X be the set of all ordinals with cardinality less than K, considered as a discrete topological space. Let $P = X^{\mathbf{N}}$ be the topological product of countably many copies of X, with projections $\pi_n: X^{\mathbf{N}} \to X$. Prove that the complete boolean algebra $R(X^{\mathbf{N}})$ is extremally U-generated by a countable set. [Show first that $R(X^{\mathbf{N}})$ is extremally U-generated by the family

$$\{\pi_n^{-1}(\xi) \mid n \in \mathbf{N},\ \xi \in X\}.$$

Secondly, show that $R(X^{\mathbf{N}})$ is extremally U-generated by the countable set

$$\{A_{m,n} \mid m, n \in \mathbf{N}\},$$

where

$$A_{m,n} = \{x \in X^{\mathbf{N}} \mid \pi_m(x) \leq \pi_n(x)\}.]$$

(e) (Gaifman-Hales-Solovay) Prove that each infinite set extremally U-generates a proper class of pairwise non-isomorphic $\mathscr{A}$-objects.
(d) Prove that $\mathscr{A}$ is not cocomplete and that U has no left adjoint.

30H. Let $(\mathscr{A}, U)$ be the usual concrete category of (commutative) C^*-algebras, and let $D: \mathscr{A} \to \mathbf{Set}$ be the "unit disc functor" that associates with each C^*-algebra X its unit disc

$$D(X) = \{x \mid x \in X \text{ and } \|x\| \leq 1\}.$$

Prove that D is representable, but that U is not.

30I. *Dualities and Representability*
Let $(\mathscr{A}, U)$ and $(\mathscr{B}, V)$ be concrete categories that are dually equivalent; i.e., there are contravariant functors $G: \mathscr{A} \to \mathscr{B}$ and $F: \mathscr{B} \to \mathscr{A}$ such that

$$F \circ G \approx 1_{\mathscr{A}}$$

and

$$G \circ F \approx 1_{\mathscr{B}}.$$

Suppose that the $\mathscr{A}$-object A represents U and the $\mathscr{B}$-object B represents V, and let $\tilde{A} = F(B)$ and $\tilde{B} = G(A)$. Prove that:
(a) $U(\tilde{A}) \approx V(\tilde{B})$.
(b) $V \circ G \approx hom_{\mathscr{A}}(\_\_, \tilde{A})$ and $U \circ F \approx hom_{\mathscr{B}}(\_\_, \tilde{B})$.
If, moreover, $\mathscr{B}$ has products, then show that

(c) for each $\mathscr{A}$-object X there exists a monomorphism

$$m_X: G(X) \to \tilde{B}^{U(X)}$$

such that $V(m_X)$ is the embedding of

$$V(G(X)) \approx hom_{\mathscr{A}}(X, \tilde{A})$$

into

$$hom_{\mathbf{Set}}(U(X), U(\tilde{A})) \approx hom_{\mathbf{Set}}(U(X), V(\tilde{B})) = (V(\tilde{B}))^{U(X)} \approx V(\tilde{B}^{U(X)}).$$

Compare these results with the examples in 10.6 and 14.18.

§31 FREE OBJECTS

31.1 DEFINITION

If $G: \mathscr{A} \to$ **Set** is a set-valued functor and (u, A) is a G-universal map for the set X, then A is called a **G-free object over** X and the morphism $u: X \to G(A)$ is called the **insertion of the generators** X **into** A. We say that $\mathscr{A}$ **has G-free objects** provided that for each set X there exists a G-free object.

31.2 EXAMPLES

If $(\mathscr{A}, U)$ is the concrete category of groups, then the U-free objects of $\mathscr{A}$ are exactly the free groups. Likewise, free R-modules, free rings, free monoids, free semigroups, free lattices, and free boolean algebras are exactly the U-free objects, in the sense of the above definition, for the corresponding concrete categories with forgetful functor U.

31.3 NOTATIONAL REMARK

Throughout the remainder of this section, we will assume that $G: \mathscr{A} \to$ **Set** is a functor, and we will simply use the term "free" rather than "G-free".

31.4 PROPOSITION

(1) *An $\mathscr{A}$-object is free over the empty set provided that it is an initial object of $\mathscr{A}$.*

(2) *An $\mathscr{A}$-object is free over a singleton set provided that it represents G.*

(3) *$\mathscr{A}$ has free objects if and only if G has a left adjoint.*

(4) *If at least one non-empty set has a free object, then G preserves limits.* □

In the preceding chapter we have seen that if (u, A) is any G-universal map, then u extremally G-generates A (26.6). However, an insertion of the generators $u: X \to G(A)$ need not be an injective function as the following simple example shows: Let $\mathscr{A}$ be the full subcategory of **Set** whose objects are the singletons and the empty set, and let $G: \mathscr{A} \to$ **Set** be the embedding functor. Then $\mathscr{A}$ has free objects, but for sets that have more than one element, the insertion of the generators is not an injective function. The following theorem shows, however, that it is rarely the case that insertions of generators are not injective.

31.5 THEOREM

If $\mathscr{A}$ contains at least one object A for which $G(A)$ has more than one element, then for each free object B, the insertion of the generators $X \xrightarrow{u} G(B)$ is an injective function.

Proof: If $u(x) = u(y)$, then for all functions $f\colon X \to G(A)$, there is some $\bar{f}\colon B \to A$ such that $f = G(\bar{f}) \circ u$; hence $f(x) = f(y)$. Since $G(A)$ has at least two elements, this implies that $x = y$. □

Every free R-module is known to be projective. A corresponding theorem does not hold for arbitrary concrete categories. For example, the discrete Hausdorff spaces are U-free, yet they are not projective in the concrete category $(\mathbf{Top}_2, U)$. A corresponding theorem *will* become true, however, if we modify somewhat our definition of projectivity.

31.6 DEFINITION

Let $\mathscr{E}$ be a class of $\mathscr{A}$-morphisms. An $\mathscr{A}$-object P is called **$\mathscr{E}$-projective** provided that the functor $hom(P, \_)\colon \mathscr{A} \to \mathbf{Set}$ sends morphisms in $\mathscr{E}$ to surjective functions.

If $\mathscr{E}$ is the class of all regular epimorphisms, then P is called **regular-projective**.

If $\mathscr{E}$ is the class of all extremal epimorphisms, then P is called **extremal-projective**.

If $\mathscr{E}$ is the class of all $\mathscr{A}$-morphisms f for which $G(f)$ is surjective, then P is called **sur-projective**.

31.7 EXAMPLE

If $\mathscr{E}$ is the class of all epimorphisms in $\mathscr{A}$, then P is $\mathscr{E}$-projective if and only if it is projective (cf. 12.14).

31.8 PROPOSITION

Each free object is sur-projective.

Proof: Let (u, A) be a G-universal map for X and let $f\colon A \to B$ and $g\colon C \to B$ be $\mathscr{A}$-morphisms such that $G(g)$ is surjective. Then, clearly, there exists a function $h\colon X \to G(C)$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{u} & G(A) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle G(f)} \\ G(C) & \xrightarrow[G(g)]{} & G(B) \end{array}$$

commutes.

Since (u, A) is a G-universal map for X, there exists a unique $\bar{h}\colon A \to C$ with $G(\bar{h}) \circ u = h$. Consequently

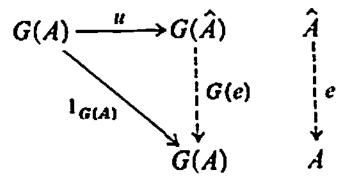
$$G(g \circ \bar{h}) \circ u = G(g) \circ G(\bar{h}) \circ u = G(g) \circ h = G(f) \circ u,$$

so that $g \circ \bar{h} = f$ (26.6), i.e., $hom(A, g)(\bar{h}) = f$. Hence, A is sur-projective. □

31.9 PROPOSITION

If $\mathscr{A}$ has free objects, then for each $\mathscr{A}$-object A there exists a free object $\hat{A}$ and a morphism $e\colon \hat{A} \to A$ such that $G(e)$ is a surjection. In other words, each $\mathscr{A}$-object *is a surjective image of some free object. Furthermore, if G is faithful, then e can be chosen so as to be an epimorphism.*

Proof: Let $(u, \hat{A})$ be a G-universal map for the set $G(A)$. Then there exists a morphism $e: \hat{A} \to A$ such that the triangle



commutes. Since $G(e)$ is a retraction, it must be a surjective function. If G is faithful, it reflects epimorphisms, so that e must be an epimorphism. □

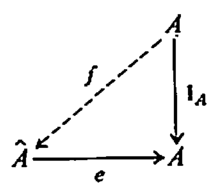
31.10 PROPOSITION

If $\mathscr{A}$ has free objects, then for each $\mathscr{A}$-object A, the following conditions are equivalent:

(1) *A is sur-projective.*

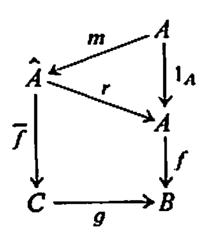
(2) *A is a "retract" of a free object; i.e., there is some free object $\hat{A}$ and some retraction $r: \hat{A} \to A$.*

Proof: To see that (1) implies (2), recall that according to the previous proposition (31.9) there is a free object $\hat{A}$ and a morphism $e: \hat{A} \to A$ such that $G(e)$ is surjective. Consequently by (1) there exists a morphism $f: A \to \hat{A}$ such that the triangle



commutes. Hence e is a retraction.

To see that (2) implies (1), let $\hat{A}$ be a free object and let $r: \hat{A} \to A$ be a retraction. Then there is a morphism m with $r \circ m = 1_A$. Let $f: A \to B$ and $g: C \to B$ be morphisms with $G(g)$ surjective. Since $\hat{A}$ is sur-projective (31.8), there exists a morphism $\bar{f}: \hat{A} \to C$ such that $g \circ \bar{f} = f \circ r$. Consequently $\bar{f} = \bar{f} \circ m$ has the property that



$g \circ \bar{f} = g \circ \bar{f} \circ m = f \circ r \circ m = f \circ 1_A = f.$ □

Now we will show that for any concrete category, each free object over a non-empty set is a separator. First, however, we establish the following more general result.

31.11 PROPOSITION

If $H: \mathscr{C} \to \mathscr{D}$ is faithful, (u, C) is an H-universal map for D, and D is a separator for $\mathscr{D}$, then C is a separator for $\mathscr{C}$.

Proof: Let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be a pair of $\mathscr{C}$-morphisms such that for each $\mathscr{C}$-morphism $k: C \to A$, $f \circ k = g \circ k$. Then for each $k: C \to A$, we have

$$H(f) \circ (H(k) \circ u) = H(g) \circ (H(k) \circ u).$$

Since (u, C) is an H-universal map for D, this implies that $H(f) \circ \hat{k} = H(g) \circ \hat{k}$, for each $\mathscr{D}$-morphism $\hat{k}: D \to G(A)$. Since D is a separator for $\mathscr{D}$, it follows that $H(f) = H(g)$. Hence, since H is faithful, $f = g$. □

31.12 COROLLARY

If $(\mathscr{A}, G)$ is concrete, then each free object over a non-empty set is a separator for $\mathscr{A}$. □

Our next consideration is the question of whether or not non-isomorphic sets must have non-isomorphic free objects. The example following 31.4 shows that this is not always the case. Indeed, Exercises 31D and 32I show that it may not even be true for "algebraic" categories. However, in many cases it is true, as the following proposition shows:

31.13 LEMMA

Suppose that $\mathscr{A}$ has an object A with $1 < card(G(A)) < \aleph_0$. If $u: X \to G(\hat{A})$ is the insertion of the generators into the free object $\hat{A}$ over X and if $g: Y \to G(\hat{A})$ G-generates $\hat{A}$, then $card\ X \leq card\ Y$.

Proof: Let $card\ X = k$, $card\ Y = m$, and $card\ G(A) = n$. Since g G-generates $\hat{A}$, we have

$$card(hom_{\mathscr{A}}(\hat{A}, A)) \leq card(hom_{Set}(Y, G(A))) = n^m.$$

Likewise, since $(u, \hat{A})$ is a G-universal map for X, it follows that

$$card(hom_{\mathscr{A}}(\hat{A}, A)) = card(hom_{Set}(X, G(A))) = n^k.$$

Consequently $n^k \leq n^m$; so that since n is finite and greater than 1, we have $k \leq m$. □

31.14 PROPOSITION

Suppose that $\mathscr{A}$ has an object A with $1 < card(G(A)) < \aleph_0$, and let $\hat{A}_1$ and $\hat{A}_2$ be free over X_1 and X_2, respectively. If $\hat{A}_1$ and $\hat{A}_2$ are isomorphic, then so are X_1 and X_2. □

EXERCISES

31A. Suppose that $\mathscr{A}$ is complete, well-powered and co-(well-powered), and that $U: \mathscr{A} \to$ **Set** is faithful and representable. Prove that $\mathscr{A}$ is cocomplete if and only if U has a left adjoint. [Use 23.17.]

31B. *Subcategories Closed Under Certain Constructions II*
Let $G: \mathscr{C} \to$ **Set** be a set-valued functor and for each full subcategory $\mathscr{A}$ of $\mathscr{C}$, let:

$P\mathscr{A}$ = the full subcategory of $\mathscr{C}$ whose objects are (the object part of) products of $\mathscr{A}$-objects.
$S\mathscr{A}$ = the full subcategory of $\mathscr{C}$ whose objects are (the object part of) subobjects of $\mathscr{A}$-objects.
$Q\mathscr{A}$ = the full subcategory of $\mathscr{C}$ whose objects are surjective images of $\mathscr{A}$-objects; i.e., all objects B for which there exists an $\mathscr{A}$-object A and an $\mathscr{A}$-morphism $e: A \to B$ where $G(e)$ is a surjective function.

Prove that:

(a) $QQ\mathscr{A} = Q\mathscr{A} \supset \mathscr{A}$.
(b) If $\mathscr{C}$ has products and G preserves them, then $PQ\mathscr{A} \subset QP\mathscr{A}$.
(c) If $\mathscr{C}$ has products and G preserves them, then $QP\mathscr{A}$ is the smallest full subcategory of $\mathscr{C}$ that contains $\mathscr{A}$ and is closed under the formation of products and surjective images.
(d) If $\mathscr{C}$ has pullbacks and G preserves them, then $SQ\mathscr{A} \subset QS\mathscr{A}$.
(e) If $\mathscr{C}$ has pullbacks and G preserves them, then $QS\mathscr{A}$ is the smallest full subcategory of $\mathscr{C}$ that contains $\mathscr{A}$ and is closed under the formation of subobjects and surjective images.
(f) If $\mathscr{C}$ is complete and G preserves limits, then $QSP\mathscr{A}$ is the smallest full subcategory of $\mathscr{C}$ that contains $\mathscr{A}$ and is closed under the formation of products, subobjects, and surjective images (see Exercise 23C).

31C. Prove that if $\mathscr{A}$ has coproducts and A is a free object in $\mathscr{A}$ over a one-element set, then B is a free object over an m-element set if and only if $B = {}^{m}A$.

31D. (Jónsson-Tarski)
Let $\mathscr{A}$ be the category that has objects all quadruples of the form $(A, \cdot, l, r)$, where A is a set, $\cdot$ is a binary operation on A; i.e.,

$$\cdot : A \times A \to A,$$

and l and r are unary operations on A such that for all $x, y \in A$,

$$l(x \cdot y) = x, \qquad r(x \cdot y) = y, \qquad l(x) \cdot r(x) = x;$$

and such that $Mor(\mathscr{A})$ consists of all functions

$$f: (A, \cdot, l, r) \to (\bar{A}, \bar{\cdot}, \bar{l}, \bar{r})$$

where

$$f: A \to \bar{A}$$
$$f(x \cdot y) = f(x) \,\bar{\cdot}\, f(y)$$
$$f(l(x)) = \bar{l}(f(x))$$
$$f(r(x)) = \bar{r}(f(x)).$$

Let $U: \mathscr{A} \to \mathbf{Set}$ be the obvious forgetful functor. Prove that:

(1) $\mathscr{A}$ has U-free objects.

(2) $(A, \cdot, l, r)$ is an $\mathscr{A}$-object if and only if

$$\cdot : A \times A \to A$$

is bijective, and

$$l = \pi_1(\cdot)^{-1}, \qquad r = \pi_2(\cdot)^{-1},$$

where

$$\pi_1, \pi_2 : A \times A \to A$$

are the projection functions.

(3) Any two U-free objects of $\mathscr{A}$ over finite non-empty sets are isomorphic. [First show that X is U-free over the set $M \uplus \{x, y\}$ implies that it is U-free over $M \cup \{x \cdot y\}$.]

§32 ALGEBRAIC CATEGORIES AND ALGEBRAIC FUNCTORS

Several of the categories usually studied in algebra—such as **SGrp**, **Mon**, **Grp**, R-**Mod**, **Rng**, R-**Alg**, **BooAlg**, and **Lat**—have many properties in common. Because of this, there have been numerous attempts to study them simultaneously (universal algebra) and to abstract the essence of their common properties via a general concept of "algebraic category".

At first it was thought that algebraic categories should be characterized by the fact that they can be defined by means of *finite* "operations" and "identities". Later it was observed that "finiteness" was not really essential and that categories such as α-complete lattices, α-complete boolean algebras, complete atomic boolean algebras, and even compact Hausdorff spaces behave in many respects like the algebraic categories mentioned above. Because of this, more general concepts have evolved—either constructively by means of "varietal structures" or "triples over **Set**" or, more recently, axiomatically. In this text we will use the axiomatic approach.

As is easily seen, each of the above "algebraic" categories is complete and cocomplete, and for each there exists a naturally associated forgetful set-valued functor U that has a left adjoint (i.e., for each set there exists a U-free object). Moreover, in each case U preserves and reflects regular epimorphisms. It is this last property (namely that each morphism, f, for which $U(f)$ is surjective, must be a regular epimorphism) that is typical for algebraic categories, and which (in conjunction with the other properties mentioned above) causes algebraic categories to behave so nicely.

Algebraic Categories

32.1 DEFINITION

A concrete category $(\mathscr{A}, U)$ is called **algebraic** provided that it satisfies the following three conditions:

(1) $\mathscr{A}$ has coequalizers.

(2) U has a left adjoint.

(3) U preserves and reflects regular epimorphisms.

32.2 EXAMPLES

(1) The following concrete categories are algebraic: **Set**, **pSet**, **SGrp**, **Mon**, **Grp**, **Rng**, R-**Mod**, R-**Alg**, torsion-free abelian groups, **Lat**, α-complete lattices, distributive lattices, **BooAlg**, α-complete boolean algebras, (commutative) C^*-algebras [together with the unit disc functor (30H)], **CompT$_2$**, compact abelian groups, and zero-dimensional compact Hausdorff spaces (see the remark that follows).

(2) The following concrete categories are not algebraic:

(i) **Top**, **POS**, and **Cat** (for these, U does not reflect regular epimorphisms).

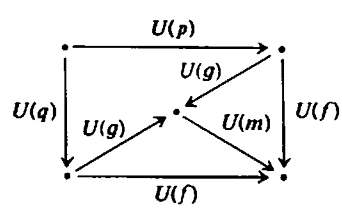
(ii) **Field**, abelian torsion groups, complete lattices, complete boolean algebras and the "usual" concrete category of (commutative) C^*-algebras (for these, U has no left adjoint).

For almost all of the examples in 32.2(1) it has already been established that $\mathscr{A}$ is complete and has coequalizers and that the forgetful functor has a left adjoint. It is also easy to see that in each case the forgetful functor preserves regular epimorphisms and reflects isomorphisms. Thus, the following theorem shows that for each, the forgetful functor reflects regular epimorphisms as well.

32.3 THEOREM

Let $(\mathscr{A}, U)$ be a concrete category that has pullbacks and coequalizers and for which U preserves pullbacks and regular epimorphisms. Then U preserves and reflects monomorphisms, $\mathscr{A}$ is uniquely (regular epi, mono)-factorizable, and U preserves these factorizations. If, in addition, U reflects isomorphisms, then U reflects regular epimorphisms and thus also reflects the (regular epi, mono)-factorizations.

Proof: Since U is faithful, it reflects monomorphisms (12.8) and since it preserves pullbacks, it must preserve monomorphisms (24.5). Let f be an $\mathscr{A}$-morphism, let (p, q) be the congruence relation of f, and let (g, G) be the coequalizer of p and q. Then there exists a unique morphism m with $f = m \circ g$. Consider the following commutative diagram in **Set**.



Since U preserves regular epimorphisms, $U(g)$ is surjective. From the fact that the outer and inner squares are pullback squares (since U preserves pullbacks), and the fact that the diagram commutes, it follows that $U(m)$ is injective. Since U reflects monomorphisms, m must be a monomorphism. Hence $f = m \circ g$ is a (regular epi, mono)-factorization of f. Clearly, U preserves the factorization and the factorization is unique (17.18). Now suppose that U reflects isomorphisms and f is an $\mathscr{A}$-morphism for which $U(f)$ is surjective. Let $f = m \circ e$ be the (regular epi, mono)-factorization of f. Then since U preserves monomorphisms, $U(m)$ is bijective so that m must be an $\mathscr{A}$-isomorphism. Hence f must be a regular epimorphism. □

32.4 THEOREM

If a concrete category $(\mathscr{A}, U)$ has the following properties:

(1) *$\mathscr{A}$ is complete and U preserves limits;*

(2) *U preserves regular epimorphisms and reflects isomorphisms;*

(3) *each set extremally U-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects; and*

(4) *$\mathscr{A}$ is well-powered and extremally co-(well-powered);*

then $(\mathscr{A}, U)$ must be algebraic.

Proof: Conditions (1), (3), and (4) imply that U has a left adjoint (28.9), and conditions (1) and (4) imply that $\mathscr{A}$ has coequalizers (23.11). Hence by the preceding Theorem (32.3) U must reflect regular epimorphisms. □

Actually, we shall see that a concrete category $(\mathscr{A}, U)$ is algebraic if and only if it satisfies the four conditions of Theorem 32.4 (see 32.5, 32.10, and 32.12). Note especially that the non-algebraic categories **Top**, **POS**, and $\mathbf{Top_2}$ satisfy all of the conditions of the above theorem except for the property that U reflects isomorphisms.

32.5 PROPOSITION

If $(\mathscr{A}, U)$ is algebraic, then U preserves and reflects monomorphisms and isomorphisms.

Proof: Since U is a functor, it preserves isomorphisms; since it has a left adjoint it preserves limits (and thus preserves monomorphisms (24.5)); and since it is faithful, it reflects monomorphisms (12.8). By definition, U reflects regular epimorphisms so that since a morphism is an isomorphism if and only if it is a regular epimorphism and a monomorphism (16.16 dual), U must reflect isomorphisms. □

32.6 COROLLARY

Let $(\mathscr{A}, U)$ be algebraic and let f be an $\mathscr{A}$-morphism. Then

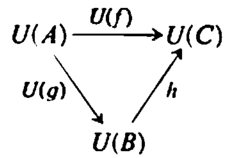
(1) *f is a monomorphism if and only if $U(f)$ is injective.*

(2) *f is a regular epimorphism if and only if $U(f)$ is surjective.*

(3) *f is an isomorphism if and only if $U(f)$ is bijective.* □

32.7 PROPOSITION

Let $(\mathscr{A}, U)$ be algebraic, $f: A \to C$ be an $\mathscr{A}$-morphism, $g: A \to B$ be a regular epimorphism in $\mathscr{A}$, and $h: U(B) \to U(C)$ be a function such that the triangle



commutes. Then there exists a unique $\mathscr{A}$-morphism $\bar{h}: B \to C$ such that $U(\bar{h}) = h$.

Proof: Since g is a regular epimorphism, (g, B) must be the coequalizer of some pair (p, q) of $\mathscr{A}$-morphisms. Now

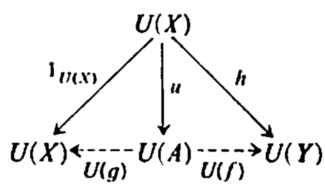
$$U(f \circ p) = U(f) \circ U(p) = h \circ U(g) \circ U(p)$$
$$= h \circ U(g) \circ U(q) = U(f) \circ U(q) = U(f \circ q).$$

Since U is faithful, we have $f \circ p = f \circ q$. Hence by the definition of coequalizer, there exists a unique morphism $\bar{h}: B \to C$ such that $f = \bar{h} \circ g$. Since $U(g)$ is an epimorphism, $U(\bar{h}) \circ U(g) = U(f) = h \circ U(g)$ implies that $U(\bar{h}) = h$. Uniqueness follows from the fact that U is faithful. □

32.8 PROPOSITION

Let $(\mathscr{A}, U)$ be algebraic, $(X, (g_i))$ and $(Y, (m_i))$ be sources in $\mathscr{A}$ such that $(U(Y), (U(m_i)))$ is a mono-source in Set, *and $h: U(X) \to U(Y)$ be a function such that $U(g_i) = U(m_i) \circ h$ for each i. Then there exists a unique $\mathscr{A}$-morphism $\bar{h}: X \to Y$ such that $U(\bar{h}) = h$.*

Proof: Let (u, A) be a U-universal map for $U(X)$. Then there exist $\mathscr{A}$-morphisms g and f such that the diagram



commutes. Thus

$$U(g_i \circ g) \circ u = U(g_i) \circ 1_{U(X)} = U(m_i) \circ h = U(m_i \circ f) \circ u,$$

so that since u U-generates A, it follows that

$$g_i \circ g = m_i \circ f \qquad \text{for each} \quad i.$$

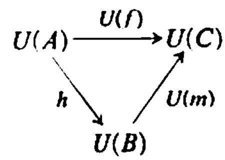
Now, for each i

$$U(m_i) \circ h \circ U(g) = U(g_i \circ g) = U(m_i \circ f) = U(m_i) \circ U(f)$$

so that since $(U(Y), (U(m_i)))$ is a mono-source, $h \circ U(g) = U(f)$. But $U(g)$ is a retraction, so that since U reflects regular epimorphisms, g must be a regular epimorphism. Hence f, g, and h satisfy the hypotheses of Proposition 32.7. Consequently there is a unique $\mathscr{A}$-morphism h such that $U(\bar{h}) = h$. □

32.9 COROLLARY

Let $(\mathscr{A}, U)$ be algebraic, $f\colon A \to C$ be an $\mathscr{A}$-morphism, $m\colon B \to C$ be a monomorphism in $\mathscr{A}$, and $h\colon U(A) \to U(B)$ be a function such that the triangle



commutes. Then there exists a unique $\mathscr{A}$-morphism $\bar{h}\colon A \to B$ such that $U(\bar{h}) = h$.

Proof: Since U preserves monomorphisms, $(U(B), (Um))$ is a mono-source. Apply the proposition. □

32.10 PROPOSITION

Every algebraic category is well-powered and regular co-(well-powered).

Proof: We already know that every concrete category is regular co-(well-powered) (16N). To show that an algebraic category $(\mathscr{A}, U)$ is well-powered, let (B, h) and (C, g) be subobjects of an $\mathscr{A}$-object A, such that $(U(B), U(h))$ and $(U(C), U(g))$ are isomorphic subobjects of $U(A)$. By the above corollary, (B, h) and (C, g) must be isomorphic subobjects of A. Thus since Set is well-powered and U preserves monomorphisms, $\mathscr{A}$ must be well-powered. □

32.11 PROPOSITION

If $(\mathscr{A}, U)$ is algebraic, then U reflects limits.

Proof: Let $D\colon I \to \mathscr{A}$ be a functor and let $(L, (l_i))$ be a source in $\mathscr{A}$ such that $(U(L), (U(l_i)))$ is a limit of $U \circ D$. Since U is faithful, $(L, (l_i))$ must be a natural source for D. If $(X, (g_i))$ is some natural source for D, then $(U(X), (U(g_i)))$ is a natural source for $U \circ D$, so that there exists a unique function $h\colon U(X) \to U(L)$ such that $U(g_i) = U(l_i) \circ h$, for each i. Since $(U(L), (U(l_i)))$ is a limit, it is a mono-source. Hence by Proposition 32.8 there is a unique $\mathscr{A}$-morphism $\bar{h}$ such that $U(\bar{h}) = h$; i.e., (since U is faithful) such that for each i, $g_i = l_i \circ \bar{h}$. Consequently, $(L, (l_i))$ is a limit of D. □

32.12 THEOREM

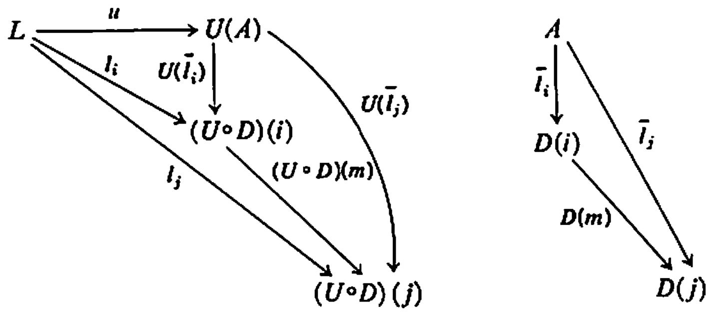
If $(\mathscr{A}, U)$ is algebraic, then $\mathscr{A}$ is complete and U preserves and reflects limits.

Proof: Since U has a left adjoint, it preserves limits, and by the above proposition (32.11) it also reflects them. Thus we need only show that $\mathscr{A}$ is complete.

Let $D\colon I \to \mathscr{A}$ be a small functor, let $(L, (l_i))$ be a limit of $U \circ D$, and let (u, A) be a U-universal map for L. Then for each i there is a unique morphism $\bar{l}_i\colon A \to D(i)$ such that $U(\bar{l}_i) \circ u = l_i$. Now for each I-morphism $m\colon i \to j$, the equality

$$U(\bar{l}_j) \circ u = l_j = (U \circ D)(m) \circ l_i = (U \circ D)(m) \circ U(\bar{l}_i) \circ u = U(D(m) \circ \bar{l}_i) \circ u$$

and the fact that u U-generates A (26.6) implies that $\bar{l}_j = D(m) \circ \bar{l}_i$.



Consequently, $(A, (\bar{l}_i))$ is a natural source for D, so that $(U(A), (U(\bar{l}_i)))$ is a natural source for $U \circ D$. Hence there exists a unique function f: $U(A) \to L$ such that for each i, $U(\bar{l}_i) = l_i \circ f$. Since $(L, (l_i))$ is a limit, it is a mono-source, so that the equations

$$l_i \circ f \circ u = U(\bar{l}_i) \circ u = l_i = l_i \circ 1_L$$

imply that $f \circ u = 1_L$. Thus f is a surjection and, as such, is a regular epimorphism. Therefore (f, L) must be the coequalizer of some pair of functions $C \overset{r}{\underset{s}{\rightrightarrows}} U(A)$. Let (v, B) be a U-universal map for C. Then there exist unique morphisms $\bar{r}$, $\bar{s}$: $B \to A$ such that

$$r = U(\bar{r}) \circ v \qquad \text{and} \qquad s = U(\bar{s}) \circ v.$$

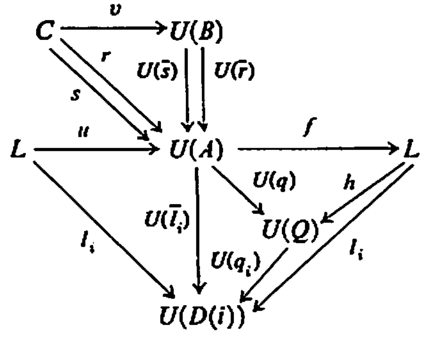
Let $(q, Q) \approx Coeq(\bar{r}, \bar{s})$. Then the equality

$$\begin{aligned} U(q) \circ r &= U(q) \circ (U(\bar{r}) \circ v) = U(q \circ \bar{r}) \circ v \\ &= U(q \circ \bar{s}) \circ v = U(q) \circ (U(\bar{s}) \circ v) = U(q) \circ s \end{aligned}$$

implies that there is a unique function h: $L \to U(Q)$ with $U(q) = h \circ f$. Since $U(q)$ is surjective, h must be surjective also. Furthermore, for each i,

$$U(\bar{l}_i \circ \bar{r}) \circ v = U(\bar{l}_i) \circ r = l_i \circ f \circ r = l_i \circ f \circ s = U(\bar{l}_i) \circ s = U(\bar{l}_i \circ \bar{s}) \circ v.$$

This, together with the fact that v U-generates B, implies that $\bar{l}_i \circ \bar{r} = \bar{l}_i \circ \bar{s}$. Consequently, for each i there exists a unique morphism q_i: $Q \to D(i)$ such that $\bar{l}_i = q_i \circ q$.



Now

$$U(q_i) \circ h \circ f = U(q_i) \circ U(q) = U(q_i \circ q) = U(l_i) = l_i \circ f,$$

so that since f is a surjection, $U(q_i) \circ h = l_i$, for each i. This provides a factorization of the (extremal mono)-source $(L, (l_i))$ (19.13), where h is an epimorphism. Hence h is an isomorphism. Thus $(U(Q), (U(q_i)))$ is a limit of $U \circ D$. Since U reflects limits (32.11), $(Q, (q_i))$ must be a limit of D. □

32.13 COROLLARY

Each algebraic category $(\mathscr{A}, U)$ is uniquely (regular epi, mono)-factorizable, and U preserves and reflects these factorizations.

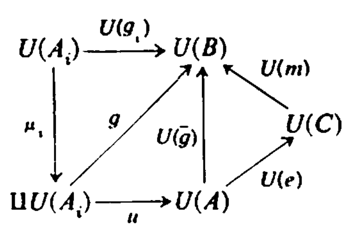
Proof: Immediate from Theorem 32.3. □

32.14 THEOREM

Every algebraic category is cocomplete.

Proof: If $(\mathscr{A}, U)$ is algebraic, then $\mathscr{A}$ has coequalizers. Thus it remains to be shown that $\mathscr{A}$ has coproducts (23.8). To do this, it is sufficient to show that for each small discrete category I, the "constant functor" functor $G: \mathscr{A} \to \mathscr{A}^I$ has a left adjoint (26I). It is easily seen that G preserves limits (28F(a)) so that since $\mathscr{A}$ is complete and well-powered, it suffices to show that each $\mathscr{A}^I$-object extremally G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects (28.9); i.e., that each I-indexed family of $\mathscr{A}$-objects $(A_i)_I$ is the domain of at most a set of pairwise non-isomorphic (extremal epi)-sinks.

Let $((\mu_i), \amalg U(A_i))$ be the coproduct in **Set** of the family $(U(A_i))_I$, and let (u, A) be a U-universal map for $\amalg U(A_i)$. We claim that if $(A_i \xrightarrow{g_i} B, B)$ is an (extremal epi)-sink with domain $(A_i)_I$, then B must be (the object part of) a regular quotient of A. (There is at most a set of such pairwise non-isomorphic regular quotients, since $\mathscr{A}$ is regular co-(well-powered).) Since $((\mu_i), \amalg U(A_i))$ is a coproduct, there is a unique function g such that the top left triangle in the following diagram commutes:



Since (u, A) is a U-universal map for $\amalg U(A_i)$, there exists a unique $\mathscr{A}$-morphism $\bar{g}: A \to B$ such that the middle triangle commutes. It remains to be shown that $(\bar{g}, B)$ is a regular quotient of A.

Let

$$A \xrightarrow{\bar{g}} B = A \xrightarrow{e} C \xrightarrow{m} B$$

be a (regular epi, mono)-factorization of $\bar{g}$. Then the commutativity of the above diagram and Corollary 32.9 imply that for each i there exists a unique

$\mathscr{A}$-morphism $h_i: A_i \to C$ with $U(h_i) = U(e) \circ u \circ \mu_i$. Hence since U is faithful, $(g_i)_I = m \circ (h_i)_I$ is a factorization of the sink $((g_i)_I, B)$. But since m is a monomorphism and $((g_i)_I, B)$ is an (extremal epi)-sink, m must be an isomorphism. Thus $\bar{g}$ must be a regular epimorphism. □

Algebraic Functors

If $(\mathscr{A}, U)$ is an algebraic category, then by definition the forgetful functor U preserves and reflects regular epimorphisms and has a left adjoint. We will now see that each functor between algebraic categories that "forgets part of the structure", such as the forgetful functor from **Rng** to **Ab** ("forgetting multiplication"), from **Rng** to **Mon** ("forgetting addition"), or from compact topological groups to **Grp** ("forgetting the topology"), has the same properties (32.20). On the other hand it will be shown that each functor that has these properties is essentially one that "forgets part of the structure"; i.e., is one that is faithful (32.17). It is this concept of an "algebraic functor" that plays a central role in categorically distinguishing algebra from other areas of mathematics.

32.15 DEFINITION

A functor is called an **algebraic functor** provided that it has a left adjoint and preserves and reflects regular epimorphisms.

32.16 PROPOSITION

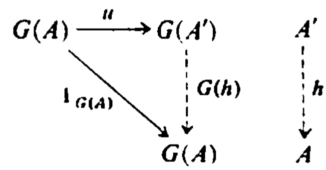
The composition of algebraic functors is algebraic.

Proof: Recall that the composition of functors that have left adjoints has a left adjoint (27.8). □

32.17 PROPOSITION

Each algebraic functor is faithful.

Proof: Suppose that $G: \mathscr{A} \to \mathscr{B}$ is algebraic and $A \underset{g}{\overset{f}{\rightrightarrows}} \hat{A}$ are $\mathscr{A}$-morphisms such that $G(f) = G(g)$. Let (u, A') be a G-universal map for $G(A)$. Then there is a unique $\mathscr{A}$-morphism $h: A' \to A$ such that the triangle



commutes.

Hence since u G-generates A', the equality

$$G(f \circ h) \circ u = G(g \circ h) \circ u$$

implies that $f \circ h = g \circ h$. Since $G(h)$ is a retraction, it is a regular epimorphism (16.15 dual), so that since G reflects regular epimorphisms, h must be an epimorphism; hence $f = g$. □

32.18 PROPOSITION

Each algebraic functor

(1) *preserves and reflects monomorphisms;*
(2) *preserves and reflects isomorphisms;*
(3) *preserves and reflects (regular epi, mono)-factorizations.* □

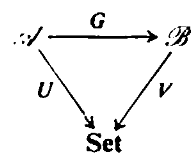
32.19 PROPOSITION

If $\mathscr{A}$ *has coequalizers and if* $U: \mathscr{A} \to \mathbf{Set}$, *then the following are equivalent:*

(1) $(\mathscr{A}, U)$ *is an algebraic category.*
(2) U *is an algebraic functor.* □

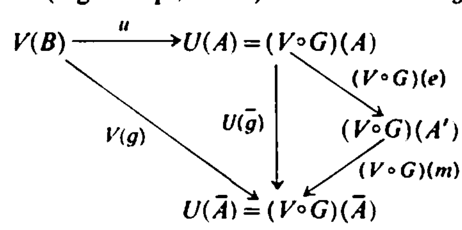
32.20 THEOREM

If $(\mathscr{A}, U)$ *and* $(\mathscr{B}, V)$ *are algebraic categories and* $G: \mathscr{A} \to \mathscr{B}$ *is any functor such that the triangle*



commutes, then G is algebraic.

Proof: Since U and V preserve and reflect regular epimorphisms and limits, G must do likewise. Thus we need only show that G has a left adjoint, and to do this it is sufficient to show that each $\mathscr{B}$-object, B, extremally G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects (28.9). Let (u, A) be a U-universal map for $V(B)$ and let $g: B \to G(\bar{A})$ extremally G-generate $\bar{A}$. Since $\mathscr{A}$ is regular co-(well-powered), it is sufficient to show that for some morphism $\bar{g}$, $(\bar{g}, \bar{A})$ is a regular quotient object of A. By the definition of universal map, there exists a unique $\mathscr{A}$-morphism $\bar{g}: A \to \bar{A}$ such that $V(g) = U(\bar{g}) \circ u$. Let $A \xrightarrow{\bar{g}} \bar{A} = A \xrightarrow{e} A' \xrightarrow{m} \bar{A}$ be the (regular epi, mono)-factorization of $\bar{g}$ (32.13).



By Corollary 32.9, there exists a unique $\mathscr{B}$-morphism $h: B \to G(A')$ such that $V(h) = (V \circ G)(e) \circ u$. Since V is faithful, this implies that

$$B \xrightarrow{g} G(\bar{A}) = B \xrightarrow{h} G(A') \xrightarrow{G(m)} G(\bar{A}).$$

Since g extremally G-generates $\bar{A}$, this implies that m is an isomorphism. Hence $\bar{g}$ must be a regular epimorphism. □

Note the similarity between this theorem and Theorem 28.12. Since algebraic categories need not be co-(well-powered), the above theorem is not a special case of the earlier one.

Above, algebraic categories have been defined "externally", i.e., as particular concrete categories. The next theorem gives an "internal" description by characterizing those (complete and cocomplete) categories $\mathscr{A}$ for which there exists a set-valued functor U such that $(\mathscr{A}, U)$ is algebraic. Since any such functor must be representable (30.20), this amounts to characterizing those $\mathscr{A}$-objects A for which $hom(A, \_): \mathscr{A} \to \mathbf{Set}$ is algebraic.

32.21 THEOREM

If $\mathscr{A}$ is complete and cocomplete, and if A is an $\mathscr{A}$-object, then the following are equivalent:

(1) *$hom(A, \_)$ is an algebraic functor.*

(2) *$(\mathscr{A}, hom(A, \_))$ is an algebraic category.*

(3) *A is a regular separator and is regular-projective.*

(4) *A is an extremal separator and is regular-projective.*

Proof:

(1) $\Rightarrow$ *(2)*. Immediate from Proposition 32.19.

(2) $\Rightarrow$ *(3)*. Let the functor $hom(A, \_)$ be denoted by U. Since U preserves regular epimorphisms, A is regular projective (31.6). Furthermore, each $\mathscr{A}$-object is a U-surjective image of some U-free object (31.9); i.e., of some copower of A (31C). Since U reflects regular epimorphisms, A must be a regular separator (19.7 dual).

(3) $\Rightarrow$ *(4)*. Immediate from the fact that each regular separator must be an extremal separator.

(4) $\Rightarrow$ *(1)*. Since $\mathscr{A}$ is cocomplete, $U = hom(A, \_)$ has a left adjoint (30.20), and since A is regular-projective, U preserves regular epimorphisms (31.6). Thus it remains to be shown that U reflects regular epimorphisms. Suppose that $f: X \to Y$ is an $\mathscr{A}$-morphism such that $hom(A, \_)(f)$ is surjective. Let

$$X \xrightarrow{f} Y = X \xrightarrow{e} Z \xrightarrow{m} Y$$

be the (regular epi, mono)-factorization of f (32.13). Then

$$hom(A, \_)(m): hom(A, Z) \to hom(A, Y)$$

must be surjective. Thus for each $g \in hom(A, Y)$ there is some $f_g \in hom(A, Z)$ such that

$$g = hom(A, m)(f_g) = m \circ f_g.$$

Since A is an extremal separator, $(hom(A, Y), Y)$ is an (extremal epi)-sink (19E). Thus since m is a monomorphism, it must be an isomorphism. Hence f is a regular epimorphism. □

32.22 COROLLARY

If $(\mathscr{A}, U)$ is an algebraic category and A is a U-free object, then $hom(A, \_)$ is an algebraic functor.

Proof: Since each free object is sur-projective (31.8) and since U preserves regular epimorphisms, each free object must be regular projective. Also it is easily seen that each $\mathscr{A}$-object is a U-surjective image of some copower of A. Since U reflects regular epimorphisms, this implies that A is a regular separator. □

32.23 COROLLARY

If $G: \mathscr{A} \to \mathscr{B}$ where $\mathscr{B}$ is complete and cocomplete and if B is a $\mathscr{B}$-object that is a regular separator, is regular-projective, and has a G-universal map; then G preserves limits.

Proof: Since $hom(B, \_\_)$ is algebraic, it reflects limits (32.11), and since $hom(B, \_\_) \circ G$ is representable (30.13), it preserves limits. Hence G must preserve limits. □

32.24 COROLLARY

If $G: \mathscr{A} \to$ Set and if B is a non-empty set that has a G-universal map, then G preserves limits. □ (Cf. Theorem 30.17.)

32.25 COROLLARY

Let $\mathscr{A}$ be a full subcategory of Set, *containing at least one non-empty set. Then the embedding functor $E: \mathscr{A} \to$ Set preserves limits.* □ (Cf. Corollary 30.18.)

EXERCISES

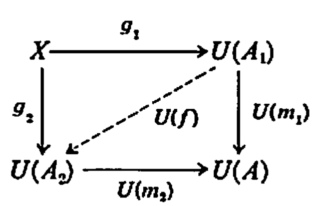
32A. Suppose that $\mathscr{A}$ is a (regular epi, mono)-factorizable category and that $U: \mathscr{A} \to \mathscr{B}$ preserves and reflects monomorphisms. Prove that U reflects regular epimorphisms if and only if it reflects isomorphisms.

32B. Prove that in any algebraic category the regular epimorphisms and the extremal epimorphisms coincide.

32C. Prove that if $(\mathscr{A}, U)$ is algebraic, $m_1: A_1 \to A$ and $m_2: A_2 \to A$ are $\mathscr{A}$-morphisms, $g_1: X \to U(A_1)$ and $g_2: X \to U(A_2)$ are functions such that

$$U(m_1) \circ g_1 = U(m_2) \circ g_2,$$

g_1 extremally U-generates A_1, and m_2 is a monomorphism, then there exists a unique $\mathscr{A}$-morphism $f: A_1 \to A_2$ such that the diagram



commutes.

32D. Prove that if $(\mathscr{A}, U)$ is algebraic, then for each function $f: X \to U(A)$, the pair (f, A) has an essentially unique (extremal U-generating, mono)-factorization.

32E. Prove that if $(\mathscr{A}, U)$ is an algebraic category and B is an $\mathscr{A}$-object, then B is regular-projective if and only if it is a retract of a U-free object.

32F. Prove that in any algebraic category the pullback of a regular epimorphism is a regular epimorphism.

32G. *Finitary Algebraic Categories*

(a) For any algebraic category $(\mathscr{A}, U)$ $[= (\mathscr{A}, hom(A, \_))]$, show that the following conditions are equivalent:

(i) $(\mathscr{A}, U)$ is finitary (see 22E).

(ii) If (l_i, L) is a direct limit, and all the l_i's are monomorphisms, then each morphism $f: B \to L$ from a finitely generated $\mathscr{A}$ object, B, into L can be factored through one of the l_i's.

(iii) If B is a finitely generated $\mathscr{A}$-object, then for each morphism $f: B \to {}^I A$ there exists a finite subset J of I such that f can be factored through the natural injection $\mu_J^I: {}^J A \to {}^I A$.

(iv) A is "abstractly finite", i.e., for each morphism $f: A \to {}^I A$ there exists a finite subset J of I such that f can be factored through the natural injection $\mu_J^I: {}^J A \to {}^I A$.

(v) If B is an $\mathscr{A}$-object and M is a subset of $U(B)$ that extremally U-generates B, and if $(B_i)_I$ is the family of all subalgebras of B that are extremally U-generated by finite subsets M_i of M, then $\{U(B_i) \mid i \in I\}$ covers $U(B)$.

(vi) If $((k_i), K)$ is a direct limit in $\mathscr{A}$, B is an $\mathscr{A}$-object, and $f: U(K) \to U(B)$ is a function such that for each i there exists a morphism f_i with $U(f_i) = f \circ (U(k_i))$ then there exists a morphism $\bar{f}: L \to B$ with $U(\bar{f}) = f$.

(b) In the case that $\mathscr{A}$ is connected, show that the above conditions are equivalent to:

(**) If B is a finitely generated $\mathscr{A}$-object, then for each morphism $f: B \to \coprod_I A_i$, there exists a finite subset K of I such that f can be factored through the natural injection:

$$\mu_K^I: \coprod_K A_i \to \coprod_I A_i.$$

(c) Show that if $(\mathscr{A}, U)$ is a finitary algebraic category, then each $\mathscr{A}$-object A is a direct limit of those subobjects of A that are extremally U-generated by finite sets.

(d) Show that the category of zero-dimensional compact Hausdorff spaces is algebraic, but not finitary.

32H. *The Dual of the Category of Sets*

Let $\mathscr{P}$: $\mathbf{Set}^{op} \to \mathbf{Set}$ be the power-set functor defined by:

$$\mathscr{P}(X) = \{A \mid A \subset X\}$$

$$\mathscr{P}(X \xrightarrow{f} Y)(A) = f^{-1}[A].$$

(a) Prove that $\mathscr{P}$ is faithful.

(b) Prove that $\mathscr{P}$ has a left adjoint. [For each set X, define $u: X \to \mathscr{P}(\mathscr{P}(X))$ by:

$$u(x) = \{A \mid x \in A \subset X\},$$

and show that $(u, \mathscr{P}(X))$ is a $\mathscr{P}$-universal map for X.]

(c) Prove that $\mathscr{P}$ preserves and reflects regular epimorphisms.

(d) Show that $(\mathbf{Set}^{op}, \mathscr{P})$ is an algebraic category.

(e) Prove that the dual of any concretizable category is concretizable (cf. Exercise 12L).

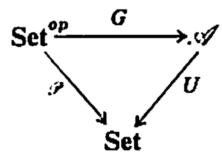
(f) Determine whether or not the dual of each "algebrizable" category is "algebrizable".

(g) Let $(\mathscr{A}, U)$ be the concrete category of complete atomic boolean algebras and complete boolean homomorphisms (see Exercise 14H). Prove that the functor G: $\mathbf{Set}^{op} \to \mathscr{A}$ defined by:

$$G(X) = \text{the complete atomic boolean algebra of subsets of } X.$$

$$G(X \xrightarrow{f} Y)(A) = f^{-1}[A]$$

is an equivalence of categories.
(h) Prove that the triangle



commutes, and conclude that $(\mathscr{A}, U)$ is algebraic.
(i) Construct a functor F: $\mathscr{A} \to \mathbf{Set}^{op}$ and natural isomorphisms η and ε such that $(G, F, \eta, \varepsilon)$ is an equivalence situation.
(j) Prove that each set is an inverse limit of finite sets.
(k) Prove that each $\mathscr{A}$-object is a direct limit of $\mathscr{A}$-objects each of which is extremally U-generated by a finite set.
(l) Prove that $(\mathscr{A}, U)$ is not finitary.

32I. Prove that the category $(\mathscr{A}, U)$ described in Exercise 31D is algebraic.

32J. Let $\mathscr{A}$ be the category of abelian torsion groups, and let U: $\mathscr{A} \to \mathbf{Set}$ be the forgetful functor. Prove that:

(1) $\mathscr{A}$ is complete and cocomplete.
(2) U preserves finite limits.
(3) U preserves and reflects regular epimorphisms.
(4) U reflects congruence-relations.
(5) U does not preserve products.

32K. Let $(\mathscr{A}, U)$ be the concrete category of compact Hausdorff spaces, and let A be an $\mathscr{A}$-object. Prove that
(a) $hom(A, \_)$ reflects regular epimorphisms if and only if $U(A) \neq \emptyset$.
(*b*) $hom(A, \_)$ is algebraic if and only if $U(A) \neq \emptyset$ and A is extremally disconnected (i.e., $int(cl(int\ B)) = cl(int\ B)$ for each subset B of A).

32L. Let $(\mathscr{A}, U)$ be the concrete category of abelian groups, let $\mathscr{B}$ be the full subcategory of $\mathscr{A}$ whose objects are those groups that can be embedded into products of finite abelian groups, let E: $\mathscr{B} \hookrightarrow \mathscr{A}$ denote the embedding functor, and let $V = U \circ E$: $\mathscr{B} \to \mathbf{Set}$ denote the forgetful functor. Prove that
(a) $(\mathscr{B}, V)$ is algebraic.
(b) $(\mathscr{B}, V)$ is finitary.
(c) $(\mathscr{B}, V)$ is not strongly finitary. [*Hint:* Q/Z is a direct limit in $\mathscr{A}$ of its finite subgroups, but does not belong to $\mathscr{B}$ since each homomorphic image of Q/Z is divisible.]